A "typical" $n \times n$ matrix has n distinct eigenvalues

In the sequel, we will denote by $\mathbb{R}^{n\times n}$ the vector space of $n \times n$ matrices with real entries. Similarly, $\mathbb{C}^{n\times n}$ will denote the vector space of $n \times n$ matrices with complex entries. Let

$$U_n = \{A \in \mathbb{C}^{n\times n} : A \text{ has } n \text{ distinct eigenvalues}\}$$

and

$$V_n = \{A \in \mathbb{R}^{n\times n} : A \text{ has } n \text{ distinct (possibly complex) eigenvalues}\}.$$

Theorem 1. *The set U_n is an open and dense subset of $\mathbb{C}^{n\times n}$.*

Proof. We first show that the set U_n is open. To this end, it suffices to show that the complement $\mathbb{C}^{n\times n} \setminus U_n$ is closed. Suppose that $\{A_k : k \in \mathbb{N}\}$ is a sequence of matrices in $\mathbb{C}^{n\times n}$ such that $A_k \notin U_n$ for all $k \in \mathbb{N}$ and $A_k \to A$ as $k \to \infty$. We need to show that $A \notin U_n$.

Since A_k does not belong to the set U_n, the matrix A_k has at most $n-1$ distinct eigenvalues. Hence, the matrix A_k has at least one eigenvalue λ_k with algebraic multiplicity at least 2. This implies $p_k(\lambda_k) = p_k'(\lambda_k) = 0$, where $p_k(\lambda) = \det(\lambda I - A_k)$ denotes the characteristic polynomial of A_k. Let $p(\lambda) = \det(\lambda I - A)$ be the characteristic polynomial of the limit matrix A. Since $A_k \to A$ as $k \to \infty$, the coefficients of $p_k(\lambda)$ converge to the corresponding coefficients of $p(\lambda)$ as $k \to \infty$. In particular, all coefficients of $p_k(\lambda)$ are uniformly bounded (independent of k). Consequently, $|p_k(\lambda) - \lambda^n| \leq M \sum_{j=0}^{n-1} |\lambda|^j$, where M is a constant independent of k. Since $p_k(\lambda_k) = 0$, it follows that $|\lambda_k|^n \leq M \sum_{j=0}^{n-1} |\lambda_k|^j$. From this it is easy to see that the sequence $\{\lambda_k : k \in \mathbb{N}\}$ is bounded, i.e. $\sup_{k\in\mathbb{N}} |\lambda_k| < \infty$.

By the Bolzano-Weierstrass theorem, we can find a subsequence $\{\lambda_{k_j} : j \in \mathbb{N}\}$ of the original sequence $\{\lambda_k : k \in \mathbb{N}\}$ such that the limit $\lim_{j\to\infty} \lambda_{k_j} = \mu$ exists. The number μ satisfies $p(\mu) = \lim_{j\to\infty} p_{k_j}(\lambda_{k_j}) = 0$ and $p'(\mu) = \lim_{j\to\infty} p_{k_j}'(\lambda_{k_j}) = 0$. Hence, μ is an eigenvalue of A with algebraic multiplicity at least 2. Therefore, A has at most $n-1$ distinct eigenvalues. Thus, $A \notin U_n$ as claimed.

We next show that the set U_n is dense. The proof is by induction on n. Assume that U_{n-1} is a dense subset of $\mathbb{C}^{(n-1)\times(n-1)}$ (i.e. every $(n-1)\times(n-1)$ matrix can be approximated by one that has $n-1$ distinct eigenvalues). We claim that U_n is a dense subset of $\mathbb{C}^{n\times n}$. To this end, we assume that a matrix $A \in \mathbb{C}^{n\times n}$ and a positive real number r are given. Our goal is to find a matrix $B \in U_n$ such that $\|A - B\| < r$. By the fundamental theorem of algebra, the matrix A has an eigenvalue $\lambda \in \mathbb{C}$. Let $v_1 \in \mathbb{C}^n$ be an eigenvector of A with eigenvalue λ. Since $v_1 \neq 0$, we can find vectors $v_2, \dots, v_n \in \mathbb{C}^n$ such that $\{v_1, v_2, \dots, v_n\}$ is linearly independent. We now define

$$S = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \in \mathbb{C}^{n\times n}.$$

The matrix A can be written in the form

$$A = S \begin{bmatrix} \lambda & Q \\ 0 & A_0 \end{bmatrix} S^{-1},$$

where $A_0 \in \mathbb{C}^{(n-1)\times(n-1)}$ and $Q \in \mathbb{C}^{1\times(n-1)}$. The induction hypothesis guarantees the existence of a matrix $B_0 \in \mathbb{C}^{(n-1)\times(n-1)}$ such that B_0 has $n-1$ distinct eigenvalues and

$$\left\| S \begin{bmatrix} 0 & 0 \\ 0 & A_0 - B_0 \end{bmatrix} S^{-1} \right\| < \frac{r}{2}.$$

Moreover, we can find a number $\delta \in \mathbb{C}$ such that $\lambda + \delta$ is not an eigenvalue of B_0 and

$$\left\| S \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix} S^{-1} \right\| < \frac{r}{2}.$$

We then define

$$B = S \begin{bmatrix} \lambda + \delta & Q \\ 0 & B_0 \end{bmatrix} S^{-1} \in \mathbb{C}^{n\times n}.$$

Every eigenvalue of B_0 is an eigenvalue of B. Moreover, $\lambda + \delta$ is another eigenvalue of B. Since B_0 has $n-1$ distinct eigenvalues and $\lambda + \delta$ is not an eigenvalue of B, we conclude that B has n distinct eigenvalues. Since $\|A - B\| < \frac{r}{2} + \frac{r}{2} = r$, the proof is complete.

Theorem 2. *The set V_n is an open and dense subset of $\mathbb{R}^{n\times n}$.*

Proof. We know from the previous theorem that U_n is an open subset of $\mathbb{C}^{n\times n}$. This, in particular, implies that V_n is an open subset of $\mathbb{R}^{n\times n}$. However, the fact that U_n is a dense subset of $\mathbb{C}^{n\times n}$ is not sufficient to conclude that V_n is a dense subset of $\mathbb{R}^{n\times n}$. (It follows from Theorem 1 that every matrix $A \in \mathbb{R}^{n\times n}$ can be approximated by a matrix $B \in \mathbb{C}^{n\times n}$ that has n distinct eigenvalues. Theorem 2 tells us that we can find an approximating matrix B, whose entries are *real numbers*.)

In order to show that V_n is a dense subset of $\mathbb{R}^{n\times n}$, we proceed by induction on n. We begin with the case $n = 2$. A 2×2 matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has two distinct eigenvalues if and only if $(a-d)^2 \neq -4bc$. The set of all 2×2 matrices satisfying this condition is clearly dense. Therefore, V_2 is a dense subset of $\mathbb{R}^{2\times 2}$.

We next assume that V_{n-1} is a dense subset of $\mathbb{R}^{(n-1)\times(n-1)}$ and V_{n-2} is a dense subset of $\mathbb{R}^{(n-2)\times(n-2)}$. Our goal is to show that V_n is a dense subset of $\mathbb{R}^{n\times n}$. To this end, we assume that a matrix $A \in \mathbb{R}^{n\times n}$ and a positive real number r are given. We need to find a matrix $B \in V_n$ such that $\|A-B\| < r$. Let λ be a (possibly complex) eigenvalue of A. We consider two cases:

Case 1: Suppose that $\lambda = \bar{\lambda}$, so that $\lambda \in \mathbb{R}$. Let $v_1 \in \mathbb{R}^n$ be an eigenvector of A with eigenvector λ. As above, we choose vectors $v_2, \ldots, v_n \in \mathbb{R}^n$ such that $\{v_1, v_2, \ldots, v_n\}$ is linearly independent, and define

$$S = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

The matrix A can be written in the form

$$A = S \begin{bmatrix} \lambda & Q \\ 0 & A_0 \end{bmatrix} S^{-1},$$

where $A_0 \in \mathbb{R}^{(n-1)\times(n-1)}$ and $Q \in \mathbb{R}^{1\times(n-1)}$. As above, the induction hypothesis implies the existence of a matrix $B_0 \in \mathbb{R}^{(n-1)\times(n-1)}$ such that B_0 has $n-1$ distinct eigenvalues and

$$\left\| S \begin{bmatrix} 0 & 0 \\ 0 & A_0 - B_0 \end{bmatrix} S^{-1} \right\| < \frac{r}{2}.$$

Moreover, we can find a number $\delta \in \mathbb{R}$ such that $\lambda + \delta$ is not an eigenvalue of B_0 and

$$\left\| S \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix} S^{-1} \right\| < \frac{r}{2}.$$

We then define

$$B = S \begin{bmatrix} \lambda + \delta & Q \\ 0 & B_0 \end{bmatrix} S^{-1} \in \mathbb{R}^{n \times n}.$$

It is easy to see that B has n distinct eigenvalues. Since $\|A - B\| < \frac{r}{2} + \frac{r}{2} = r$, the assertion follows.

Case 2: Suppose that $\lambda \neq \bar{\lambda}$. Let $v \in \mathbb{C}^n$ be an eigenvector of A with eigenvalue λ. We define $w_1 = \mathrm{Re}(v) \in \mathbb{R}^n$ and $w_2 = \mathrm{Im}(v) \in \mathbb{R}^n$. Since $\mathrm{Im}(\lambda) \neq 0$, the vectors w_1 and w_2 are linearly independent. We choose vectors $w_3, \dots, w_n \in \mathbb{R}^n$ such that $\{w_1, w_2, w_3, \dots, w_n\}$ is linearly independent, and define

$$S = \begin{bmatrix} | & | & | & & | \\ w_1 & w_2 & w_3 & \cdots & w_n \\ | & | & | & & | \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

Since $Av = \lambda v$, we have

$$A = S \begin{bmatrix} E & Q \\ 0 & A_0 \end{bmatrix} S^{-1},$$

where

$$E = \begin{bmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{bmatrix} \in \mathbb{R}^{2 \times 2},$$

$A_0 \in \mathbb{R}^{(n-2) \times (n-2)}$ and $Q \in \mathbb{R}^{2 \times (n-2)}$. It follows from the induction hypothesis that there exists a matrix $B_0 \in \mathbb{R}^{(n-2) \times (n-2)}$ such that B_0 has $n - 2$ distinct eigenvalues and

$$\left\| S \begin{bmatrix} 0 & 0 \\ 0 & A_0 - B_0 \end{bmatrix} S^{-1} \right\| < \frac{r}{2}.$$

Moreover, we can find a real number δ such that neither $\lambda + \delta$ nor $\bar{\lambda} + \delta$ is an eigenvalue of B_0 and

$$\left\| S \begin{bmatrix} \delta\, I_{2 \times 2} & 0 \\ 0 & 0 \end{bmatrix} S^{-1} \right\| < \frac{r}{2}.$$

We then define

$$B = S \begin{bmatrix} E + \delta\, I_{2 \times 2} & Q \\ 0 & B_0 \end{bmatrix} S^{-1} \in \mathbb{R}^{n \times n}.$$

As above, it is not difficult to see that every eigenvalue of B_0 is also an eigenvalue of B. Moreover, the numbers $\lambda + \delta$ and $\overline{\lambda} + \delta$ are eigenvalues of $E + \delta I_{2\times 2}$. Therefore, the numbers $\lambda + \delta$ and $\overline{\lambda} + \delta$ are eigenvalues of B. Note that $\lambda + \delta$ and $\overline{\lambda} + \delta$ are two distinct complex numbers, none of which is an eigenvalue of B_0. Therefore, B has n distinct eigenvalues. Moreover, it is easy to see that $\|A - B\| < r$. This completes the proof.