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Quasiconformal homeomorphisms and dynamics
I. Solution of the Fatou-Julia problem on wandering domains

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Introduction

If one perturbs the analytic dynamical system $z \xrightarrow{R} z^2$ on the Riemann sphere $\overline{\mathbf{C}}$ to $z \xrightarrow{R_a} z^2 + az$ for small a, the following happens: Before perturbation the round unit circle C is invariant under iteration of R and R is expanding on C ($|R'(z)| > 1$), R has dense orbits, and is even ergodic on C relative to linear measure. After perturbation R_a now preserves a unique Jordan curve C_a close to C and again R is expanding and has dense orbits on C_a. Now C_a is not a rectifiable curve. It is a fractal curve with Hausdorff dimension > 1 which increases with $|a|$. (Figure 1). The intricacies of C are of a self-similar nature

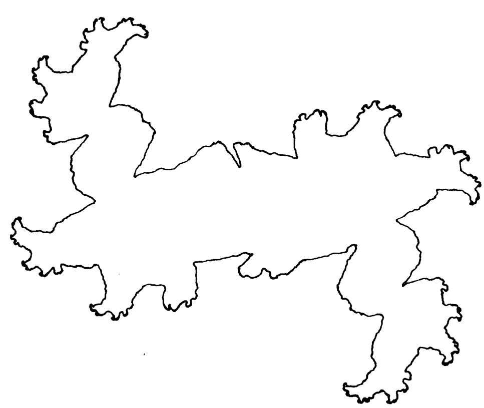

FIGURE 1

because of the expanding property of R_a on C_a. In particular C_a is a quasi-circle in the sense of Ahlfors and R_a is ergodic with respect to its Hausdorff measure which is finite and positive [19].

This situation strongly reminds one of Poincaré's original perturbations of Fuchsian groups $\Gamma \subset \mathrm{PSL}(2, \mathbf{R})$ into quasi-Fuchsian groups in $\mathrm{PSL}(2, \mathbf{C})$. The Poincaré limit set then changes from a round circle to a non-differentiable Jordan curve (Figure 2). These Poincaré deformations (1883) have been treated and

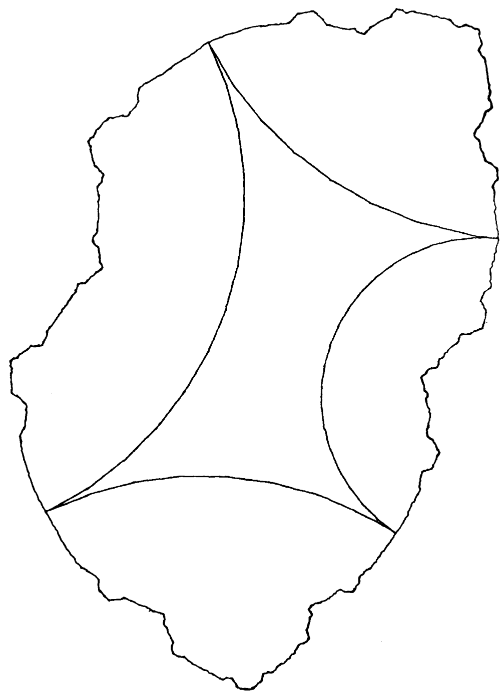

FIGURE 2

globalized using quasi-conformal homeomorphisms by Ahlfors and Bers ([3]).

Fatou and Julia ([8], [11]), who constructed the iteration theory of analytic mappings, were well aware of the analogy with Poincaré's work. We continue this analogy by injecting the modern theory of quasiconformal mappings into the dynamical theory of iteration of complex analytical mappings. In the above example we will find (Part III) that all the R_a are quasiconformal deformations of one another on the entire sphere for $0 < |a| < 1$.

One is familiar in differentiable dynamics with non-differentiable homeomorphisms φ conjugating one system R_1 to another R_2, $\varphi R_1 = R_2 \varphi$. Such φ cannot even be Lipschitz if eigenvalues of corresponding periodic points have different sizes. An important point of this work is that *nearby complex analytic dynamical systems tend to be conjugate using homeomorphisms which, although not Lipschitz, are nevertheless quasiconformal.*

In addition, many corollaries will follow from the remarkable fact that quasiconformal homeomorphisms in $\dim_{\mathbf{C}} = 1$ can be canonically constructed from their *measurable* distortion of the conformal geometry by use of the measurable Riemann mapping theorem (see §7). Since the distortions are only required to be measurable, qc (quasiconformal) homeomorphisms work well in dynamical contexts where invariant functions are rarely continuous but are always measurable.

In this paper we prove a dynamical result about the iteration of any complex analytic mapping of the Riemann sphere. The proof uses dynamics, Riemann surface theory, planar topology-prime ends, and the theory of quasiconformal mappings. The same argument gives a new proof of the Ahlfors finiteness theorem for finitely generated Kleinian groups (§11).

Let $R\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be a complex analytic self-mapping of the Riemann sphere. We are interested in the dynamical structure of the iterates $R, R^2, R^3, \ldots$. Fatou and Julia ([8], [11]) showed that here one finds a coherent and attractive mathematical theory of some depth. More recently, computer studies have revealed the incredible beauty of these examples (for an example of this see the frontispiece of Mandelbrot's new book) as well as many unexplained patterns (see Figures 1, 2, 3, 4 of Part III). Also studying these complex maps gives a powerful tool for understanding deeper questions about the iteration of real non-linear maps of the line which are treated in many parts of science and applied mathematics [9]. Finally, another motive is to profit from the interplay of the surprisingly good dictionary connecting complex iteration theory and Kleinian groups. See below and [20].

A point z in $\overline{\mathbf{C}} = \mathbf{C} \cup \infty$ is *stable* for the iterates $R, R^2, \ldots$ if on some neighborhood of z these mappings form an equicontinuous family of mappings into $\overline{\mathbf{C}}$. The *stable set* D_R of stable points consists of countably many open connected sets D_α called *stable regions*. These are transformed among themselves by R and each $D_\alpha \xrightarrow{R} D_\beta$ is a finite-to-one surjective branched covering. (In the literature D_R is also called the set of *normal* points for R because $R, R^2, \ldots$ form a normal family there. In Siegel's paper [16] the more descriptive term stable was employed because of the dynamical orbit interpretation.) The definition of D_R was Fatou's starting point [8].

The complement J_R of all these stable regions is a limit set (often called the Julia set) similar to that of Poincaré for Kleinian groups. Julia began with the definition of J_R as the closure of the expanding periodic points ($z = R^n(z)$ and $|(R^n)'(z)| > 1$).

Heuristically, the dynamics of R on J_R is chaotic and on D_R it is rigid or dissipative (or both). More precisely, Fatou and Julia showed that for any point z of J_R the inverse orbit $\{\bigcup_n R^{-n}z\}$ is dense in J_R. Near any point in D_R Fatou and Julia constructed limiting analytic functions on neighborhoods using normality of the family $R, R^2, \ldots$. They found convergence to constants (corresponding to fixed points or periodic points of R), to the identity (corresponding to regions where R is analytically equivalent to an irrational rotation) and one other possibility which they could not analyze. This last possibility, the wandering region, is taken care of here after 65 years. We show it never occurs.

THEOREM 1 (eventual periodicity). *Let D_α be any stable region for a rational mapping R of the Riemann sphere. The sequence of successive regions D_α, RD_α, $R^2D_\alpha, \dots$ is always eventually periodic.*

The quasiconformal part of the proof shows a purely ergodic-theory fact for the action of R on the Julia limit set. Define the large *orbits* of R as the equivalence classes $x \sim y$ if and only if $R^n x = R^m y$ for some n, m greater than or equal to zero.

THEOREM 2 (Julia recurrence). *If A is any Borel subset of the Julia limit set which is a fundamental set, that is A intersects any large orbit of R in at most one point, then the Lebesgue measure of A is zero.*

Both of these theorems have analogues for Kleinian groups due originally to Ahlfors [2] and Sullivan [18].

We can use Theorem 1 to study the equivalence relation $x \approx y$ if and only if $R^n x = R^n y$ some $n \geq 0$. The quotient D_∞ of R: $D \to D$ (any setting) by this relation is called the direct limit of the direct system $\cdots \to D \overset{R}{\to} D \overset{R}{\to} D \to \cdots$ and there is an induced map R_∞: $D_\infty \to D_\infty$ which is bijective. By Theorem 1 all equivalence classes in $U = (\mathbf{C} -$ limit set) are represented by R restricted to the periodic stable regions.

In Part III, Section 2, it is shown that on $D = U - \{$large orbits of periodic points$\}$ the map R is a finite union of maps where:

Either the direct limit is a Riemann surface and (R_∞, D_∞) is a finite union of

(attracting) i) $z \to \lambda z$ on the punctured plane $|\lambda| < 1$;
(parabolic) ii) $z \to z + 1$ on the plane;
(Siegel disk) iii) $z \to \lambda z$ on the punctured disk $|\lambda| = 1$;
(Herman ring) iv) $z \to \lambda z$ on the annulus $|\lambda| = 1$;

Or the direct limit is not a Riemann surface but the direct system is equivalent to a finite union of

(superattracting) v) $z \to z^k$ on the punctured disk, $k = 2, 3, \dots$.

The total number of units i), ..., v), is proved to be less than 8 (degree $R - 1$) but the sharp bound may be 2 (degree $R - 1$).

In Part III the classification of periodic stable regions and dynamical information on the Fatou-Julia limit set is used to prove quasiconformal conjugacy theorems. Besides using Theorems 1 and 2 of this paper, III uses the holomorphic motions results of [12] and [22]. A proof of the density of structurally stable models was nearly completed in [12] and finally completed in III using [22]. In Part II of the series (submitted to Acta Math.) the holomorphic motions and dynamical ideas of [12] are applied to finitely generated Kleinian groups to show that structural stability implies hyperbolicity.

The main outstanding problems are:

For Kleinian groups: to find the analogue of the critical point-attracting regions part of the proof of density of structurally stable models from [12].

For rational maps: to find the analogue of the analysis of the measurable conformal structures result [18] for limit sets of Kleinian groups.

We close with a sample of the dictionary between analytic iteration and discrete subgroups of PSL(2, C) which lies behind this series of papers.

Complex analytic iteration	*Discrete subgroups of* PSL(2, C)
entire mapping	arbitrary Kleinian group
Blaschke product	arbitrary Fuchsian group
rational mapping, R	finitely generated Kleinian group, Γ
degree of mapping, d	number of generators, n
$(2d-2)$ analytic parameters	$(3n-3)$ analytic parameters
$(2d-2)$ critical points	(?) ends of hyperbolic 3 manifolds
Fatou-Julia limit set ([8], [11])	Poincaré limit set (1880)
stable regions	domain of discontinuity
periodic points of R	fixed points of elements of Γ
dense in limit set	dense in limit set
Riemann surface of R	Riemann surface of Γ
eventual periodicity theorem, $d < \infty$	Ahlfors finiteness theorem, $n < \infty$
attracting region	cocompact stabilizer of a discontinuous component
parabolic region	cofinite area stabilizer with cusp
Siegel disk	(?) limit group by qc deformation in arithmetically good "direction"
non-linearizable indifferent periodic point	(?) limit group by qc deformation in a "Liouville" direction
Hermann ring	(?) similar to Siegel disk analogy
super attracting region	?
invariant line fields on limit set (restricted examples known)	no invariant line fields on limit set $(n < \infty)$ [18]

measure of limit set?	measure of limit set?
there are non-locally connected limit sets (Liouville limits)	"arithmetically good" limit groups have locally connected limit sets
expanding or hyperbolic on limit set	geometrically finite without cusps
Hausdorff dimension of limit set and pressure of $\log\lvert R'\rvert$	Hausdorff dimension of limit set and lowest eigenvalue of Laplacian on H^3/Γ
all critical points preperiodic (Thurston)	cocompact Γ, i.e., compact hyperbolic three manifolds

Acknowledgements. Conversations with John Guckenheimer in 1978 encouraged me to study the geometry of the Julia limit curves of $\bar{z} \to z^2 + az$. Michel Herman introduced me to the appearance of small denominators and Siegel's work for rational maps. He also pointed out the Fatou and Julia question about wandering domains in 1978. All this occurred during "the mappings of the interval" year at IHES. During the Sullivan-Thurston geometry seminar at Colorado 1980–81 the new proof of Ahlfors' theorem was developed.

More recently in the fall of 1981 the seminar of Douady and Hubbard at the University of Paris at Orsay on the quadratic family brought to these questions a fresh and stimulating light. The time was ripe to introduce quasiconformal mappings into analyic iteration problems.

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1. Direct limits of Riemann surfaces

Let $R_1 \overset{f_1}{\to} R_2 \overset{f_2}{\to} R_3 \overset{f_3}{\to} \cdots$ be a sequence of surjective analytic maps of Riemann surfaces. We say the Riemann surface R_∞ represents the direct limit of the sequence $\{f_n\}$ if there are compatible analytic surjective maps $R_n \overset{\pi_n}{\to} R_\infty$, namely $\pi_{n+1} \circ f_n = \pi_n$, so that some π_n identifies two points if and only if they are identified in the sequence: $\pi_n x = \pi_n y$ if and only if $g_k(x) = g_k(y)$ for some k where $g_k = \prod_{i=0}^k f_{n+i}$.

A Riemann surface is *hyperbolic* if the universal covering surface is the disk. A Riemann surface is *non-elementary* if the fundamental group (i.e. the covering group acting on the disk) is not Abelian. We suppose the R_i are hyperbolic.

PROPOSITION 1. *If the maps $f_1, f_2, \ldots$ are unbranched coverings and if one of the R_i is non-elementary, then the direct limit R_∞ exists. Furthermore, either*

i) *the fundamental group of R_∞ is not finitely generated or*

ii) *eventually the f_i are isomorphisms.*

Proof. If $D \overset{\pi}{\to} R_1$ is the universal cover of R_1, then the composition $D \to R_1 \overset{f_1}{\to} \cdots \to R_n$ determines the universal covering of R_n. We obtain an increasing union of covering groups $\Gamma_1 \subset \Gamma_2 \subset \cdots$ representing the sequence. The union group Γ_∞ contains no elliptics because the coverings are unbranched. Also Γ_∞ is non-elementary because one of the Γ_i is non-elementary. Thus Γ_∞ is a discrete group(*) and $R_\infty = D/\Gamma_\infty$ represents the direct limit.

If Γ_∞ is finitely generated then after some n, $\Gamma_n = \Gamma_\infty$.

2. The case of the wandering annulus

Let A_0 be a stable region of the rational map R which is an annulus and define $A_{n+1} = R(A_n)$, $n = 0, 1, 2, \ldots$. We assume R: $A_n \to A_{n+1}$ is a covering map and let d_n be the degree of R: $A_n \to A_{n+1}$.

PROPOSITION 2. *If the A_k are pairwise disjoint then for n sufficiently large* $d_n = 1$.

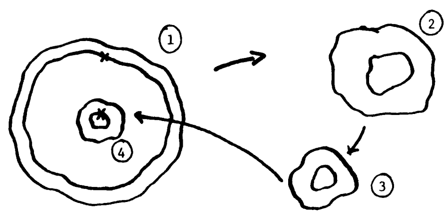
Proof. i) Let $n_1 < n_2 < \cdots$ be a sequence of indices so that $d_{n_i} > 1$. Then R has a critical point in each complementary component of $A_{n_1}, A_{n_2}, \ldots$ (by topology and the openness of R). Since there are only finitely many critical points of R, infinitely many of the A_{n_i} must nest around one critical point of R.

(*)One knows a subgroup Γ of PSL(2, **R**) is either elementary, contains elliptics or is discrete. This follows easily from the closure of Γ and the fact that elliptics are open to show the component of the identity of the closure is trivial. See Sullivan [20].

ii) Let S be a smooth closed curve imbedded in A which separates the boundary components of A_0. Since $\{R^n\}$ restricted to S forms an equicontinuous family, the arc length of $R^n(S)$ is bounded in the spherical metric. Now the winding number of $R^n(S)$ in A_n becomes arbitrarily large with n (being the product $d_0 d_2 \dots d_{n-1}$). Since the arc length of $R^n(S)$ is bounded, it follows that one component of the complement of A_n must have arbitrarily small spherical diameter for n sufficiently large.

iii) In the spherical metric $|R'z|$ is bounded by M, say. So when n is sufficiently large the R image of the small complementary component of A_n maps to the small complementary component of A_{n+1} (and does not explode onto the sphere—the only other possibility by the openness property of R).

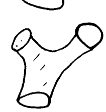
iv) Now some power k of R carries one member A_{n_0} of the nested family of annuli described in i) to another member with the non-explosion property of iii). But then a point on the small boundary component of A_{n_0} (which is of course in the Julia limit set) becomes stable for the iterates of R^k. This is a contradiction.



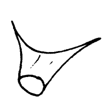
3. Riemann surfaces of infinite topological type

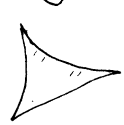
Let S be a Riemann surface covered by the disk provided with its complete hyperbolic (Poincaré) metric. Let $\mathscr{C}$ be a maximal collection of disjoint simple closed geodesics in S. Then the closure of each component of $S - \mathscr{C}$ is geodesically convex and must be conformally of the form sphere minus

0) two disks:  "funnel"

i) three disks:  "pair of pants"

ii) two disks and one point:  "degenerated pair of pants"

iii) one disk and two points:  "degenerated pair of pants"

iv) three points:  "degenerated pair of pants"

Otherwise, one could add a new simple closed geodesic to the collection $\mathscr{C}$. See Thurston [23].

The surface S has infinite topological type if and only if $\mathscr{C}$ is infinite. For each *unit* 0), i), ii), or iii) considered by itself, the hyperbolic lengths of the boundary curves may be specified arbitrarily in $(0, \infty)$. When S is assembled from its *units* the only condition required to build a hyperbolic surface is that lengths of curves along which units are glued be equal. There is also a rotation or twist parameter to be specified at each gluing curve, except for unit 0).

PROPOSITION 3. *If S has infinite topological type there are arbitrarily large dimensional families of deformations of the hyperbolic structures so that each pair of surfaces in the family is quasiconformally (even quasi-isometrically) isomorphic.*

Proof. Deform the length of a large finite number of curves in $\mathscr{C}$ slightly but independently. A finite number of units change slightly giving deformations of S which are nearly isometric to it. It is easy to construct an "almost-isometry" between S and the deformed surface which is actually an isometry outside a finite number of units.

4. The structure of a wandering stable region

Let U_0 be a stable region so that if $U_0, U_1, U_2, \ldots$ are defined by $U_{n+1} = RU_n$, the U_i are pairwise disjoint. Since R has only finitely many critical points we may discard finitely many of the first U_i's to eliminate critical points of R. Now we have (after relabeling) each $U_n \xrightarrow{R} U_{n+1}$ is a finite unbranched covering.

PROPOSITION 4. *Either* 1) *from some n on, U_{n+i} has finite topological type and each $U_{n+i} \xrightarrow{R} U_{n+i+1}$ is an isomorphism $i \geq 0$, or*

2) *the direct limit U_∞ of $U_0 \xrightarrow{R} U_1 \xrightarrow{R} \cdots$ exists and has infinite topological type.*

Proof. 1) If some U_n is simply connected all subsequent U_{n+i} are also simply connected because an oriented Riemann surface cannot have a finite non-trivial fundamental group. The covering maps $U_m \xrightarrow{R} U_{m+1}$ must then be isomorphisms.

2) If some U_n is an annulus, then all subsequent U_{n+i} are because R is an unbranched covering. The eventual injectivity follows from Section 2.

3) The remaining case is covered by Section 1.

5. Rigidity of conjugacy on the Julia limit set

For a rational mapping R let C_R denote the subgroup of homeomorphisms of the Julia limit set J_R which commute with R. Note that C_R is closed in the compact open topology on the group of homeomorphisms of J_R.

PROPOSITION 5. C_R *(with the induced compact open topology) admits a continuous injective homomorphism into a totally disconnected Cantor group (an inverse limit of finite groups).*

Proof. Consider A_1 the finite set of periodic points in J_R with lowest possible period. Let $A_{n+1} = R^{-1}A_n \cup A_n$. Then if $gR = Rg$, $gA_n = A_n$. But $A = \bigcup_n A_n$ is dense. So $g = \text{id}$ if $g|A = \text{id}$. Thus C_R injects into the Cantor group = inverse limit {permutations A_n}, and this representation is clearly continuous.

COROLLARY. *A topological conjugacy between two rational maps is unique on the Julia limit set up to an element in a totally disconnected group of homeomorphisms.*

Example (oral communication of Michel Herman). C_R can be infinite, e.g. in the Lattes example:

$$z \to 2z \quad \text{on} \quad \mathbf{C} \Big/ \left\{ \begin{matrix} z \to -z \\ z \to z+1 \\ z \to z+i \end{matrix} \right\} = \overline{\mathbf{C}}.$$

Here $J_R = \overline{\mathbf{C}}$ (Lattes (1918) and C_R contains $\mathrm{GL}(2, \mathbf{Z})/\{\pm 1\}$. One can show this is all of C_R (Sullivan [21], Shub [15]).

6. Prime ends [14]

Let U be a simply connected domain in the sphere with frontier ∂U. A *cross cut* in U is a simple arc c in U with endpoints in ∂U. A *chain* is a sequence of cross cuts c_n satisfying

i) diameter $c_n \to 0$ (spherical metric),
ii) the closures of the c_n are disjoint,
iii) (relative to a fixed base point b in U) c_n separates c_{n+1} from b.

One says two chains c_n and $\overline{c_n}$ are *equivalent* if each c_n separates all but finitely many $\overline{c_n}$ from b and vice versa.

A *prime end of* U is an equivalence class of chains. Carathéodory (1913) showed that a conformal equivalence between two U's determines an isomorphism between the set of prime ends, and for the open unit disk the prime ends are in one-to-one correspondence with points of the circle boundary. The *impression* of a prime end is defined to be $\bigcap_n \overline{B_n}$ where $\overline{B_n}$ is the closure of the component of $U - c_n$ not containing b. The *impression* of a prime end is clearly *connected*. We define the *fibre* of a point x in ∂U to be the set of prime ends represented by chains c_n where $c_n \to x$.

Clearly any *prime end belongs to at least one fibre* (of some point in its impression). The name *fibre* was chosen because the Riemann map φ of the unit disk D to U has non-tangential limits except on a set E of measure zero (Fatou) and the limit function $\partial D - E \underset{\partial\phi}{\to} \partial U$ cannot be constant on a set of positive measure (F. and M. Riesz). If c_n is a chain converging to a in ∂U where c_n determines the prime end ξ, and if φ has a radial limit at ξ then this radial limit must equal a. *Thus the fibre of a in ∂U is contained in the union of* {*the preimage of a of the boundary of the Riemann map $\partial\varphi$: $\partial D - E \to \partial U$*} *and* {*the exceptional set E*}. *Thus we have*:

Proposition 6. *The fibre of any point in ∂U has angular measure zero. In particular fibres are totally disconnected in the topology on the unit circle.*

The discussion above extends immediately to any domain U in the sphere whose complement has finitely many connected components-and to the Riemann map connecting U with a domain obtained from the sphere by removing finitely many disks. For details and references see the survey of Piranian [14]. Using work of Beurling [5], one can replace measure by capacity, which implies fibres are in fact much smaller. We only need the total disconnectivity here which may also be derived by a purely topological argument due to Epstein and Mather.

7. The measurable Riemann mapping theorem with parameters

A quasiconformal mapping φ of the Riemann sphere $\overline{\mathbf{C}}$ converts the standard conformal structure into a bounded measurable conformal structure $\mu(\varphi)$. A bounded measurable conformal structure μ is a measurable field of metrics on the tangent spaces to $\overline{\mathbf{C}}$ which is defined almost everywhere and only up to scale and so that the eccentricities of the unit circles are almost everywhere uniformly bounded from that of the degenerate ellipse. (See Ahlfors's book "Quasiconformal Mappings" for discussion of qc homeomorphisms.)

A remarkable theorem is that $\varphi \to \mu(\varphi)$ is onto the space of bounded measurable conformal structures. If we normalize φ to fix three points, the inverse correspondence {*bounded measurable conformal structures*} $\overset{\text{AB}}{\to}$ {*normalized quasiconformal homeomorphisms of $\overline{\mathbf{C}}$*} *is also well-defined and bijective.*

Ahlfors and Bers [3] showed AB was in a natural sense continuous, real analytic, even holomorphic. This is the measurable Riemann mapping theorem with parameters. In particular,

PROPOSITION 7. *Any finite dimensional real analytic family* $\{\alpha\}$ *of bounded measurable conformal structures* $\{\mu_\alpha\}$ *on* $\overline{\mathbf{C}}$ *determines a continuous and real analytic family of normalized quasiconformal homeomorphisms* $\{\varphi_\alpha\}$ *in the sense that for each* z *in* $\overline{\mathbf{C}}$, $\varphi_\alpha(z)$ *varies real analytically in* $\overline{\mathbf{C}}$ *and as a function of* α, φ_α *is a continuous map of* $\{\alpha\}$ *into the homeomorphisms of* $\overline{\mathbf{C}}$ *with the compact open topology.*

Proof. See Ahlfors-Bers [3].

8. Quasiconformal deformations of rational mappings

Let μ be a measurable conformal structure which is preserved a.e. (almost everywhere) by the rational mapping R. Let φ_μ be the normalized quasiconformal homeomorphism of the sphere carrying μ to the standard conformal structure a.e. (see §7).

PROPOSITION 8. *Let* $R_\mu = \varphi_\mu R \varphi_\mu^{-1}$. *Then* R_μ *is a rational mapping of the sphere (the* μ*-quasiconformal deformation of* R*).*

Proof. Locally away from the branched points of R, R_μ is by the formula a quasiconformal homeomorphism of $\overline{\mathbf{C}}$. Since R preserves μ a.e., R_μ preserves the standard conformal structure a.e. Thus, locally R_μ must be conformal (see Ahlfors's book). The branch points of R transform by φ_μ to removable singularities and thus to branch points of R_μ. Thus R_μ is a continuous map of the entire Riemann sphere, which is locally analytic and thus a rational mapping.

Remark. The same discussion applies to give quasiconformal deformations of entire mappings, subgroups of PSL(2, C), or any dynamical pseudogroup of complex analytic transformations in $\dim_{\mathbf{C}} = 1$.

9. The arc argument for simply connected wandering domains

Suppose U is a simply connected stable region so that the $U, RU, R^2U, \ldots$ are pairwise disjoint and $R^nU \xrightarrow{R} R^{n+1}U$, $n > 0$, is injective. We will give here the argument showing this is impossible. Later we will indicate the modifications required for the general case.

Let ψ: $D \to U$ be a fixed Riemann map with distinct radial limits $\bar{a}, \bar{b}, \bar{c}$ in ∂U at the points a, b, c of ∂D. We make use of a real analytic family of diffeomorphisms of ∂D fixing a, b, c of dimension greater than the dimension of the space of rational maps of degree d. Let $W_0 \times \partial D \overset{\partial\varphi}{\to} \partial D$ be a real analytic map where W_0 is an open neighborhood of the origin in Euclidean space of $\dim > 4d + 2$, $W \subset W_0$ is a compact neighborhood chosen therein, and each $\partial\varphi(w, \theta)$ is a diffeomorphism of ∂D, and $\partial\varphi(0, \theta)$ is the identity. *We suppose for* $w_1 \neq w_2$ *there exists* θ *so that* $\partial\varphi(w_1, \theta) \neq \partial\varphi(w_2, \theta)$. Call this the *injective property of* W_0. Let φ: $W_0 \times D \to D$ be defined by coning $\partial\varphi$, $\varphi(w; (r, \theta)) = (r, \partial\varphi(w, \theta))$. The $\varphi(w; \cdot)$ for w in W are uniformly Lipschitz so that they are K-quasiconformal for some K. Let $\bar{\mu}(w)$ denote the conformal distortion of $\varphi(w; \cdot)$ transported by the Riemann map ψ to U.

By our wandering assumption on U an equivalence class for the $\sim$ relation ($x \sim y \Leftrightarrow$ for some n, m, $R^n x = R^m y$) only intersects U in at most one point. Thus using R we can spread each $\bar{\mu}(w)$ over $\tilde{U}$, the union of the $\sim$ classes of points of U, to get a conformal structure preserved by R. Using the standard structure on (the sphere-$\tilde{U}$) we have a measurable conformal structure $\mu(w)$ defined on all of $\bar{\mathbf{C}}$ invariant by R and depending real analytically on w.

By the measurable Riemann mapping theorem (§7) we obtain $W \times \bar{\mathbf{C}} \overset{\sigma}{\to} \bar{\mathbf{C}}$ a real analytic family of (normalized to fix $\bar{a}$, $\bar{b}$ and $\bar{c}$) quasiconformal homeomorphisms of the sphere so that $\varphi_w = \varphi(w, \cdot)$ converts $\mu(w)$ to the standard conformal structure.

Then $R_w = \varphi_w R \varphi_w^{-1}$ is a real analytic family of quasiconformal deformations of R (Proposition 8).

If $\mathbf{C}P_0^{2d+1}$ is the space of rational maps of degree d, $W \overset{\pi}{\to} \mathbf{C}P_0^{2d+1}$ defined by $w \to R_w$ is real analytic and 0 in W maps to R. Since W has dimension larger than that of $\mathbf{C}P_0^{2d+1}$, some fibre of π has positive dimension. So let R_t, $t \in [0, 1]$, be a *nontrivial* arc in the fibre of π over some g in $\mathbf{C}P_0^{2d+1}$. Then we have $R_t = g$ or $\varphi_t R \varphi_t^{-1} = g$ or $\bar{\varphi}_t R \bar{\varphi}_t^{-1} = R$ where $\bar{\varphi}_t = \varphi_0^{-1}\varphi_t$ since $\varphi_0^{-1} g \varphi_0 = R$.

But $\bar{\varphi}_t R = R\bar{\varphi}_t$ on the Julia limit set implies $\bar{\varphi}_t$ is the identity on the Julia limit set because $\bar{\varphi}_0$ is the identity and as we have shown (§5) the centralizer of R on the Julia limit set does not contain any nontrivial arcs (since it injects into a totally disconnected group).

But then $\bar{\varphi}_t$ also acts trivially on the prime ends of U because (cf. §6):

i) The fibres are totally disconnected;

ii) $\bar{\varphi}_t$ is the identity on boundary $U \subset$ Julia set so that $\bar{\varphi}_t$ could only permute the points in various fibres (which cover all the prime ends).

We get that the φ_t: $U \to U$, when transported by the Riemann map to maps $D \xrightarrow{\bar{\bar{\varphi}}_t} D$, all give the same map on the boundary (the prime ends of U).

But this contradicts the injective property of W because the $\bar{\bar{\varphi}}_t$ must agree with the φ_t having the same normalization and the same conformal distortion.

Remark. The argument shows such a wandering disk yields, for R, a map from (normalized) quasi-symmetric maps of ∂D (the universal Teichmüller space) to rational maps of degree d which cannot map a non-trivial arc to a point. There are entire mappings (e.g. $z \xrightarrow{E} z - 1 + e^{-z} + 2\pi i$) with wandering, simply connected stable regions. The map E is just Newton's method for finding the zeroes of $e^z - 1$ composed with verticals translation. One uses this interpretation to verify the assertion about E. The wandering disk here is not eventually injective; so the arc argument above does not apply.

10. Demonstration of Theorems 1 and 2

Now we collect together the various tools to prove the main theorem. Namely,

THEOREM. *If R: $\bar{\mathbf{C}} \to \bar{\mathbf{C}}$ is a rational mapping of degree d, then every stable region (a component of the complement of the Julia set of R) is eventually periodic.*

Proof. If U_0 is such a region, then the forward images $U_1, U_2, \ldots$ are disjoint and one can apply Section 4, to get either

i) $U_n \xrightarrow{f} U_{n+1}$ is eventually bijective with finite topological type or

ii) the direct limit U_∞ exists and has infinite topological type.

For case i) we have described in detail the argument in Section 9 leading to a contradiction in the simply connected case. The finitely connected eventually bijective case is virtually identical. Now one has a Riemann map between the region and the sphere minus a finite number of round disks and this boundary of circles is described by prime ends in the region. The discussion and results of Section 6 are the same. Finally, the arc argument is the same.

For case ii), we make use of the proposition of Section 3 that $\dim \mathscr{C}(S) = \infty$ where S has infinite topological type and $\mathscr{C}(S)$ is the space of conformal equivalence classes of conformal structures on S which are quasiconformally equivalent to S. We use these structures ν to construct quasiconformal deformations R_ν of R so that the direct limit of the corresponding wandering component of R_ν is S_ν (see §§8, 9).

Thus we get a continuous injection from arbitrarily large finite dimensional manifolds of $\mathscr{C}(S)$ into the space of rational maps of degree d. Since the dimension of the latter is $2d + 1$ we have a contradiction showing that case ii) is impossible. This completes the proof of the theorem.

Proof of Theorem 2. If a Borel set A had positive measure, we could construct on J_R an infinite dimensional family $\{\mu\}$ of measurable conformal structures preserved by R. These determine quasiconformal deformations $R_\mu = \varphi_\mu R \varphi_\mu^{-1}$ where all the φ_μ are different on J_R by construction. Thus by Section 5 we again have a contradiction.

11. New proof of the Ahlfors finiteness theorem

If $\Gamma \subset \mathrm{PSL}(2, \mathbf{C})$ is a discrete subgroup, then Γ acts on the Riemann sphere $\mathbf{C}$ by Moebius transformations $z \to (az + b)/(cz + d)$. Under this action $\mathbf{C}$ is equivariantly divided into a closed set Λ (the Poincaré limit set) where every orbit is dense and a complementary open set Ω where the action is properly discontinuous. (This is due to Poincaré.) The quotient U/Γ is thus a Riemann surface.

There is, due to Ahlfors, a celebrated finiteness theorem concerning U/Γ when Γ is finitely generated.

THEOREM. *Each component of U/Γ is of conformal finite type (that is compact with finitely many punctures). There are only finitely many components of U/Γ.*

Most of the 1965 proof is easy to understand but there is one step which is more mysterious.

We can give a new proof of this theorem which is conceptually transparent and closely parallels the eventual periodicity theorem proved here for rational maps.

The idea of the proof is simply this: Since Γ is finitely generated the space of representations of Γ into $\mathrm{PSL}(2, \mathbf{C})$ is finite dimensional so that the given representation has only a finite dimensional space of deformations.

Now it is well understood since Ahlfors and Bers that the quasiconformal deformations of U/Γ determine (using the measurable Riemann mapping theorem just as in §§7, 8) deformations of Γ. Thus it follows formally that each component of U/Γ has finite topological type (see §3) and that all but finitely many components must be conformally rigid. Thus there are only two questions whose answers are not obvious:

i) Can there be infinitely many components of U/Γ which are conformally rigid?

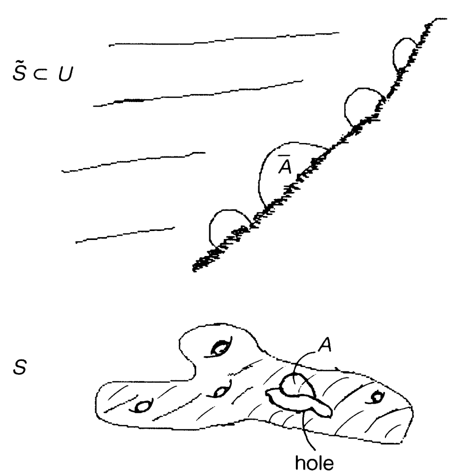
ii) Can a component of U/Γ be a Riemann surface of finite conformal type with an additional disk (or disks) removed?

The point of our discussion is to explain the analytical difficulty ii) in a new way. There are several satisfying ways to treat i), either algebraically or topologically, due to Greenberg, Bers, and Kulkarni-Shalen. In a postscript below we mention a new way due indirectly to Ahlfors in 1965.

Let us return to the more fundamental second point. The conceptual remark required to treat ii) is that if S is a component of U/Γ then not only does a qc deformation of S determine a deformation of Γ but *a* qc *deformation of* S *modulo its ideal boundary determines a deformation of* Γ.

This is so because 1) in analogy with Section 5, any homeomorphism of the limit set commuting with Γ is the identity. 2) The lift of the ideal boundary of S is contained in the ideal boundary of that part of U lying over $S \subset U/\Gamma$. 3) This ideal boundary of U can be "attached" to the limit set using prime ends as in Sections 6, 9.

To explain 3) a bit more we need only see how a hole or removed disk in a component of U/Γ yields an infinite dimensional space of deformations of Γ.



To this end consider a half disc A in S union its ideal boundary so that ∂A intersect ideal boundary S is an arc. Let $\bar{A}$ be one component of the lift of A to U.

Now we construct deformations of Γ, just as in Section 9, which are generated entirely in A and whose boundary correspondences are represented faithfully in the arc of ∂A intersect ideal boundary S. These deformations in A are lifted to U to give different deformations of Γ. To see this we use prime ends of $\bar{A}$ just as in Section 9 to overcome the technical difficulty that the frontier of $\bar{A}$ and the ideal boundary need not be homeomorphic.

Note. It is bad boundary possibilities which require the unobvious log log mollifier step in Ahlfors' original 1965 proof.

This completes our discussion of this new way to treat the Ahlfors finiteness theorem. This and the eventual periodicity theorem for rational maps are just two of many finiteness theorems possible for conformal dynamical systems based on the measurable Riemann mapping thereom.

Postscript. To treat point i) we may first pass to a subgroup Γ' of Γ of finite index without torsion (Selberg's lemma). A conformally rigid component of U/Γ' must be a triply punctured sphere. Each of these determines a cusp. Now each such cusp determines an orbit in the limit set, where the sum of spherical derivatives squared along the orbit is finite, and there can only be finitely many of these by the argument of Ahlfors in the Tulane Conference on Kleinian groups 1965 (see Sullivan, "Finiteness of cusps," Acta Math. **147** (1981), 289–299 for details).

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