ARTICLES

The Dynamics of Circle Homeomorphisms: A Hands-on Introduction

JAMES A. WALSH
Oberlin College
Oberlin, OH 44074-1019

Introduction

The dynamics of circle homeomorphisms is a deep, beautiful, and surprisingly accessible topic for students in an advanced calculus or introductory real analysis course. The remainder of this article should be considered a proof of this claim. It is structured as a sequence of connected exercises providing the reader with an introduction to the theory. These exercises could be interspersed throughout a semester course in junior-level real analysis, or used collectively as a capstone experience (solutions to the exercises are on the web **[25]** or available by writing the author). We begin with historical background.

Henri Poincaré **[22]** introduced the study of the dynamics of circle homeomorphisms in his attempt to classify solutions to ordinary differential equations (or *flows*) defined on the two-dimensional torus $\mathbb{T}^2$. We think of $\mathbb{T}^2$ as being obtained by identifying points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ if $(x_1, y_1) = (x_2, y_2) + (m, n)$ for some integer pair (m, n). Flows defined on $\mathbb{T}^2$ thus correspond to vector fields $V: \mathbb{R}^2 \to \mathbb{R}^2$, $V(x, y) = (V_1(x, y), V_2(x, y))$, which are 1-periodic in each coordinate as in FIGURE 1 (**[5]**, Ch. 17], **[11]**, §6.1], **[16]**, §1.5, §14.2]). Poincaré was interested in the role the topology of $\mathbb{T}^2$ plays in determining the long-term behavior of solutions.

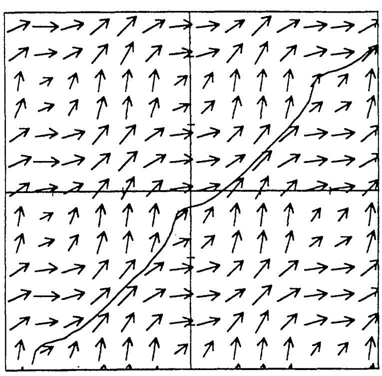


FIGURE 1
The vector field $V(x, y) = (1.1 + \sin(2\pi y), 1.1 - \cos(2\pi x))$ is invariant under integer translations.

Beginning with the simplest case, Poincaré considered differential equations having no equilibrium points (points (x_0, y_0) for which $V(x_0, y_0) = (0, 0)$). These flows exist on $\mathbb{T}^2$ because the Euler characteristic is zero **[10**, §3.5]. Within this class of differential equations, he restricted his study to vector fields with $V_1(x, y) > 0$ and $V_2(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2$—all vectors point "northeast" in such vector fields (see FIGURE 1).

Now consider a meridianal circle $\mathbb{T}$ on the torus and let $\theta \in \mathbb{T}$ (see FIGURE 2). Note that, due to the class of vector fields under consideration, the trajectory beginning at θ must wrap around the torus and intersect $\mathbb{T}$ on the "other side." Poincaré thus

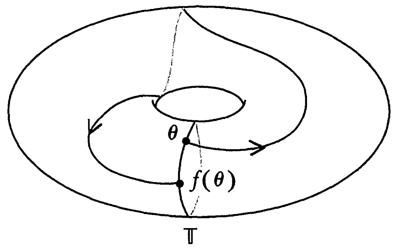


FIGURE 2
Defining the return map $f: \mathbb{T} \to \mathbb{T}$.

defined a map $f: \mathbb{T} \to \mathbb{T}$ which assigns to $\theta \in \mathbb{T}$ the first point of intersection of the trajectory of θ with $\mathbb{T}$. Assuming the vector field V has continuously differentiable components, the existence and uniqueness theorem for ODE's (see, e.g., **[21**, §2.2]) implies the map f is an orientation-preserving (circle) homeomorphism. The qualitative study of ODE's defined on $\mathbb{T}^2$ is thus reduced to the study of iteration of the return map f.

The map f is an example of what is now called a *Poincaré return map*. More generally, such maps allow for the qualitative study of flows in $\mathbb{R}^n$ by considering the Poincaré return map on an appropriately chosen hyperplane in $\mathbb{R}^n$ ([**9**, §1.5], [**11**, §6.1], [**16**, §0.3], [**21**, §3.4], [**24**, §5.8]). What now seems such a natural technique is one of Poincaré's many deep and significant contributions to the field of dynamical systems. For an insightful discussion of Poincaré's work in dynamical systems ("creation of" is perhaps more apt) see the introduction in [**23**].

The study of the dynamics of circle homeomorphisms provides a host of wonderful exercises at the advanced undergraduate level. In the following sections we present an introduction to the subject, but rather than include proofs of statements we provide exercises (with hints where appropriate) which the reader is encouraged to complete. So, with pencil and paper in hand, please read on!

Preliminaries

Much of the following can be found in [**12**]. Let $\mathbb{T}$ denote the real numbers mod 1, i.e., $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We think of $\mathbb{T}$ as a circle of circumference one due to the identification of 0 mod 1 and 1 mod 1 in $\mathbb{R}/\mathbb{Z}$. The value $1/4 \in \mathbb{T}$, for example, represents one-fourth of a "turn" around $\mathbb{T}$. Let $f: \mathbb{T} \to \mathbb{T}$ be an orientation-preserving circle homeomorphism, so that each of f and f^{-1} is continuous, and f preserves the (cyclic) order of points on $\mathbb{T}$. The simplest type of circle homeomorphism is a *rigid rotation* $r_\omega(\theta) = \theta + \omega$, where ω is a fixed real number mod 1.

Given $\theta \in \mathbb{T}$, define $f^0(\theta) = \theta$ and, for integers $i > 0$, $f^i(\theta) = f(f^{i-1}(\theta))$. For $i < 0$, let $f^i(\theta) = f^{-1}(f^{i+1}(\theta))$. We seek to understand the behavior of *orbits* of f

$$o(\theta, f) = o(\theta) = \{f^i(\theta) : i \in \mathbb{Z}\}.$$

The simplest type of orbit is a *periodic orbit*, one for which $f^n(\theta) = \theta$ for some $n \in \mathbb{Z}$. Such a θ-value is called a *periodic point* of period n. Periodic orbits for f correspond to periodic trajectories on $\mathbb{T}^2$ for the class of differential equations considered by Poincaré in **[22]**.

Exercise 1. Consider rigid rotation $r_\omega : \mathbb{T} \to \mathbb{T}$.

(a) Let $\omega = p/q \in \mathbb{Q}$ (throughout, we assume $\gcd(p, q) = 1$). Show that all orbits of r_ω are periodic with the same period.

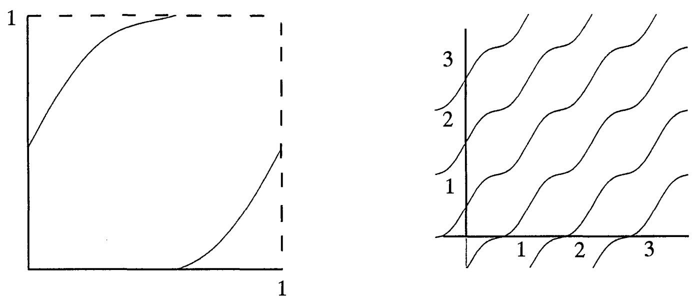
(b) Suppose $\omega \notin \mathbb{Q}$. Show that all orbits of r_ω are dense in $\mathbb{T}$. Hint: For $\theta \in \mathbb{T}$, show that $o(\theta)$ is an infinite set. Use the compactness of $\mathbb{T}$ or the Bolzano-Weierstrass Theorem to conclude that $o(\theta)$ has a limit point in $\mathbb{T}$. Given $\epsilon > 0$, deduce the existence of integers m and n with $r_\omega^n(\theta)$ and $r_\omega^m(\theta)$ less than ϵ apart. Now use the fact that r_ω is rigid rotation.

For $x \in \mathbb{R}$, let $\langle \cdot \rangle$ denote the fractional part of x. Let $\pi : \mathbb{R} \to \mathbb{T}$, $x \mapsto \langle x \rangle$, be the projection map from the reals onto $\mathbb{T}$. Note that π wraps any interval of the form $[x, x+1)$ once around $\mathbb{T}$ since it identifies x and $x+1$. A *lift* of f is a function $F : \mathbb{R} \to \mathbb{R}$ such that $\pi F(x) = f \pi(x)$ for all $x \in \mathbb{R}$, i.e., such that the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\;F\;} & \mathbb{R} \\ \pi \big\downarrow & & \big\downarrow \pi \\ \mathbb{T} & \xrightarrow{\;f\;} & \mathbb{T} \end{array}$$

commutes. Given that $f : \mathbb{T} \to \mathbb{T}$ is continuous, one can choose a continuous lift $F : \mathbb{R} \to \mathbb{R}$ (**[5**, §17.1], [**17**, §5.6]). We will assume that all lifts in this article are continuous.

The graph of a lift can be drawn as follows. Consider the graph of f as a subset of $\mathbb{T} \times \mathbb{T}$; the graph can be drawn in the unit square with opposite edges identified, as in FIGURE 3(a). Now tile the plane with integer translates of this square. Each curve in



The graph of $f : \mathbb{T} \to \mathbb{T}$
(a)

Graphs of lifts $F : \mathbb{R} \to \mathbb{R}$
(b)

FIGURE 3

FIGURE 3(b) is then the graph of a lift of f. We see that f has infinitely many lifts, any two of which differ by an integer. Each lift is strictly increasing on $\mathbb{R}$, and $F(x+1) = F(x) + 1$ for all $x \in \mathbb{R}$. (You'll soon be invited to prove these facts!)

To understand the behavior of orbits of f, we often study orbits of F:

$$o(x, F) = o(x) = \{F^i(x) : i \in \mathbb{Z}\},$$

which can be thought of as orbits of f "laid out" in the real line. For example, if $o(\theta, f)$ has passed $\pi(0) \in \mathbb{T}$ p times after q iterates, and if $x \in [0, 1)$, $\pi(x) = \theta$ and F is the lift satisfying $F(0) \in [0, 1)$, then $p \le F^q(x) < p + 1$.

A lift of a circle homeomorphism is particularly useful as it allows us to study iteration of a map defined on $\mathbb{R}$—for which we have the full arsenal of results from real analysis at our disposal.

Exercise 2. Show that for all $k \in \mathbb{Z}$, $R_\omega(x) = x + \omega + k$ is a lift of the rigid rotation r_ω. (For the remainder of this paper, we will set $k = 0$, so $R_\omega(x) = x + \omega$.)

Exercise 3. Show that a lift of an orientation-preserving circle homeomorphism is strictly increasing. Hint: Show F is strictly increasing on $[n, n+1)$ for all $n \in \mathbb{Z}$. Then use the continuity of F.

Exercise 4. Let F be a lift of an orientation-preserving circle homeomorphism f. Show that, for all integers n, $\pi F^n(x) = f^n \pi(x)$. That is, F^n is a lift of f^n for all $n \in \mathbb{Z}$.

Exercise 5. Show that any two lifts of a circle homeomorphism differ by a fixed integer k. Hint: To show that the *same* k works for all x, use the fact that continuous functions take connected sets to connected sets.

Exercise 6. (a) Show that a lift F of an orientation-preserving circle homeomorphism must satisfy $F(x+1) = F(x) + 1$ for all $x \in \mathbb{R}$. Hint: Given $x \in \mathbb{R}$, show $F(x+1) = F(x) + k$ for some $k \in \mathbb{Z}$. If $k \neq 1$, use the intermediate value theorem to contradict the fact f is a homeomorphism.
(b) Show $\forall x \in \mathbb{R}$, $\forall n, k \in \mathbb{Z}$, $F(x+k) = F(x) + k$ and $F^n(x+k) = F^n(x) + k$.

Exercise 7. Let F be a lift of an orientation-preserving circle homeomorphism.

(a) Suppose there exist $x \in \mathbb{R}$ and integers p and q such that $F^q(x) = x + p$. Show that $\pi(x)$ is a periodic point of period q for f. Such an x is called a *p/q-periodic point*.
(b) Suppose $\theta \in \mathbb{T}$ is a periodic point of period q. Show that there is an integer p such that, for all $x \in \pi^{-1}(\theta) = \{x \in \mathbb{R} : \pi(x) = \theta\}$, $F^q(x) = x + p$. Hint: Use the fact that if $x, y \in \pi^{-1}(\theta)$, then $x = y + k$ for some $k \in \mathbb{Z}$. Then use Exercise 6(b).

Knowing that $x \in \mathbb{R}$ is a p/q-periodic point yields information beyond the fact $\pi(x)$ is a period q point for f. As indicated in FIGURE 4, p counts the number of times the

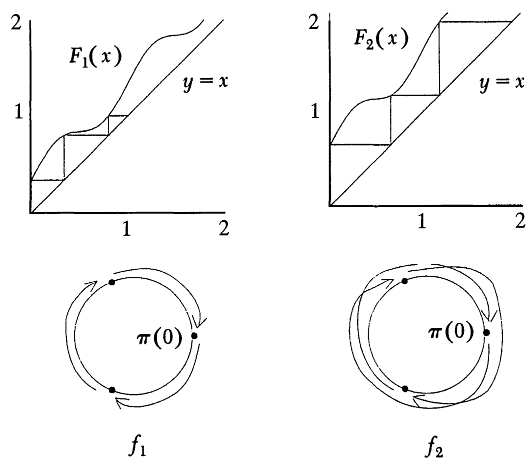


FIGURE 4
$\pi(0)$ is a period 3 point for f_1 and f_2. 0 is a $1/3$-periodic point for F_1 and 0 is a $2/3$-periodic point for F_2.

orbit of $\pi(x)$ traverses $\mathbb{T}$ every q iterates. The horizontal and vertical lines in FIGURE 4 (the "cobweb" diagram) represent a graphical interpretation of iteration [8, §1.3].

The *extended orbit* of $x \in \mathbb{R}$ under F is defined by

$$eo(x, F) = eo(x) = \{F^i(x) + j : i, j \in \mathbb{Z}\}.$$

Exercise 8. (a) Show that $eo(x) = \pi^{-1}(o(\pi(x), f))$. (b) Let $\omega \notin \mathbb{Q}$. Show that $eo(x, R_\omega)$ is dense in $\mathbb{R}$.

The rotation number

DEFINITION. Let F be a lift of an orientation-preserving circle homeomorphism $f:\mathbb{T} \to \mathbb{T}$. The *rotation number* $\rho(f)$ is defined, for any $x \in \mathbb{R}$, as the fractional part of

$$\lim_{n \to \infty} \frac{F^n(x) - x}{n}. \tag{1}$$

Remarks. The fractional part is chosen so that $\rho(f)$ is independent of the lift F (see Exercise 5). Henceforth, when the limit in (1) is used we will assume the fractional part has been taken.

The rotation number measures the average distance a point x travels per iterate of F, or, projecting onto $\mathbb{T}$, the average rotation per iterate of f.

Exercise 9. Show that, for all $x \in \mathbb{R}$, $\rho(r_\omega) = \omega$.

PROPOSITION 1. *Let F be a lift of an orientation-preserving circle homeomorphism. The rotation number $\rho(f)$ exists and its value is independent of x. That is, $\rho(f)$ is well-defined.*

Exercise 10. Show that $\rho(f)$, if it exists, is independent of x by completing the following steps:

(a) Let $k \in \mathbb{Z}$. Use Exercise 6 to show

$$\lim_{n \to \infty} \frac{F^n(x+k) - (x+k)}{n} = \lim_{n \to \infty} \frac{F^n(x) - x}{n}.$$

Hence to show $\rho(f)$ is the same for $x_1 \neq x_2$, we may assume x_1, $x_2 \in [0, 1)$, so that

$$x_2 - 1 \leq x_1 \leq x_2 + 1. \tag{2}$$

(b) Show

$$\lim_{n \to \infty} \frac{F^n(x_2) - x_2}{n} \leq \lim_{n \to \infty} \frac{F^n(x_1) - x_1}{n} \leq \lim_{n \to \infty} \frac{F^n(x_2) - x_2}{n}.$$

(Use each of inequality (2) and Exercises 3 and 6 twice.)

Exercise 11. Show that $\rho(f)$ exists by completing the following steps ([8, §1.14]).

(a) Show that if there is a p/q-periodic point x, then $\rho(f) = p/q$. Hint: Use x to compute $\rho(f)$. For a fixed integer q show there is a constant M such that $|F^r(y) - y| \leq M$ for all $y \in \mathbb{R}$ and all $0 \leq r < q$.

(b) Suppose there are no p/q-periodic points. (i) Use the intermediate value theorem to show that, for any $n > 0$, there is an integer k_n such that $k_n < F^n(x) - x < k_n + 1$ for all $x \in \mathbb{R}$. (ii) Apply this inequality m times with x-values 0, $F^n(0)$,

$F^{2n}(0), \ldots, F^{(m-1)n}(0)$, respectively. Add the resulting m inequalities to find $mk_n < F^{mn}(0) < m(k_n + 1)$. (iii) Show that

$$\left| \frac{F^{mn}(0)}{mn} - \frac{F^n(0)}{n} \right| < \frac{1}{n} \quad \text{and} \quad \left| \frac{F^{mn}(0)}{mn} - \frac{F^m(0)}{m} \right| < \frac{1}{m}.$$

Deduce that

$$\left| \frac{F^n(0)}{n} - \frac{F^m(0)}{m} \right| < \frac{1}{n} + \frac{1}{m},$$

and hence that $\left\{ \frac{F^n(0) - 0}{n} \right\}_{n=1}^{\infty}$ is a Cauchy sequence. Conclude that $\rho(f)$ exists.

Rational rotation numbers

We have seen that if F has a p/q-periodic point then the rotation number equals p/q. Surprisingly, the converse is also true.

THEOREM 1 **[22]**. *Suppose F is a lift of an orientation-preserving circle homeomorphism. Then $\rho(f) = p/q$ if and only if F has a p/q-periodic point.*

This theorem provides an example of a true success story in dynamical systems theory: simply compute a limit to gain significant insight into the dynamics of the given map. If $\rho(f) \in \mathbb{Q}$, f has a periodic point (and, indeed, the long-term behavior of all orbits is completely determined—see Theorem 2). If $\rho(f) \notin \mathbb{Q}$, f has no periodic points. A natural jump to make in this case would be to conclude that all orbits are dense in $\mathbb{T}$. We will see, however, that this is not always the case.

Exercise 12. Prove Theorem 1. Hint: Exercise 11(a) provides one direction. For the other, suppose F has no p/q-periodic point. Argue that, for some $\epsilon > 0$ and for all $x \in \mathbb{R}$, either $F^q(x) > x + p + \epsilon$ or $F^q(x) < x + p - \epsilon$ (use the intermediate value theorem, compactness of $[0, 1]$, and Exercise 6). Now find a contradiction.

THEOREM 2. *Let F be a lift of an orientation-preserving circle homeomorphism with $\rho(f) = p/q$. Then, for all $x \in \mathbb{R}$, either*

(i) x is a p/q-periodic point, or

(ii) there is a p/q-periodic point $x_0 \in \mathbb{R}$ with $|F^n(x) - F^n(x_0)| \to 0$ as $n \to \infty$.

Case (ii) of Theorem 2 implies that asymptotically the orbit of $\pi(x)$ tends to a periodic orbit that traverses $\mathbb{T}$ p times every q iterates of f.

Exercise 13. Let $G: \mathbb{R} \to \mathbb{R}$ be continuous and strictly increasing. For $x \in \mathbb{R}$, show that either $|G^n(x)| \to \infty$ or $G^n(x) \to x_0$ as $n \to \infty$, where x_0 is a fixed point of G. Hint: Use properties of monotonic sequences of real numbers.

Exercise 14. Prove Theorem 2. Hint: Use Theorem 1 to conclude that $G(x) = F^q(x) - p$ has a fixed point x_0. Show that G has infinitely many fixed points via Exercise 6. Now use Exercise 13.

Irrational rotation numbers

The dynamics of a circle homeomorphism with irrational rotation number involves more delicate and much deeper mathematics (**[7]**, **[12]**, **[13]**, **[14]**, **[27]**), so only part of the story will be told here. Yet even this partial investigation leads to several surprising results.

We assume F is a lift of an orientation-preserving circle homeomorphism with $\rho(f) = \alpha \notin \mathbb{Q}$. Recall that all orbits under rigid rotation r_α are dense in $\mathbb{T}$. Must the same condition also hold for f? To answer this, we need the notion of a conjugacy.

DEFINITION. Two orientation-preserving circle homeomorphisms f and g are *conjugate* if there exists an orientation-preserving circle homeomorphism $h: \mathbb{T} \to \mathbb{T}$ such that $hf(\theta) = gh(\theta)$ for all $\theta \in \mathbb{T}$, i.e., such that the diagram

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{f} & \mathbb{T} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{T} & \xrightarrow{g} & \mathbb{T} \end{array}$$

commutes.

Remark. The definition of topological conjugacy (see, e.g., [**24**, §2.6]) requires that h be a homeomorphism satisfying $hf = gh$. We require in addition that h is orientation-preserving; this assures that conjugate orientation-preserving circle homeomorphisms have the same rotation number ([**12**], [**16**, §11.1]).

Exercise 15. (a) Show that two conjugate orientation-preserving circle homeomorphisms have the same dynamics. That is, show $h(o(\theta, f)) = o(h(\theta), g)$ for all $\theta \in \mathbb{T}$. Hence, if θ is a periodic point of period n for f, for example, then $h(\theta)$ is a periodic point of period n for g. Likewise, $h^{-1}(o(\theta, g)) = o(h^{-1}(\theta), f)$.

(b) Show that f has a dense orbit if and only if g has a dense orbit. (This is an example of the fact that topological properties of orbits are also preserved by the conjugacy.)

We can now rephrase our question as follows: Given $\rho(f) = \alpha \notin \mathbb{Q}$, under what condition(s) is f conjugate to r_α? If the conjugacy exists, then all orbits of f are dense in $\mathbb{T}$. This problem is surprisingly difficult; Poincaré, for example, was unable to resolve it [**2**]. But we can make progress towards its solution via the following propositions.

PROPOSITION 2. *Let F be a lift of an orientation-preserving circle homeomorphism with $\rho(f) = \alpha \notin \mathbb{Q}$. Fix $x_0 \in \mathbb{R}$ and define $\overline{H}: eo(x_0) \to \mathbb{R}$ by*

$$\overline{H}\big(F^i(x_0) + j\big) = i\alpha + j \quad \textit{for all} \quad i, j \in \mathbb{Z}.$$

If $x_1,\ x_2 \in eo(x_0)$ and $x_1 < x_2$, then $\overline{H}(x_1) < \overline{H}(x_2)$. That is, $\overline{H}$ preserves the "$<$" ordering on $eo(x_0)$.

Proof idea: Suppose the conclusion is false. Then there exist i, j, k, $l \in \mathbb{Z}$ satisfying $F^i(x_0) + j < F^k(x_0) + l$ and $i\alpha + j \geq k\alpha + l$, or $(i - k)\alpha \geq l - j$. Note that $i \neq k$.

Case 1. Assume $i - k > 0$. Then $\alpha \geq (l - j)/(i - k)$. By Exercises 3 and 6, $F^i(x_0) + j < F^k(x_0) + l$ implies $F^{(i-k)}(x_0) - x_0 < l - j$. Now show this implies $\alpha \leq (l - j)/(i - k)$, and deduce a contradiction.

Exercise 16. Complete the argument for the $i - k < 0$ case.

Hence if $\rho(f) = \alpha \notin \mathbb{Q}$, the ordering of points in $eo(x_0, F)$ is the same as the ordering of points in $eo(0, R_\alpha)$. This would seem to kindle hope that a conjugacy between F and R_α exists, with an appropriate extension of $\overline{H}$ to the real line as the choice for the conjugacy. Although this is not always possible, we can say the following:

PROPOSITION 3. *Let F be a lift of an orientation-preserving circle homeomorphism $f: \mathbb{T} \to \mathbb{T}$ with $\rho(f) = \alpha \notin \mathbb{Q}$. There exists a map $H: \mathbb{R} \to \mathbb{R}$ such that*

(*i*) *H is continuous and increasing;*
(*ii*) *$\forall x \in \mathbb{R}$, $H(x+1) = H(x) + 1$, and H is a lift of a map $h: \mathbb{T} \to \mathbb{T}$ such that the diagram*

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{f} & \mathbb{T} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{T} & \xrightarrow{r_\alpha} & \mathbb{T} \end{array}$$

commutes.

Remark. By "increasing" we mean nondecreasing; thus the map H need not be a conjugacy, as it may not be strictly increasing.

Proof. Fix $x_0 \in \mathbb{R}$, and define $\overline{H}: eo(x_0) \to \mathbb{R}$ as in Proposition 2. Extend $\overline{H}$ to a map $H: \mathbb{R} \to \mathbb{R}$, $y \mapsto \sup\{\overline{H}(x): x \in eo(x_0),\ x < y\}$.

Exercise 17. Prove that $H: \mathbb{R} \to \mathbb{R}$ is increasing. Hint: Use Proposition 2.

Exercise 18. Prove that $H: \mathbb{R} \to \mathbb{R}$ is continuous. Hint: Suppose H is not continuous at $y_0 \in \mathbb{R}$. Use Exercises 17 and 8(b) to derive a contradiction.

Exercise 19. Prove that for all $x \in \mathbb{R}$, $H(x+1) = H(x) + 1$. Hint: First show that $eo(x_0, F) + 1 = eo(x_0, F)$, and that $H: eo(x_0, F) \to \mathbb{R}$ satisfies $H(x+1) = H(x) + 1$ for all $x \in eo(x_0, F)$.

To complete the proof of Proposition 3, note that for $i, j, k, l \in \mathbb{Z}$,

$$\begin{aligned} H\big(F\big(F^i(x_0) + j\big)\big) &= H\big(F^{i+1}(x_0) + j\big) = (i+1)\alpha + j = H\big(F^i(x_0) + j\big) + \alpha \\ &= R_\alpha\big(H\big(F^i(x_0) + j\big)\big), \end{aligned}$$

so that $H \circ F = R_\alpha \circ H$ on $eo(x_0)$. Now for $y \in \mathbb{R}$,

$$\begin{aligned} HF(y) &= \sup\big\{\overline{H}(x) : x \in eo(x_0),\ x < F(y)\big\} \\ &= \sup\big\{\overline{H}(F(x)) : x \in eo(x_0),\ x < y\big\} \quad \text{(Exercise 6)} \\ &= \sup\big\{R_\alpha\big(\overline{H}(x)\big) : x \in eo(x_0),\ x < y\big\} \\ &= R_\alpha H(y). \quad \square \end{aligned}$$

If the map $H: \mathbb{R} \to \mathbb{R}$ in Proposition 3 is not a homeomorphism, there must be an interval $J \subset \mathbb{R}$ on which H is constant, i.e., $H(J) = \{y_0\}$ for some $y_0 \in \mathbb{R}$.

Exercise 20. (a) Show that for such an interval J, H is constant on $F^i(J) + j$ for all $i, j \in \mathbb{Z}$. Hint: Proposition 3.

(b) Show that H is locally constant on an open, dense subset of $\mathbb{R}$. Thus H is an example of a *Cantor function*; see FIGURE 5 for an approximation of the graph of a Cantor function arising in the next section. Hint: Use Proposition 3 and Exercise 8(b).

Exercise 21. Suppose f is an orientation-preserving circle homeomorphism with $\rho(f) = \alpha \notin \mathbb{Q}$. Show that if f has a dense orbit, then f is conjugate to r_α, and hence all orbits of f are dense in $\mathbb{T}$.

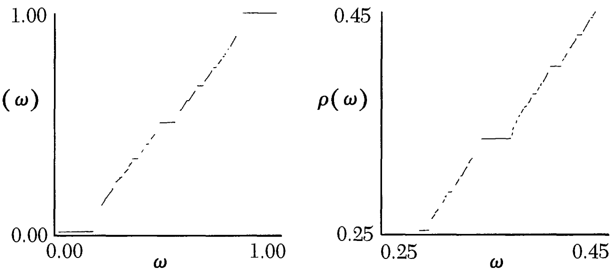


FIGURE 5
A plot of $\rho(\omega) = \rho(f_{\omega,1})$.

We see via Exercise 21 that the question of whether the map H is a homeomorphism depends upon the existence of a dense orbit for f. It is indeed curious to find that, in the absence of a dense orbit for f, the mapping H is a pathological Cantor function.

Not until 1932 did A. Denjoy provide a sufficient condition for determining when a circle homeomorphism with irrational rotation number α is conjugate to rigid rotation r_α. Surprisingly, the degree of smoothness of f is the deciding factor. For proofs of Denjoy's theorem see **[5]**, **[7]**, **[9]**, **[12]**, **[16]**, **[18]**, **[19]**, **[24]**.

Denjoy's Theorem **[7]**. *Let $f:\mathbb{T}\to\mathbb{T}$ be an orientation-preserving circle homeomorphism with $\rho(f)=\alpha\notin\mathbb{Q}$. If f is a C^2-diffeomorphism, then f is conjugate to r_α.*

Remarks. The requirement that f be twice continuously differentiable is sufficient but not necessary: the conjugacy exists if f is C^1 with f' of bounded variation (**[5]**, **[7]**, **[9]**, **[16]**). See **[9]** for a discussion of the role played by the bounded variation of f'. There also exist C^1 orientation-preserving circle diffeomorphisms with irrational rotation number that are not conjugate to r_α (**[7]**, **[8]**, **[12]**, **[13]**, **[27]**).

We note that a natural analogue of rotation number exists for maps and flows defined on higher dimensional tori. For an introduction to *rotation vectors* and the role they play in understanding the dynamics of such maps and flows, see **[26]**.

An example

We now link ideas from the preceding sections by considering the two-parameter *standard family* of circle maps

$$F:\mathbb{R}\to\mathbb{R},\ F_{\omega,\epsilon}(x) = x+\omega+\frac{\epsilon}{2\pi}\sin(2\pi x), \quad \omega,\epsilon\ge 0. \tag{3}$$

Note that if $\epsilon = 0$ we recover R_ω, so family (3) represents a perturbation of rigid rotation. The standard family arises in many physical systems as a model for periodically forced nonlinear oscillators (**[3]**, **[6]**, **[15]**, **[20]**).

Exercise 22. Show that for $\epsilon\le 1$, $F_{\omega,\epsilon}$ is a lift of an orientation-preserving circle homeomorphism $f_{\omega,\epsilon}:\mathbb{T}\to\mathbb{T}$.

We would like to investigate how $\rho(f_{\omega,\epsilon})$ changes as the parameters ω and ϵ are varied. To this end, temporarily fix ϵ and let $F_\omega(x) = F_{\omega,\epsilon}(x)$. Define a function $\rho:[0,1)\to\mathbb{R}$, $\rho(\omega)=\rho(f_\omega)$.

Exercise 23. (a) Show that $\omega_1<\omega_2$ implies $\rho(\omega_1)\le\rho(\omega_2)$, so that $\rho(\omega)$ is an increasing function. Hint: Exercise 3.

(b) Show $\rho(0)=0$ and $\lim_{\omega \to 1} \rho(\omega)=1$. Hint: Find a fixed point for F_0 and a $1/1$-periodic point for $F_{1-\delta}$ for δ sufficiently small.

(c) Show $\rho:[0,1) \to \mathbb{R}$ is a continuous function of ω. Hint: Let $\omega_0 \in [0,1)$ and $\epsilon > 0$ be given. Pick $n \in \mathbb{Z}$ with $2/n < \epsilon$, and $k \in \mathbb{Z}$ satisfying $k-1 < F_{\omega_0}^n(0) < k+1$. Using Exercises 3 and 6, show $m(k-1) < F_{\omega_0}^{mn}(0) < m(k+1)$. Pick $\delta > 0$ so that $|\omega - \omega_0| < \delta$ implies $k-1 < F_\omega^n(0) < k+1$, and show $|F_\omega^{mn}(0) - F_{\omega_0}^{mn}(0)| < 2m$. Now deduce $|\rho(\omega) - \rho(\omega_0)| < \epsilon$.

Suppose $\rho(f_{\omega_0}) = p/q$ for some ω_0. By Theorem 1, the function $G(x) = F_{\omega_0}^q(x) - p$ has a fixed point x_0. Let $\lambda = G'(x_0)$. If $\lambda \neq 1$ (so that the graph of G is not tangent to the line $y = x$ at $x = x_0$), the implicit function theorem yields a $\delta > 0$ so that for $\omega \in (\omega_0 - \delta, \omega_0 + \delta)$, $F_\omega^q(x) - p$ has a fixed point. That is, for $\omega \in (\omega_0 - \delta, \omega_0 + \delta)$, F_ω has a p/q-periodic point and $\rho(\omega) = p/q$. If $\lambda = 1$ there again exists such a δ, but the argument is more complicated ([**8**, §1.14], [**16**, §11.1]). In either case we have that if $\rho(\omega_0) = p/q$, then $\rho(\omega) = p/q$ on some interval of ω-values containing ω_0.

V.I. Arnol'd ([**1**], [**16**, §11.1]) showed that adding an arbitrarily small constant to an orientation-preserving circle homeomorphism with $\rho(f) \notin \mathbb{Q}$ changes the rotation number. In summary, we have that $\rho(\omega)$ is a continuous, increasing function, with $\rho(0) = 0$ and $\rho(1) = 1$. Moreover, to each rational in $[0,1)$ corresponds an interval on which ρ is constant. Since the rationals are dense in $[0,1)$, we see that $\rho(\omega)$ (surprisingly) provides another example of a Cantor function (see FIGURE 5).

To see how $\rho(f_{\omega,\epsilon})$ varies as a function of both parameters ω and ϵ, set $A_r = \{(\omega,\epsilon): \rho(f_{\omega,\epsilon}) = r\}$. (The A_r are level sets of the function $\rho:(\omega,\epsilon) \mapsto \rho(f_{\omega,\epsilon})$.)

Exercise 24. (a) Show that $A_r = \{(r,0)\}$ for $\epsilon = 0$. (b) Show that $A_{r_1} \cap A_{r_2} = \varnothing$ for $\epsilon \le 1$ and $r_1 \neq r_2$.

We have seen that $\rho(f_{\omega_0,\epsilon}) = p/q$ implies the existence of an interval of ω-values for which $\rho(f_{\omega,\epsilon}) = p/q$. The sets $A_{p/q}$ therefore have nonempty interior, and are known as *p/q-resonance zones* or *Arnol'd tongues*. For $r \notin \mathbb{Q}$, A_r will have empty interior due to Arnol'd's result. In this case A_r is a curve extending from $\epsilon = 0$ to $\epsilon = 1$ in the ω,ϵ-plane [**4**]. Note that in general no two of the A_r regions can intersect (Exercise 24(b)). We have then an amazingly intricate *bifurcation plot* for the standard family in the ω,ϵ-plane. A computer-generated plot of the boundaries of certain resonance zones for $0 \le \epsilon \le 3$ is shown in FIGURE 6. Note that for $\epsilon > 1$ these

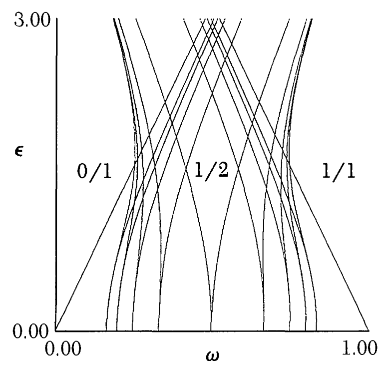


FIGURE 6

Boundaries of p/q-resonance zones for the standard family $F_{\omega,\epsilon}$ for $0 \le \epsilon \le 3$. The boundaries of the 0/1, 1/6, 1/5, 1/4, 1/3, 1/2, 2/3, 3/4, 4/5, 5/6, and 1/1 horns are pictured from left to right.

regions can intersect ($F_{\omega,\epsilon}$ is no longer a homeomorphism); it turns out that for rationals r_1 and r_2 and $(\omega,\epsilon) \in A_{r_1} \cap A_{r_2}$, $F_{\omega,\epsilon}$ is chaotic in the sense that it has positive topological entropy **[4]**.

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