8 Periodic Linear Differential Equations - Floquet Theory

The general theory of time varying linear differential equations $\dot{x}(t) = A(t)x(t)$ is still amazingly incomplete. Only for certain classes of functions $A : \mathbb{R} \longrightarrow gl(d, \mathbb{R})$ do we have a satisfactory understanding of the qualitative behavior of the solutions. Historically the first complete theory for a class of time-varying linear systems was initiated by Floquet [18] in 1883 for the periodic case. In this section we briefly review Floquet's theory and relate it to the idea of Lyapunov exponents and Lyapunov spaces as introduced in Section 2. Details supporting our discussion here can be found in Amann [2], Guckenheimer and Holmes [22], Hahn [23], Stoker [35], and Wiggins [37]. Partly, we follow the careful exposition in Chicone [13, Section 2.4].

Definition 8.1 *A **periodic linear differential equation** $\dot{x} = A(t)x$ is given by a matrix function $A : \mathbb{R} \longrightarrow gl(d, \mathbb{R})$ that is continuous and periodic (of period $T > 0$). Similarly as in Example 7.3, we use the shift $\theta(t, \tau) = t + \tau \mod T$. Then we may write $\dot{x} = A(\theta(t, 0))x$ and the solutions define a dynamical system via $\Phi : \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}^d \longrightarrow \mathbb{S}^1 \times \mathbb{R}^d$, if we identify $\mathbb{R} \mod T$ with the circle $\mathbb{S}^1$.*

Our first results concern the fundamental matrix of a periodic linear system.

We will need the following lemma which can be derived using the Jordan canonical form and the scalar logarithm (see e.g. Amann [2, Lemma 20.7] or Chicone [13, Theorem 2.47]). The difference between the real and the complex situation becomes already evident by looking at $-1 = e^{i\pi}$.

Lemma 8.2 *For every invertible matrix $S \in Gl(d, \mathbb{C})$ there is a matrix $R \in gl(d, \mathbb{C})$ such that $S = e^R$. For every invertible matrix $S \in Gl(d, \mathbb{R})$ there is a real matrix $Q \in gl(d, \mathbb{R})$ such that $S^2 = e^Q$. The eigenvalues of R and Q are mapped onto the eigenvalues of S and S^2, respectively.*

Proof. Observe that in both cases it suffices to consider a (complex or real) Jordan block. For the first statement write $S = \lambda I + N = \lambda(I + \frac{1}{\lambda}N)$ with nilpotent N, i.e., $N^m = 0$ for some $m \in \mathbb{N}$, and consider the series expansion for $t \longmapsto \ln(1 + t)$. Then $S = e^R$ with

$$R = (\ln \lambda)I + \sum_{j=1}^{m} \frac{(-1)^{j+1}}{j\lambda^j} N^j.$$

For the second assertion define $Q := R + \bar{R} \in gl(d, \mathbb{R})$. Then $S^2 = e^R e^{\bar{R}} = e^{R+\bar{R}} = e^Q$, since $S = e^R = e^{\bar{R}}$.

The proof above also shows that the eigenvalues of R and Q, respectively, are mapped onto the eigenvalues of e^R. ■

Remark 8.3 *Another way to construct Q is to observe that the real parts of the eigenvalues of S^2 are all positive. Then the real logarithm of these real parts exist and one can discuss the Jordan blocks similarly as above noting that a real logarithm of*

$$r \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \text{ is } (\ln r)I + \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

Remark 8.4 *By construction, the real parts of the eigenvalues of R and Q, respectively, are uniquely determined by S. The imaginary parts are unique up to addition of $2k\pi i, k \in k\mathbb{Z}$. In particular, several eigenvalues of R and Q may be mapped to the same eigenvalue of e^R and e^Q, respectively.*

The principal fundamental solution $X(t), t \in \mathbb{R}$, is the unique solution of the matrix differential equation

$$\dot{X}(t) = A(t)X(t) \text{ with initial value } X(0) = I. \tag{8.1}$$

Then the solutions of $\dot{x} = A(t)x,\ x(0) = x_0$, are given by $x(t) = X(t)x_0$. The following lemma shows consequences of the periodicity assumption for $A(t)$ for the fundamental solution.

Lemma 8.5 *The principal fundamental solution $X(t)$ of $\dot{x} = A(t)x$ with T-periodic $A(\cdot)$ satisfies*

$$X(kT + t) = X(t)X(T)^k \text{ for all } t \in \mathbb{R} \text{ and all } k \in \mathbb{N}.$$

Proof. The assertion is clear for $k = 0$. Suppose it holds for $k - 1 \in \mathbb{N}$. Then

$$X(kT) = X((k-1)T + T) = X(T)X(T)^{k-1} = X(T)^k. \tag{8.2}$$

Define

$$Y(t) := X(t + kT)X(kT)^{-1}, t \in \mathbb{R}.$$

Then $Y(0) = I$ and differentiation yields, using periodicity of $A(\cdot)$

$$\frac{d}{dt}Y(t) = \dot{X}(kT + t)X(kT)^{-1} = A(kT + t)X(kT + t)X(kT)^{-1} = A(t)Y(t).$$

Since the solution of this initial value problem is unique, $Y(t) = X(t)$ and hence, by (8.2),

$$X(t + kT) = X(t)X(kT) = X(t)X(T)^k \text{ for } t \in \mathbb{R}.$$

■

Proposition 8.6 *There is a matrix $Q \in gl(d, \mathbb{R})$ such that the fundamental solution $X(\cdot)$ satisfies*

$$X(2T) = e^{2TQ}.$$

The real parts λ_j of the eigenvalues of Q are uniquely determined by this condition, and are called ***Floquet exponents****. Furthermore, the eigenvalues α_j of $X(2T) = X(T)^2$ satisfy $|\alpha_j| = e^{\lambda_j}$.*

Proof. By Lemma 8.2 a matrix Q with $X(2T) = e^{2TQ}$ exists and the real parts of the eigenvalues are unique. The eigenvalues of Q are mapped to the eigenvalues α_j of $X(2T)$ and their imaginary part does not contribute to the absolute values of the α_j. ■

Next we relate the Floquet exponents to the Lyapunov exponents $\lambda(x_0) = \limsup_{t\to\infty} \frac{1}{t} \ln \|\varphi(t, x_0)\|$, where $\varphi(t, x_0)$ denotes the solution of $\dot{x} = A(t)x$ with $\varphi(0, x_0) = x_0$ (compare Definition 2.12).

The following theorem for periodic linear differential equations is analogous to Theorem 2.13.

Theorem 8.7 *Let $\Phi = (\theta, \varphi) : \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}^d \longrightarrow \mathbb{S}^1 \times \mathbb{R}^d$ be the flow associated with a periodic linear differential equation $\dot{x} = A(t)x$. The system has a finite number of Lyapunov exponents and they coincide with the Floquet exponents λ_j, $j = 1, ..., l \leq d$. For each exponent λ_j and each $\tau \in \mathbb{S}^1$ there exists a splitting $\mathbb{R}^d = \bigoplus_{j=1}^{l} L(\lambda_j, \tau)$ of $\mathbb{R}^d$ into linear subspaces with the following properties:*

(i) The subspaces $L(\lambda_j, \tau)$ have the same dimension independent of τ, i.e. for each $j = 1, ..., l$ it holds that $\dim L(\lambda_j, \sigma) = \dim L(\lambda_j, \tau) =: d_i$ for all $\sigma, \tau \in \mathbb{S}^1$,

(ii) the subspaces $L(\lambda_j, \tau)$ are invariant under the flow Φ, i.e. for each $j = 1, ..., l$ it holds that $\varphi(t, \tau)L(\lambda_j, \tau) = L(\lambda_j, \theta(t, \tau)) = L(\lambda_j, t + \tau)$ for all $t \in \mathbb{R}$ and $\tau \in \mathbb{S}^1$,

(iii) $\lambda(x, \tau) = \lim_{t\to\pm\infty} \frac{1}{t} \ln \|\varphi(t, \tau, x)\| = \lambda_j$ if and only if $x \in L(\lambda_j, \tau)\backslash\{0\}$.

Proof. By their definition in Proposition **??**, the Floquet exponents λ_j are the real parts of the eigenvalues of $Q \in gl(d, \mathbb{R})$.

First we show that the Floquet exponents are the Lyapunov exponents. By Lemma 8.5 we can write

$$X(kT + s) = X(kT)X(s) \text{ for all } k \in \mathbb{Z} \text{ and } t, s \in \mathbb{R}.$$

and recall that

$$X(2T) = e^{2TQ}.$$

For the autonomous linear differential equation $\dot{y} = Qy$ Theorem 2.13 yields a decomposition of $\mathbb{R}^d$ into subspaces $L(\lambda_j)$ which are characterized by the property that the Lyapunov exponents for $t \to \pm\infty$ are given by the real parts λ_j of the eigenvalues.

The continuously differentiable matrix function $Z(t) := X(t)e^{-Qt}, t \in \mathbb{R}$, maps the solution $e^{Qt}x_0$ of $\dot{y} = Qy, y(0) = x_0 \in \mathbb{R}^d$, to the solutions of $\dot{x} = A(t)x, x(0) = x_0$, since

$$X(t)x_0 = X(t)e^{-Qt}e^{Qt}x_0 = Z(t)\left[e^{Qt}x_0\right]. \tag{8.3}$$

Observe that $Z(t)$ is $2T$-periodic, since

$$Z(2T+s) = X(2T+s)e^{-(2T+s)Q} = X(s)X(2T)e^{-2TQ}e^{-Qs} = X(s)e^{-sQ} = Z(s)$$

Since $Z(\cdot)$ is continuous, it follows that $Z(t)$ and $Z(t)^{-1}$ are bounded on $\mathbb{R}$.

The exponential growth rates remain constant under multiplication by the bounded matrix $Z(t)$ with bounded inverse $Z(t)^{-1}$. Hence we get a corresponding decomposition of $\mathbb{R}^d$ which is characterized by the property that the exponential growth rates for a solution starting at time $t = 0$ in the corresponding subspace $L(\lambda_j, 0) := L(\lambda_j)$ has exponential growth rate equal to a given Floquet exponent λ_j. Then

$$L(\lambda_j, \tau) := X(\tau)L(\lambda_j, 0) \;\; \tau \in \mathbb{R},$$

are subspaces which yields a splitting of $\mathbb{R}^d$ into subspaces characterized by the property that the exponential growth rates for a solution starting at time $t = \tau$ in the corresponding subspace $L(\lambda_j, \tau)$ has exponential growth rate for $t \to \pm\infty$ equal to λ_j. But the exponential growth rate of the solution $x(t, x_0)$ with $x(0) = x_0$ is equal to the exponential growth rate of the solution $y(t)$ with $y(T) = x_0$. In fact, for $t \in \mathbb{R}$

$$x(t, x_0) = X(t)x_0 = X(t)x_0$$

and

$$x_0 = X(T)y(0), \text{ i.e., } y(0) = X(T)^{-1}x_0$$

implies

$$y(t) = X(t)X(T)^{-1}x_0.$$

Hence for $t \in \mathbb{R}$

$$y(t+T) = X(t+T)X(T)^{-1}x_0 = X(t)x_0 = x(t, x_0).$$

and the exponential growth rates for $t \to \pm\infty$ coincide. This shows that the decomposition above is T-periodic and, clearly, it also depends continuously on τ. ■

Corollary 8.8 *For each $j = 1, ..., l \leq d$ the map $L_j : \mathbb{S}^1 \longrightarrow \mathbb{G}_{d_j}$ defined by $\tau \longmapsto L(\lambda_j, \tau)$ is continuous. The linear subspaces $L(\lambda_j, \cdot)$ are called the Lyapunov spaces (or sometimes the Floquet spaces) of the periodic matrix function $A(t)$.*

Proof. This follows from the construction of the spaces $L(\lambda_j, \tau)$ and the corresponding properties of the Lyapunov spaces of the autonomous equation $\dot{x} = Qx$. ■

These facts show that for periodic matrix functions $A : \mathbb{R} \longrightarrow gl(d, \mathbb{R})$ the Floquet exponents and Floquet spaces replace the real parts of eigenvalues and the Lyapunov spaces, concepts that are so useful in the linear algebra of (constant) matrices $A \in gl(d, \mathbb{R})$. The number of Lyapunov exponents and the dimensions of the Lyapunov spaces are independent of $\tau \in \mathbb{S}^1$, while the Lyapunov spaces themselves depend on the time parameter τ of the periodic matrix function $A(t)$, and they form periodic orbits in the Grassmannians $\mathbb{G}_{d_j}$ and in the corresponding flag.

Remark 8.9 *Transformations as $Z(t)$ are known as Lyapunov transformations, see [23], Chapters 61-63.*

Periodic linear differential equations yield periodic differential equations in projective space: As in Lemma 5.1, the flow $\Phi : \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}^d \longrightarrow \mathbb{S}^1 \times \mathbb{R}^d$ corresponding to $\dot{x} = A(t)x$ projects onto a flow $\mathbb{P}\Phi$ on $\mathbb{S}^1 \times \mathbb{P}^{d-1}$ where again the first component is the shift by $\theta(t, \tau) = t + \tau \mod T$ and the second component is given by the solutions of the periodic differential equation

$$\dot{s} = (A(t) - s^T A(t)s\; I)\; s \text{ with } s \in \mathbb{P}^{d-1}.$$

The next corollary characterizes the Lyapunov spaces for periodic linear differential equations by this projected flow. It is the analogue of Theorem 5.2.

Corollary 8.10 *Let $\mathbb{P}\Phi$ be the projection onto $\mathbb{S}^1 \times \mathbb{P}^{d-1}$ of a periodic linear flow as defined above. Then the following assertions hold.*

(i) $\mathbb{P}\Phi$ has l chain recurrent components $\{\mathcal{M}_1, ..., \mathcal{M}_l\}$, where l is the number of different Lyapunov exponents.

(ii) For each Lyapunov exponent λ_i one has that $\mathcal{M}_i = \{(\tau, \mathbb{P}x),\ x \in L_i(\lambda_i, \tau) \text{ and } \tau \in \mathbb{S}^1\}$, the projection of the i-th Lyapunov space $L_i(\lambda_i, \tau)$ onto $\mathbb{P}^{d-1}$. Furthermore $\{\mathcal{M}_1, ..., \mathcal{M}_l\}$ defines the finest Morse decomposition of $\mathbb{P}\Phi$ and $\mathcal{M}_i \prec \mathcal{M}_j$ if and only if $\lambda_i < \lambda_j$.

(iii) For the sets $\mathcal{M}_i$ in the finest Morse decomposition, the sets

$$\mathcal{V}_i^\tau := \{x \in \mathbb{R}^d,\ (\tau, \mathbb{P}x) \in \mathcal{M}_i\},\ \tau \in \mathbb{S}^1,$$

coincide with the Lyapunov spaces and hence yield decompositions of $\mathbb{R}^d$ into linear subspaces

$$\mathbb{R}^d = \mathcal{V}_1^\tau \oplus\ ...\ \oplus \mathcal{V}_l^\tau,\ \tau \in \mathbb{S}^1. \tag{8.4}$$

Proof. For the autonomous linear equation $\dot{x} = Qx$ we have a decomposition of $\mathbb{R}^d$ into the Lyapunov spaces $L(\lambda_j, 0)$ which by Theorem 5.2 correspond to the Morse sets in the finest Morse decomposition. By (8.3) the matrix function $Z(t)$ maps the solution of $\dot{x} = Qy, y(0) = x_0 \in \mathbb{R}^d$, to the solution of $\dot{x} = A(t)x, x(0) = x_0$.Since these maps and their inverses are uniformly bounded by compactness of $\mathbb{S}^1 \times \mathbb{P}^{d-1}$, one can show that the maximal chain transitive sets in $\mathbb{P}^{d-1}$ are mapped onto the maximal chain transitive sets in $\mathbb{S}^1 \times \mathbb{P}^{d-1}$. Then the assertions follow. ■

As an application of these results, consider the problem of stability of the zero solution of $\dot{x}(t) = A(t)x(t)$ with period $T > 0$. The following definition generalizes the last part of Definition 2.12.

Definition 8.11 *The **stable, center, and unstable subspaces** associated with the periodic matrix function $A : \mathbb{R} \longrightarrow gl(d, \mathbb{R})$ are defined as $L^-(\tau) = \bigoplus\{L(\lambda_j, \tau),\ \lambda_j < 0\}$, $L^0(\tau) = \bigoplus\{L(\lambda_j, \tau),\ \lambda_j = 0\}$, and $L^+(\tau) = \bigoplus\{L(\lambda_j, \tau),\ \lambda_j > 0\}$, respectively, for $\tau \in \mathbb{S}^1$.*

With these preparations we can state the main result regarding stability of periodic linear differential equations.

Theorem 8.12 *The zero solution $x(t, 0) \equiv 0$ of the periodic linear differential equation $\dot{x} = A(t)x$ is asymptotically stable if and only if it is exponentially stable if and only if all Lyapunov exponents are negative if and only if $L^-(\tau) = \mathbb{R}^d$ for some (and hence for all) $\tau \in \mathbb{S}^1$.*

To show the power of Floquet's approach we discuss two classical examples.

Example 8.13 *Hamiltonian systems: Let H be a continuous quadratic form in $2d$ variables $x_1, ..., x_d$, $y_1, ..., y_d$ and consider the Hamiltonian system*

$$\dot{x}_i = \frac{\partial H}{\partial y_i},\ \dot{y}_i = -\frac{\partial H}{\partial x_i},\ i = 1, ..., d.$$

Using $\mathbf{z}^T = [x^T, y^T]$ we can set $H(x, y, t) = \mathbf{z}^T A(t)\mathbf{z}$, where $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$ with A_{11} and A_{22} symmetric, and hence the equation takes the form

$$\dot{\mathbf{z}} = \begin{bmatrix} A_{12}^T(t) & A_{22}(t) \\ -A_{11}(t) & -A_{12}(t) \end{bmatrix} \mathbf{z} =: P(t)\mathbf{z}.$$

Note that $-P^T(t) = QP(t)Q^{-1}$ with $Q = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ where I is the $d \times d$ identity matrix. Assume that H is T-periodic, then the equation for z and its adjoint have the same Floquet exponents and for each exponent λ its negative $-\lambda$ is also a Floquet exponent. Hence the fixed point $0 \in \mathbb{R}^{2d}$ cannot be exponentially stable, compare [23], Chapter 60.

Example 8.14 *Hill-Mathieu equations: Consider the periodic linear oscillator*

$$\ddot{y} + q_1(t)\dot{y} + q_2(t)y = 0.$$

Using the substitution $y = z\exp(-\frac{1}{2}\int q_1(u)du)$ *one obtains Hill's differential equation*

$$\ddot{z} + p(t)z = 0,\ p(t) := q_2(t) - \frac{1}{4}q_1(t)^2 - \frac{1}{2}\dot{q}_1(t).$$

Its characteristic equation is $\lambda^2 - 2a\lambda + 1 = 0$, *with* a *still to be determined. The multipliers satisfy the relations* $\alpha_1\alpha_2 = 1$ *and* $\alpha_1 + \alpha_2 = 2a$. *The exponential stability of the system can be analyzed using the parameter* a: *If* $a^2 > 1$, *then one of the multipliers has absolute value* > 1, *and hence the system has an unbounded solution. If* $a^2 = 1$, *then the system has a non-trivial periodic solution according to Example 1. If* $a^2 < 1$, *then the system is stable. The parameter* a *can often be expressed in form of a power series, see [23], Chapter 62, for more details. A special case of Hill's equation is the Mathieu equation*

$$\ddot{z} + (\beta_1 + \beta_2 \cos 2t)z = 0,$$

with β_1, β_2 *real parameters. For this equation numerically computed stability diagrams are available, see, e.g., [35], Chapters VI.3 and 4.*