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Quasiconformal Homeomorphisms and Dynamics III. The Teichmüller Space of a Holomorphic Dynamical System

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Contents.

1. INTRODUCTION

This article completes in a definitive way the picture of rational mappings begun in [30]. It also provides new proofs for and expands upon an earlier version [46] from the early 1980s. In the meantime the methods and conclusions have become widely used, but rarely stated in full generality. We hope the present treatment will provide a useful contribution to the foundations of the field.

Let $f: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map on the Riemann sphere $\widehat{\mathbb{C}}$, with degree $d > 1$, whose iterates are to be studied.

The sphere is invariantly divided into the compact *Julia set* J, which is dynamically chaotic and usually fractal, and the *Fatou set* $\Omega = \widehat{\mathbb{C}} - J$ consisting of countably many open connected *stable regions* each of which

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is eventually periodic under forward iteration (by the main theorem of [44]). The cycles of stable regions are of five types: parabolic, attractive and superattractive basins; Siegel disks; and Herman rings. For completeness we provide a fresh proof of this classification here.

The goal of this paper is to use the above picture of the dynamics to obtain a concrete description of the *Teichmüller space* of quasiconformal deformations of f, in terms of a "quotient Riemann surface" (consisting of punctured spheres and tori, and foliated disks and annuli), and the measurable dynamics on the Julia set. This development parallels the deformation theory of Kleinian groups and many features can be carried through for other holomorphic dynamical systems.

We show the quotient $\mathrm{Teich}(\hat{\mathbb{C}}, f)/\mathrm{Mod}(\hat{\mathbb{C}}, f)$ of Teichmüller space by its *modular group* of symmetries is a complex orbifold which maps injectively into the finite-dimensional space of rational maps of degree d. This gives another route to the no-wandering-domains theorem, and recovers the finiteness of the number of cycles of stable regions.

It turns out that quasiconformal deformations are ubiquitous: for any family $f_\lambda(z)$ of rational maps parameterized by a complex manifold X, there exists an open dense subset X_0 for which each component consists of a single quasiconformal conjugacy class. In particular, *such open conjugacy classes are dense in the space of all rational maps*. We complete here the proof of this statement begun in [30], using extensions of holomorphic motions [47, 7].

We also complete the proof that *expanding* rational maps are quasiconformally structurally stable (these maps are also called *hyperbolic* or *Axiom A*). Thus the Teichmüller space introduced above provides a holomorphic parameterization of the open set of hyperbolic rational maps in a given topological conjugacy class. In particular the topological conjugacy class of a hyperbolic rational map is *connected*, a rather remarkable consequence of the measurable Riemann mapping theorem.

Conjecturally, these expanding rational maps are open and dense among all rational maps of a fixed degree. We show this conjecture follows if one can establish that (apart from classical examples associated to complex tori), a rational map carries no measurable invariant line field on its Julia set. The proof depends on the Teichmüller theory developed below and the instability of Herman rings [29].

In the last section we illustrate the general theory with the case of quadratic polynomials.

An early version of this article [46] was circulated in preprint form in 1983. The present work achieves the foundations of the Teichmüller theory of general holomorphic dynamical systems in detail, recapitulates a self-contained account of Ahlfors' finiteness theorem (Section 4.3), the no wandering domains theorem (Section 6.3) and the density of structurally stable rational maps (Section 7), and completes several earlier arguments as mentioned above.

2. STATEMENT OF RESULTS

Let f be a rational map of degree $d > 1$. By the main result of [44], every component of the Fatou set Ω of f maps to a periodic component Ω_0 after a finite number of iterations. The following theorem is essentially due to Fatou; we give a short modern proof in Section 3. That the last two possibilities actually occur in rational dynamics depends on work of Siegel, Arnold, and Herman.

THEOREM 2.1 (Classification of Stable Regions). *A component Ω_0 of period p in the Fatou set of a rational map f is of exactly one of the following five types* (*see Fig. 1*).

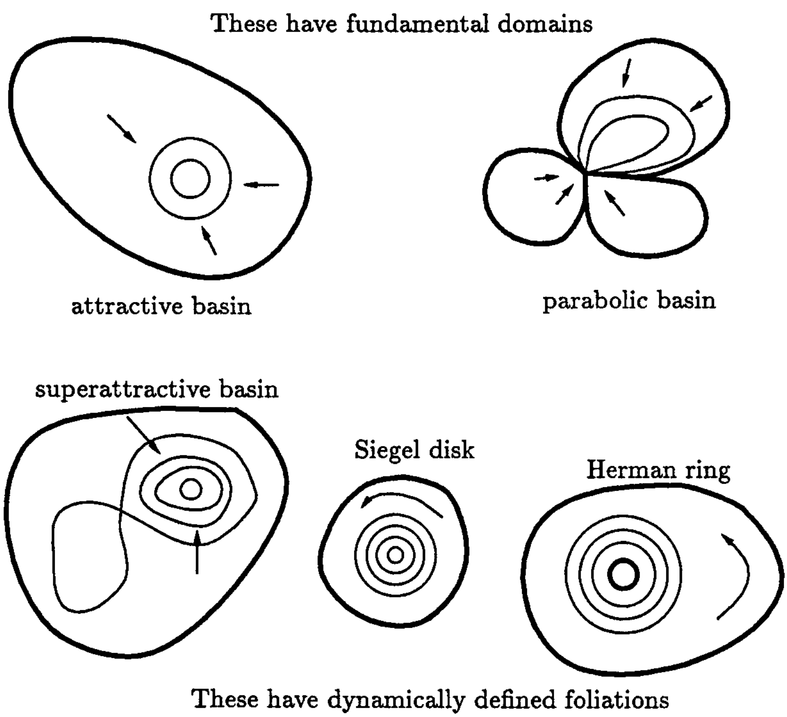

Fig. 1. The five types of stable regions.

1. *An* attractive basin: *there is a point* x_0 *in* Ω_0, *fixed by* f^p, *with* $0 < |(f^p)'(x_0)| < 1$, *attracting all points of* Ω_0 *under iteration of* f^p.

2. *A* superattractive basin: *as above, but* x_0 *is a critical point of* f^p, *so* $(f^p)'(x_0) = 0$.

3. *A* parabolic basin: *there is a point* x_0 *in* $\partial\Omega_0$ *with* $(f^p)'(x_0) = 1$, *attracting all points of* Ω_0.

4. *A* Siegel disk: Ω_0 *is conformably isomorphic to the unit disk, and* f^p *acts by an irrational rotation.*

5. *A* Herman ring: Ω_0 *is isomorphic to an annulus, and* f^p *acts again by an irrational rotation.*

A *conjugacy* $\phi: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ between two rational maps f and g is a bijection such that $\phi \circ f = g \circ \phi$. A conjugacy may be measurable, topological, quasiconformal, conformal, etc., depending upon the quality of ϕ.

To describe the rational maps g quasiconformally conjugate to a given map f, it is useful (as in the case of Riemann surfaces) to introduce the *Teichmüller space* $\mathrm{Teich}(\hat{\mathbb{C}}, f)$ of *marked* rational maps (Section 4). This space is provided with a natural metric and complex structure.

Our main results (Section 6) describe $\mathrm{Teich}(\hat{\mathbb{C}}, f)$ as a product of three simpler spaces:

> First, there is an open subset Ω^{dis} of Ω on which f acts discretely. The quotient $\Omega^{\mathrm{dis}}/f = Y$ is a complex 1-manifold of finite area whose components are punctured spheres or tori, and the Teichmüller space of Y forms one factor of $\mathrm{Teich}(\hat{\mathbb{C}}, f)$. Each component is associated to an attracting or parabolic basin of f.
>
> Second, there is a disjoint open set Ω^{fol} of Ω admitting a dynamically defined foliation, for which the Teichmüller space of $(\Omega^{\mathrm{fol}}, f)$ is isomorphic to that of a finite number of foliated annuli and punctured disks. This factor comes from Siegel disks, Herman rings and superattracting basins. Each annulus contributes one additional complex modulus to the Teichmüller space of f.
>
> Finally, there is a factor $M_1(J, f)$ spanned by measurable invariant complex structures on the Julia set. This factor vanishes if the Julia set has measure zero. There is only one class of examples for which this factor is known to be nontrivial, namely the maps of degree n^2 coming from multiplication by n on a complex torus whose structure can be varied (Section 9).

We sum up this description as:

THEOREM 2.2 (The Teichmüller Space of a Rational Map). *The space* Teich$(\hat{\mathbb{C}}, f)$ *is canonically isomorphic to a* connected *finite-dimensional complex manifold, which is the product of a polydisk* (*coming from the foliated annuli and invariant line fields on the Julia set*) *and traditional Teichmüller spaces associated to punctured spheres and tori.*

In particular, the obstruction to deforming a quasiconformal conjugacy between two rational maps to a conformal conjugacy is measured by finitely many complex moduli.

The *modular group* Mod$(\hat{\mathbb{C}}, f)$ is defined as $QC(\hat{\mathbb{C}}, f)/QC_0(\hat{\mathbb{C}}, f)$, the space of isotopy classes of quasiconformal automorphisms ϕ of f (Section 4). In Section 6.4 we will show:

THEOREM 2.3 (Discreteness of the Modular Group). *The group* Mod$(\hat{\mathbb{C}}, f)$ *acts properly discontinuously by holomorphic automorphisms of* Teich$(\hat{\mathbb{C}}, f)$.

Let Rat_d denote the space of all rational maps $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree d. This space can be realized as the complement of a hypersurface in projective space $\mathbb{P}^{2d+1}$ by considering $f(z) = p(z)/q(z)$ where p and q are relatively prime polynomials of degree d in z. The group of Möbius transformations $\mathrm{Aut}(\hat{\mathbb{C}})$ acts on Rat_d by sending f to its conformal conjugates.

A *complex orbifold* is a space which is locally a complex manifold divided by a finite group of complex automorphisms.

COROLLARY 2.4 (Uniformization of Conjugacy Classes). *There is a natural holomorphic injection of complex orbifolds*

$$\mathrm{Teich}(\hat{\mathbb{C}}, f)/\mathrm{Mod}(\hat{\mathbb{C}}, f) \to \mathrm{Rat}_d/\mathrm{Aut}(\hat{\mathbb{C}})$$

parameterizing the rational maps g quasiconformal conjugate to f.

COROLLARY 2.5. *If the Julia set of a rational map is the full sphere, then the group* Mod$(\hat{\mathbb{C}}, f)$ *maps with finite kernel into a discrete subgroup of* $PSL_2(\mathbb{R})^n \rtimes S_n$ (*the automorphism group of the polydisk*).

Proof. The Teichmüller space of f is isomorphic to $\mathbb{H}^n$. ∎

COROLLARY 2.6 (Finiteness theorem). *The number of cycles of stable regions of f is finite.*

Proof. Let d be the degree of f. By Corollary 2.4, the complex dimension of $\mathrm{Teich}(\hat{\mathbb{C}}, f)$ is at most $2d-2$. This is also the number of critical points of f, counted with multiplicity.

By Theorem 2.2 f has at most $2d-2$ Herman rings, since each contributes at least a one-dimensional factor to $\mathrm{Teich}(\hat{\mathbb{C}}, f)$ (namely the Teichmüller space of a foliated annulus (Section 5)). By a classical argument, every attracting, superattracting or parabolic cycle attracts a critical point, so there are at most $2d-2$ cycles of stable regions of these types. Finally the number of Siegel disks is bounded by $4d-4$. (The proof, which goes back to Fatou, is that a suitable perturbation of f renders at least half of the indifferent cycles attracting; cf. [35, Section 106].)

Consequently the total number of cycles of stable regions is at most $8d-8$. ∎

Remark. The sharp bound of $2d-2$ (conjectured in [44]) has been achieved by Shishikura and forms an analogue of Bers' area theorem for Kleinian groups [39, 5].

Open Questions. The behavior of the map $\mathrm{Teich}(\hat{\mathbb{C}}, f)/\mathrm{Mod}(\hat{\mathbb{C}}, f) \to \mathrm{Rat}_d/\mathrm{Aut}(\hat{\mathbb{C}})$ (Corollary 2.4) raises many questions. For example, is this map an immersion? When is the image locally closed? Can it accumulate on itself like a leaf of a foliation?[1]

We now discuss the abundance of quasiconformally conjugate rational maps. There are two main results. First, in any holomorphic family of rational maps, the parameter space contains an open dense set whose components are described by quasiconformal deformations. Second, any two topologically conjugate *expanding* rational maps are in fact quasiconformally conjugate, and hence reside in a connected holomorphic family.

DEFINITIONS. Let X be a complex manifold. A *holomorphic family of rational maps* parameterized by X is a holomorphic map $f: X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. In other words, a family f is a rational map $f_\lambda(z)$ defined for $z \in \hat{\mathbb{C}}$ and varying analytically with respect to $\lambda \in X$.

A family of rational maps is *quasiconformally constant* if f_α and f_β are quasiconformally conjugate for any α and β in the same component of X.

A family of rational maps has *constant critical orbit relations* if any coincidence $f_\lambda^n(c) = f_\lambda^m(d)$ between the forward orbits of two critical points c and d of f_λ persists under perturbation of λ.

[1] The answer to this last question is *yes*. The Teichmüller space of certain cubic polynomials with one escaping critical point is dense in the *Fibonacci solenoid*; see [9, Section 10.2].

THEOREM 2.7 (Quasiconformal Conjugacy). *A family $f_\lambda(z)$ of rational maps with constant critical orbit relations is quasiconformally constant.*

Conversely, two quasiconformally conjugate rational maps are contained in a connected family with constant critical orbit relations.

COROLLARY 2.8 (Density of Structural Stability). *In any family of rational maps $f_\lambda(z)$, the open quasiconformal conjugacy classes form a dense set.*

In particular, structurally stable maps are open and dense in the space Rat_d *of all rational maps of degree d.*

See Section 7 for discussion and proofs of these results.

Remarks. In Theorem 2.7, the conjugacy ϕ between two nearby maps f_α and f_β can be chosen to be nearly conformal and close to the identity. The condition "constant critical orbit relations" is equivalent to "number of attracting or superattracting cycles constant and constant critical orbit relations among critical points outside the Julia set."

The deployment of quasiconformal conjugacy classes in various families (quadratic polynomials, Newton's method for cubics, rational maps of degree two, etc.) has been the subject of ample computer experimentation (see, e.g., [28, 12, 15] for early work in this area) and combinatorial study ([13, 10, 38], etc.).

There is another way to arrive at quasiconformal conjugacies.

DEFINITION. A rational map $f(z)$ is *expanding* (also known as *hyperbolic* or *Axiom A*) if any of the following equivalent conditions hold (compare [11, Chapter V], [33, Section 3.4]).

- There is a smooth metric on the sphere such that $\|f'(z)\| > c > 1$ for all z in the Julia set of f.
- All critical points tend to attracting or superattracting cycles under iteration.
- There are no critical points or parabolic cycles in the Julia set.
- The Julia set J and the postcritical set P are disjoint.

Here the *postcritical* set $P \subset \hat{\mathbb{C}}$ is the closure of the forward orbits of the critical points: i.e.,

$$P = \overline{\{f^n(z) : n > 0 \text{ and } f'(z) = 0\}}.$$

The Julia set and postcritical set are easily seen to be invariant under topological conjugacy, so the expanding property is too.

THEOREM 2.9 (Topological Conjugacy). *A topological conjugacy ϕ_0 between expanding rational maps admits an isotopy ϕ_t through topological conjugacies such that ϕ_1 is quasiconformal.*

COROLLARY 2.10. *The set of rational maps in* Rat_d *topologically conjugate to a given expanding map f agrees with the set of rational maps quasiconformally conjugate to f and is thus connected.*

The proofs are given in Section 8.

Conjectures. A central and still unsolved problem in conformal dynamics, due to Fatou, is the density of expanding rational maps among all maps of a given degree.

In Section 9 we will describe a classical family of rational maps related to the addition law for the Weierstrass $\wp$-function. For such maps the Julia set (which is the whole sphere) carries an invariant line field. We then formulate the *no invariant line fields* conjecture, stating that these are the only such examples. Using the preceding Teichmüller theory and dimension counting, we establish:

THEOREM 2.11. *The no invariant line fields conjecture implies the density of hyperbolicity.*

A version of the no invariant line fields conjecture for finitely-generated Kleinian groups was established in [42], but the case of rational maps is still elusive (see [34, 33] for further discussion.)

Finally in Section 10 we illustrate the general theory in the case of quadratic polynomials. The well-known result of [13] that the Mandelbrot set is connected is recovered using the general theory.

3. CLASSIFICATION OF STABLE REGIONS

Our goal in this section is to justify the description of cycles of stable regions for rational maps. For the reader also interested in entire functions and other holomorphic dynamical systems, we develop more general propositions when possible. Much of this section is scattered through the classical literature (Fatou, Julia, Denjoy, Wolf, etc.).

DEFINITION. A Riemann surface is *hyperbolic* if its universal cover is conformally equivalent to the unit disk. Recall the disk with its conformal geometry is the Poincaré model of hyperbolic or non-Euclidean geometry. The following is the geometric formulation of the Schwarz lemma.

THEOREM 3.1. *Let $f: X \to X$ be a holomorphic endomorphism of a hyperbolic Riemann surface. Then either*

- *f is a covering map and a local isometry for the hyperbolic metric, or*
- *f is a contraction ($\|df\| < 1$ everywhere).*

Under the same hypotheses, we will show:

THEOREM 3.2. *The dynamics of f are described by one of the following (mutually exclusive) possibilities.*

1. *All points of X tend to infinity: namely, for any x and any compact set K in X, there are only finitely many $n > 0$ such that $f^n(x)$ is in K.*
2. *All points of X tend towards a unique attracting fixed point x_0 in X: that is, $f^n(x) \to x_0$ as $n \to \infty$, and $|f'(x)| < 1$.*
3. *The mapping f is an irrational rotation of a disk, punctured disk or annulus.*
4. *The mapping f is of finite order: there is an $n > 0$ such that $f^n = \mathrm{id}$.*

LEMMA 3.3. *Let $\Gamma \subset \mathrm{Aut}(\mathbb{H})$ be a discrete group of orientation-preserving hyperbolic isometries. Then either Γ is abelian, or the semigroup*

$$\mathrm{End}(\Gamma) = \{g \in \mathrm{Aut}(\mathbb{H}) : g\Gamma g^{-1} \subset \Gamma\}$$

is discrete.

Proof. Suppose $\mathrm{End}(\Gamma)$ is indiscrete, and let $g_n \to g_\infty$ be a convergent sequence of distinct elements therein. Then $h_n = g_\infty^{-1} \circ g_n$ satisfies

$$h_n \Gamma h_n^{-1} \subset \Gamma' = g_\infty^{-1} \Gamma g_\infty.$$

Since Γ' is discrete and $h_n \to \mathrm{id}$, any $\alpha \in \Gamma$ commutes with h_n for all n sufficiently large. Thus the centralizers of any two elements in Γ intersect nontrivially, which implies they commute. ∎

Proof of Theorem 3.2. Throughout we will work with the hyperbolic metric on X and appeal to the Schwarz lemma as stated in Theorem 3.1.

Let x be a point of X. If $f^n(x)$ tends to infinity in X, then the same is true for every other point y, since $d(f^n(x), f^n(y)) \leqslant d(x, y)$ for all n.

Otherwise the forward iterates $f^n(x)$ return infinitely often to a fixed compact set K in X (i.e., f is *recurrent*).

If f is a contraction, we obtain a fixed point in K by the following argument. Join x to $f(x)$ by a smooth path α of length R. Then $f^n(\alpha)$ is contained infinitely often in an R-neighborhood of K, which is also compact. Since f contracts *uniformly* on compact sets, the length of $f^n(\alpha)$, and hence $d(f^n x, f^{n+1} x)$, tends to zero. It follows that any accumulation point x_0 of $f^n(x)$ is a fixed point. But a contracting mapping has at most one fixed point, so $f^n(x) \to x_0$ for all x.

Finally assume f is a recurrent self-covering map. If $\pi_1(X)$ is abelian, then X is a disk, punctured disk or annulus, and it is easy to check that the only recurrent self-coverings are rotations, which are either irrational or of finite order.

If $\pi_1(X)$ is nonabelian, write X as a quotient $\mathbb{H}/\Gamma$ of the hyperbolic plane by a discrete group $\Gamma \cong \pi_1(X)$, and let $g_n : \mathbb{H} \to \mathbb{H}$ denote a lift of f^n to a bijective isometry of the universal cover of X. Then $g_n \in \mathrm{End}(\Gamma)$, and by the preceding lemma $\mathrm{End}(\Gamma)$ is discrete. Since f is recurrent the lifts g_n can be chosen to lie in a compact subset of $\mathrm{Aut}(\mathbb{H})$, and thus $g_n = g_m$ for some $n > m > 0$. Therefore $f^n = f^m$ which implies $f^{n-m} = \mathrm{id}$. ∎

To handle the case in which $f^n(x)$ tends to infinity in a stable domain U, we will need to compare the hyperbolic and spherical metrics.

PROPOSITION 3.4. *Let U be a connected open subset of $\hat{\mathbb{C}}$ with $|\hat{\mathbb{C}} - U| > 2$ (so U is hyperbolic). Then the ratio $\sigma(z)/\rho(z)$ between the spherical metric σ and the hyperbolic metric ρ tends to zero as z tends to infinity in U.*

Proof. Suppose $z_n \in U$ converges to $a \in \partial U$. Let b and c be two other points outside U, and let ρ' be the hyperbolic metric on the triply-punctured sphere $\hat{\mathbb{C}} - \{a, b, c\}$. Since inclusions are contracting, it suffices to show $\sigma(z_n)/\rho'(z_n) \to 0$, and this follows from well-known estimates for the triply-punctured sphere [2]. ∎

PROPOSITION 3.5 (Fatou). *Suppose $U \subset \hat{\mathbb{C}}$ is a connected region, $f: U \to U$ is holomorphic, $f^n(x) \to y \in \partial U$ for all $x \in U$, and f has an analytic extension to a neighborhood of y. Then $|f'(y)| < 1$ or $f'(y) = 1$.*

See [17, p. 242; 11, Lemma IV.2.4].

Proof of Theorem 2.1 (Classification of Stable Regions). Let U be a periodic component of Ω. Replacing f by f^p (note this does not change the Julia or Fatou sets), we can assume $f(U) = U$. Since the Julia set is perfect, U is a hyperbolic Riemann surface and we may apply Theorem 3.2. If U contains an attracting fixed point for f then it is an attracting or super-attracting basin. The case of an irrational rotation of a disk or annulus gives a Siegel disk or Herman ring; note that U cannot be a punctured disk since the puncture would not lie in the Julia set. Since the degree of f is greater than one by assumption, no iterate of f is the identity on an open set and the case of a map of finite order cannot arise either.

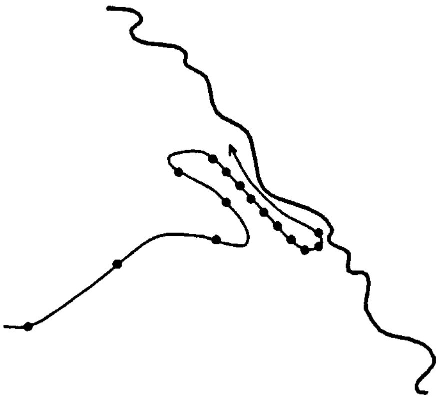

Fig. 2. Orbit tending to the Julia set.

Finally we will show that U is a parabolic basin if $f^n(x)$ tends to infinity in U.

Let α be a smooth path joining x to $f(x)$ in U. In the hyperbolic metric on U, the paths $\alpha_n = f^n(\alpha)$ have uniformly bounded length (by the Schwarz lemma). Since the ratio of spherical to hyperbolic length tends to zero at the boundary of U, the accumulation points $A \subset \partial U$ of the path $\bigcup \alpha_n$ form a continuum of fixed points for f^n. (See Fig. 2.) Since $f^n \neq \mathrm{id}$, the set $A = \{x_0\}$ for a single fixed point x_0 of f. By Fatou's result above, this fixed point is parabolic with multiplier 1 (it cannot be attracting since it lies in the Julia set), and therefore U is a parabolic basin. ∎

4. THE TEICHMÜLLER SPACE OF A HOLOMORPHIC DYNAMICAL SYSTEM

In this section we will give a fairly general definition of a *holomorphic dynamical system* and its *Teichmüller space*. Although in this paper we are primarily interested in iterated rational maps, it seems useful to consider definitions that can be adapted to Kleinian groups, entire functions, correspondences, polynomial-like maps and so on.

4.1. *Static Teichmüller Theory*

Let X be a 1-dimensional complex manifold. For later purposes it is important to allow X to be disconnected.[2] We begin by briefly recalling the traditional Teichmüller space of X.

[2] On the other hand, we follow the usual convention that a *Riemann surface* is connected.

The *deformation space* $\mathrm{Def}(X)$ consists of equivalence classes of pairs (ϕ, Y) where Y is a complex 1-manifold and $\phi: X \to Y$ is a quasiconformal map; here (ϕ, Y) is equivalent to (Z, ψ) if there is a conformal map $c: Y \to Z$ such that $\psi = c \circ \phi$.

The group $\mathrm{QC}(X)$ of quasiconformal automorphisms $\omega: X \to X$ acts on $\mathrm{Def}(X)$ by $\omega((\phi, Y)) = (\phi \circ \omega^{-1}, Y)$.

The *Teichmüller space* $\mathrm{Teich}(X)$ parameterizes complex structures on X *up to isotopy*. More precisely, we will define below a normal subgroup $\mathrm{QC}_0(X) \subset \mathrm{QC}(X)$ consisting of self-maps isotopic to the identity in an appropriate sense; then

$$\mathrm{Teich}(X) = \mathrm{Def}(X)/\mathrm{QC}_0(X).$$

To define $\mathrm{QC}_0(X)$, first suppose X is a hyperbolic Riemann surface; then X can be presented as the quotient $X = \mathbb{H}/\Gamma$ of the upper halfplane by the action of a Fuchsian group. Let $\Omega \subset S^1_\infty = \mathbb{R} \cup \{\infty\}$ be the complement of the limit set of Γ. The quotient $\bar{X} = (\mathbb{H} \cup \Omega)/\Gamma$ is a manifold with interior X and boundary Ω/Γ which we call the *ideal boundary* of X (denoted ideal-∂X).

A continuous map $\omega: X \to X$ is *the identity on* S^1_∞ if there exists a lift $\tilde{\omega}: \mathbb{H} \to \mathbb{H}$ which extends to a continuous map $\bar{\mathbb{H}} \to \bar{\mathbb{H}}$ pointwise fixing the circle at infinity. Similarly, a homotopy $\omega_t: X \to X$, $t \in [0, 1]$ is *rel* S^1_∞ if there is a lift $\tilde{\omega}_t: \mathbb{H} \to \mathbb{H}$ which extends to a homotopy pointwise fixing S^1_∞. A homotopy is *rel ideal boundary* if it extends to a homotopy of $\bar{X}$ pointwise fixing ideal-∂X. An isotopy $\omega_t: X \to X$ is *uniformly quasiconformal* if there is a K independent of t such that ω_t is K-quasiconformal.

THEOREM 4.1. *Let X be a hyperbolic Riemann surface and let $\omega: X \to X$ be a quasiconformal map. Then the following conditions are equivalent*:

1. *ω is the identity on S^1_∞.*
2. *ω is homotopic to the identity rel the ideal boundary of X.*
3. *ω admits a uniformly quasiconformal isotopy to the identity rel ideal boundary.*

The isotopy can be chosen so that $\omega_t \circ \gamma = \gamma \circ \omega_t$ for any automorphism γ of X such that $\omega \circ \gamma = \gamma \circ \omega$.

See [16, Theorem 1.1]. For a hyperbolic Riemann surface X, the normal subgroup $\mathrm{QC}_0(X) \subset \mathrm{QC}(X)$ consists of those ω satisfying the equivalent conditions above.

The point of this theorem is that several possible definitions of $\mathrm{QC}_0(X)$ coincide. For example, the first condition is most closely related to Bers' embedding of Teichmüller space, while the last is the best suited to our dynamical applications.

For a general complex 1-manifold X (possibly disconnected, with hyperbolic, parabolic or elliptic components), the ideal boundary of X is given by the union of the ideal boundaries of X_α over the hyperbolic components X_α of X. Then the group $\mathrm{QC}_0(X)$ consists of those $\omega \in \mathrm{QC}_0(X)$ which admit a uniformly quasiconformal isotopy to the identity *rel* ideal-∂X.

Such isotopies behave well under coverings and restrictions. For later reference we state:

THEOREM 4.2. *Let $\pi: X \to Y$ be a covering map between hyperbolic Riemann surfaces, and let $\tilde{\omega}_t: X \to X$ be a lift of an isotopy $\omega_t: Y \to Y$. Then ω_t is an isotopy* rel ideal-∂Y *if and only if $\tilde{\omega}_t$ is an isotopy* rel ideal-∂X.

Proof. An isotopy is *rel* ideal boundary if and only if it is *rel* S^1_∞ [16, Corollary 3.2], and the latter property is clearly preserved when passing between covering spaces.

THEOREM 4.3. *Let $\omega_t: X \to X$, $t \in [0, 1]$ be a uniformly quasiconformal isotopy of an open set $X \subset \hat{\mathbb{C}}$. Then ω_t is an isotopy* rel ideal-∂X *if and only if ω_t is the restriction of a uniformly quasiconformal isotopy of $\hat{\mathbb{C}}$ pointwise fixing $\hat{\mathbb{C}} - X$.*

See [16, Theorem 2.2 and Corollary 2.4].

Teichmüller theory extends in a natural way to *one-dimensional complex orbifolds*, locally modeled on Riemann surfaces modulo finite groups; we will occasionally use this extension below. For more details on the foundations of Teichmüller theory, see [18, 36, 23].

4.2. *Dynamical Teichmüller Theory*

DEFINITIONS. A *holomorphic relation* $R \subset X \times X$ is a countable union of 1-dimensional analytic subsets of $X \times X$. Equivalently, there is a complex 1-manifold $\tilde{R}$ and a holomorphic map

$$v: \tilde{R} \to X \times X$$

such that $v(\tilde{R}) = R$ and v is injective outside a countable subset of $\tilde{R}$. The surface $\tilde{R}$ is called the *normalization* of R; it is unique up to isomorphism over R [21, Vol. II]. Note that we do not require R to be locally closed. If $\tilde{R}$ is connected, we say R is *irreducible*.

NOTE. It is convenient to *exclude* relations (like the constant map) such that $U \times \{x\} \subset R$ or $\{x\} \times U \subset R$ for some nonempty open set $U \subset X$, and we will do so in the sequel.

Holomorphic relations are composed by the rule

$$R \circ S = \{(x, y) : (x, z) \in R \text{ and } (z, y) \in S \text{ for some } z \in X\},$$

and thereby give rise to dynamics. The *transpose* is defined by $R^t = \{(z, y) : (y, z) \in R\}$; this generalizes the inverse of a bijection.

EXAMPLES. The graph of a holomorphic map $f : X \to X$ is a holomorphic relation. So is the set

$$R = \{(z_1, z_2) : z_2 = z_1^{\sqrt{2}} = \exp(\sqrt{2} \log z_1) \text{ for some branch of the logarithm}\} \subset \mathbb{C}^* \times \mathbb{C}.$$

In this case the normalization is given by $\tilde{R} = \mathbb{C}$, $v(z) = (e^z, e^{\sqrt{2}z})$.

A (one-dimensional) *holomorphic dynamical system* $(X, \mathscr{R})$ is a collection $\mathscr{R}$ of holomorphic relations on a complex 1-manifold X.

Let $M(X)$ be the complex Banach space of L^∞ *Beltrami differentials* μ on X, given locally in terms of a holomorphic coordinate z by $\mu = \mu(z) d\bar{z}/dz$ where $\mu(z)$ is a measurable function and with

$$\|\mu\| = \operatorname*{ess.sup}_X |\mu|(z) < \infty.$$

Let π_1 and π_2 denote projection onto the factors of $X \times X$, and let $v : \tilde{R} \to R$ be the normalization of a holomorphic relation R on X. A Beltrami differential $\mu \in M(X)$ is *R-invariant* if

$$(\pi_1 \circ v)^* \mu = (\pi_2 \circ v)^* \mu$$

on every component of $\tilde{R}$ where $\pi_1 \circ v$ and $\pi_2 \circ v$ both are nonconstant. (A constant map is holomorphic for any choice of complex structure, so it imposes no condition on μ). Equivalently, μ is R-invariant if $h^*(\mu) = \mu$ for every holomorphic homeomorphism $h : U \to V$ such that the graph of h is contained in R. The invariant Beltrami differentials will turn out to be the tangent vectors to the deformation space $\operatorname{Def}(X, \mathscr{R})$ defined below.

Let

$$M(X, \mathscr{R}) = \{\mu \in M(X) : \mu \text{ is } R\text{-invariant for all } R \text{ in } \mathscr{R}\}.$$

It is easy to see that $M(X, \mathscr{R})$ is a closed subspace of $M(X)$; let $M_1(X, \mathscr{R})$ be its open unit ball.

A *conjugacy* between holomorphic dynamical systems $(X, \mathscr{R})$ and $(Y, \mathscr{S})$ is a map $\phi : X \to Y$ such that

$$\mathscr{S} = \{(\phi \times \phi)(R) : R \in \mathscr{R}\}.$$

Depending on the quality of ϕ, a conjugacy may be conformal, quasiconformal, topological, measurable, etc.

The *deformation space* $\mathrm{Def}(X, \mathscr{R})$ is the set of equivalence classes of data $(\phi, Y, \mathscr{S})$ where $(Y, \mathscr{S})$ is a holomorphic dynamical system and $\phi: X \to Y$ is a quasiconformal conjugacy. Here $(\phi, Y, \mathscr{S})$ is equivalent to $(\psi, Z, \mathscr{T})$ if there is a conformal isomorphism $c: Y \to Z$ such that $\psi = c \circ \phi$. Note that any such c is a conformal conjugacy between $(Y, \mathscr{S})$ and $(Z, \mathscr{T})$.

THEOREM 4.4. *The map* $\phi \mapsto (\bar\partial\phi/\partial\phi)$ *establishes a bijection between* $\mathrm{Def}(X, \mathscr{R})$ *and* $M_1(X, \mathscr{R})$.

Proof. Suppose $(\phi, Y, \mathscr{S}) \in \mathrm{Def}(X, \mathscr{R})$, and let $\mu = (\bar\partial\phi/\partial\phi)$. For every holomorphic homeomorphism $h: U \to V$ such that the graph of h is contained in $R \in \mathscr{R}$, the graph of $g = \phi \circ h \circ \phi^{-1}$ is contained in $S = \phi \times \phi(R) \in \mathscr{S}$. Since S is a holomorphic relation, g is holomorphic and therefore

$$h^*(\mu) = \frac{\bar\partial(\phi \circ h)}{\partial(\phi \circ h)} = \frac{\bar\partial(g \circ \phi)}{\partial(g \circ \phi)} = \mu.$$

Thus μ is R-invariant, so we have a map $\mathrm{Def}(X, \mathscr{R}) \to M_1(X, \mathscr{R})$. Injectivity of this map is immediate from the definition of equivalence in $\mathrm{Def}(X, \mathscr{R})$. Surjectivity follows by solving the Beltrami equation (given μ we can construct Y and $\phi: X \to Y$); and by observing that for each $R \in \mathscr{R}$, the normalization of $S = (\phi \times \phi)(R)$ is obtained by integrating the complex structure $(\pi_i \circ v)^* \mu$ on $\tilde{R}$ (for $i = 1$ or 2). ∎

Remark. The deformation space is naturally a *complex manifold*, because $M_1(X, \mathscr{R})$ is an open domain in a complex Banach space.

The *quasiconformal automorphism group* $\mathrm{QC}(X, \mathscr{R})$ consists of all quasiconformal conjugacies ω from $(X, \mathscr{R})$ to itself. Note that ω may permute the elements of $\mathscr{R}$. This group acts biholomorphically on the deformation space by

$$\omega: (\phi, Y, \mathscr{S}) \mapsto (\phi \circ \omega^{-1}, Y, \mathscr{S}).$$

The normal subgroup $\mathrm{QC}_0(X, \mathscr{R})$ consists of those ω_0 admitting a uniformly quasiconformal isotopy ω_t *rel* the ideal boundary of X, such that $\omega_1 = \mathrm{id}$ and $(\omega_t \times \omega_t)(R) = R$ for all $R \in \mathscr{R}$. (In addition to requiring that ω_t is a conjugacy of $\mathscr{R}$ to itself, we require that it induces the identity permutation on $\mathscr{R}$. Usually this extra condition is automatic because $\mathscr{R}$ is countable.)

The *Teichmüller space* $\mathrm{Teich}(X, \mathscr{R})$ is the quotient $\mathrm{Def}(X, \mathscr{R})/\mathrm{QC}_0(X, \mathscr{R})$. A point in the Teichmüller space of $(X, \mathscr{R})$ is a holomorphic dynamical system $(X, \mathcal{S})$ together with a *marking* $[\phi]$ by $(X, \mathcal{R})$, defined up to isotopy through quasiconformal conjugacies rel ideal boundary.

The *Teichmüller premetric* on $\mathrm{Teich}(X, \mathcal{R})$ is given by

$$d(([\phi], Y, \mathcal{S}), ([\psi], Z, \mathcal{T})) = \tfrac{1}{2} \inf \log K(\phi \circ \psi^{-1}),$$

where $K(\cdot)$ denotes the dilatation and the infimum is over all representatives of the markings $[\phi]$ and $[\psi]$. (The factor of 1/2 makes this metric compatible with the norm on $M(X)$.)

For rational maps, Kleinian groups and most other dynamical systems, the Teichmüller premetric is actually a metric. Here is a general criterion that covers those cases.

DEFINITION. Let S be an irreducible component of a relation in the semigroup generated by $\mathcal{R}$ under the operations of composition and transpose. Then we say S belongs to the *full dynamics* of $\mathcal{R}$. If S intersects the diagonal in $X \times X$ in a countable or finite set, the points x with $(x, x) \in S$ are *fixed points* of S and *periodic points* for $\mathcal{R}$.

THEOREM 4.5. *Suppose every component of X isomorphic to $\hat{\mathbb{C}}$ (respectively, $\mathbb{C}$, $\mathbb{C}^*$ or a complex torus) contains at least three (respectively, 2, 1, or 1) periodic points for $\mathcal{R}$. Then the Teichmüller premetric on* $\mathrm{Teich}(X, \mathcal{R})$ *is a metric.*

Proof. We need to obtain a conformal conjugacy from a sequence of isotopic conjugacies ψ_n whose dilatations tend to one. For this, it suffices to show that a sequence ϕ_n in $QC_0(X, \mathcal{R})$ with bounded dilatation has a convergent subsequence, since we can take $\phi_n = \psi_1^{-1} \circ \psi_n$.

On the hyperbolic components of X, the required compactness follows by basic results in quasiconformal mappings, because each ϕ_n is isotopic to the identity rel ideal boundary.

For the remaining components, consider any relation S in the full dynamics generated by $\mathcal{R}$. A self-conjugacy permutes the intersections of S with the diagonal, so a map deformable to the identity through self-conjugacies fixes every periodic point of $\mathcal{R}$. Each non-hyperbolic component is isomorphic to $\hat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*$ or a complex torus, and the same compactness result holds for quasiconformal mappings normalized to fix sufficiently many points on these Riemann surfaces (as indicated above). ∎

The *modular group* (or mapping class group) is the quotient

$$\mathrm{Mod}(X, \mathcal{R}) = \mathrm{QC}(X, \mathcal{R}) / \mathrm{QC}_0(X, \mathcal{R});$$

it acts isometrically on the Teichmüller space of $(X, \mathscr{R})$. The stabilizer of the natural basepoint $(\mathrm{id}, X, \mathscr{R}) \in \mathrm{Teich}(X, \mathscr{R})$ is the *conformal automorphism group* $\mathrm{Aut}(X, \mathscr{R})$.

THEOREM 4.6. *Let* $\mathscr{R} = \emptyset$. *Then* $\mathrm{Teich}(X, \mathscr{R})$ *coincides with the traditional Teichmüller space of* X.

Remarks. We will have occasion to discuss Teichmüller space not only as a metric space but as a complex manifold; the complex structure (if it exists) is the unique one such that the projection $M_1(X, \mathscr{R}) \to \mathrm{Teich}(X, \mathscr{R})$ is holomorphic.

At the end of this section we give an example showing the Teichmüller premetric is not a metric in general.

4.3. *Kleinian Groups*

The Teichmüller space of a rational map is patterned on that of a Kleinian group, which we briefly develop in this section (cf. [24]).

Let $\Gamma \subset \mathrm{Aut}(\hat{\mathbb{C}})$ be a torsion-free *Kleinian group*, that is a discrete subgroup of Möbius transformations acting on the Riemann sphere. The *domain of discontinuity* $\Omega \subset \hat{\mathbb{C}}$ is the maximal open set such that $\Gamma \mid \Omega$ is a normal family; its complement Λ is the *limit set* of Γ. The group Γ acts freely and properly discontinuously on Ω, so the quotient Ω/Γ is a complex 1-manifold.

Let $M_1(\Lambda, \Gamma)$ denote the unit ball in the space of Γ-invariant Beltrami differentials on the limit set; in other words, the restriction of $M_1(\hat{\mathbb{C}}, \Gamma)$ to Λ. (Note that $M_1(\Lambda, \Gamma) = \{0\}$ if Λ has measure zero.)

THEOREM 4.7. *The Teichmüller space of* $(\hat{\mathbb{C}}, \Gamma)$ *is naturally isomorphic to*

$$M_1(\Lambda, \Gamma) \times \mathrm{Teich}(\Omega/\Gamma).$$

Proof. It is obvious that

$$\begin{aligned} \mathrm{Def}(\hat{\mathbb{C}}, \Gamma) &= M_1(\hat{\mathbb{C}}, \Gamma) = M_1(\Lambda, \Gamma) \times M_1(\Omega, \Gamma) \\ &= M_1(\lambda, \Gamma) \times M_1(\Omega/\Gamma) = M_1(\lambda, \Gamma) \times \mathrm{Def}(\Omega/\Gamma). \end{aligned}$$

Since Γ is countable and fixed points of elements of Γ are dense in Λ, $\omega \mid \Lambda = \mathrm{id}$ for all $\omega \in \mathrm{QC}_0(\hat{\mathbb{C}}, \Gamma)$. By Theorem 4.3 above, $\mathrm{QC}_0(\hat{\mathbb{C}}, \Gamma) \cong \mathrm{QC}_0(\Omega, \Gamma)$ by the restriction map, and $\mathrm{QC}_0(\Omega, \Gamma) \cong \mathrm{QC}_0(\Omega/\Gamma)$ using Theorem 4.2. Thus the trivial quasiconformal automorphisms for the two deformation spaces correspond, so the quotient Teichmüller spaces are naturally isomorphic.

COROLLARY 4.8 (Ahlfors' Finiteness Theorem). *If Γ is finitely generated, then the Teichmüller space of Ω/Γ is finite-dimensional.*

Proof. Let

$$\eta\colon \mathrm{Teich}(\hat{\mathbb{C}}, \Gamma) \to V = \mathrm{Hom}(\Gamma, \mathrm{Aut}(\hat{\mathbb{C}}))/\text{conjugation}$$

be the map that sends $(\phi, \hat{\mathbb{C}}, \Gamma')$ to the homomorphism $\rho\colon \Gamma \to \Gamma'$ determined by $\phi \circ \gamma = \rho(\gamma) \circ \phi$. The proof is by contradiction: if Γ is finitely generated, then the representation variety V is finite dimensional, while if $\mathrm{Teich}(\Omega/\Gamma)$ is infinite dimensional, we can find a polydisk $\Delta^n \subset \mathrm{Def}(\hat{\mathbb{C}}, \Gamma)$ such that Δ^n maps injectively to $\mathrm{Teich}(\hat{\mathbb{C}}, \Gamma)$ and $n > \dim V$. (Such a polydisk exists because the map $\mathrm{Def}(\Omega/\Gamma) \to \mathrm{Teich}(\Omega/\Gamma)$ is a holomorphic submersion; see, e.g., [36, Section 3.4; 23, Theorem 6.9].) The composed mapping $\Delta^n \to V$ is holomorphic, so its fiber over the basepoint $\rho = \mathrm{id}$ contains a 1-dimensional analytic subset, and hence an arc. This arc corresponds to a 1-parameter family $(\phi_t, \hat{\mathbb{C}}, \Gamma_t)$ in $\mathrm{Def}(\hat{\mathbb{C}}, \Gamma)$ such that $\Gamma_t = \Gamma$ for all t. But as t varies the marking of $(\hat{\mathbb{C}}, \Gamma)$ changes only by a uniformly quasiconformal isotopy, contradicting the assumption that Δ^n maps injectively to $\mathrm{Teich}(\hat{\mathbb{C}}, \Gamma)$.

Consequently $\mathrm{Teich}(\Omega/\Gamma)$ is finite dimensional. ∎

Remarks. The proof above, like Ahlfors' original argument [1], can be improved to show Ω/Γ is of *finite type*; that is, it is obtained from compact complex 1-manifold by removing a finite number of points. This amounts to showing Ω/Γ has finitely many components. For example Greenberg shows Γ has a subgroup Γ' of finite index such that each component of Ω/Γ' contributes at least one to the dimension of $\mathrm{Teich}(\Omega/\Gamma')$ [20]. See also [41].

Kleinian groups with torsion can be treated similarly (in this case Ω/Γ is a complex orbifold.)

It is known that $M_1(\Lambda, \Gamma) = 0$ if Γ is finitely generated, or more generally if the action of Γ on the limit set is conservative for Lebesgue measure [42].

4.4. *A counterexample*

In this section we give a simple (if artificial) construction showing the Teichmüller premetric is *not* a metric in general: there can be distinct points whose distance is zero.

For any bounded Lipschitz function $f\colon \mathbb{R} \to \mathbb{R}$ let the domain $U_f \subset \mathbb{C}$ be the region above the graph of f:

$$U_f = \{z = x + iy\colon y > f(x)\}.$$

Given two such functions f and g, the shear mapping

$$\phi_{f,\,g}(x+iy)=x+i(y-f(x)+g(y))$$

is a quasiconformal homeomorphism of the complex plane sending U_f to U_g. Moreover the dilatation of $\phi_{f,\,g}$ is close to one when the Lipschitz constant of $f-g$ is close to zero.

Let $\mathscr{R}_f$ be the dynamical system on $\mathbb{C}$ consisting solely of the identity map on U_f. Then a conjugacy between $(\mathbb{C}, \mathscr{R}_f)$ and $(\mathbb{C}, \mathscr{R}_g)$ is just a map of the plane to itself sending U_f to U_g. Therefore g determines an element $(\phi_{f,\,g}, \mathbb{C}, \mathscr{R}_g)$ in the deformation space of $(\mathbb{C}, \mathscr{R}_f)$.

A quasiconformal isotopy of conjugacies between $\mathscr{R}_f$ and $\mathscr{R}_g$, starting at $\phi_{f,\,g}$, is given by

$$\psi_t(x+iy)=x+t+i(y-f(x)+g(x+t)).$$

Now suppose we can find f and g and $c_n \to \infty$ such that

(a) the Lipschitz constant of $f(x)-g(x+c_n)$ tends to zero, but

(b) $f(x) \neq \alpha + g(\beta x+\gamma)$ for any real numbers α, β and γ with $\beta > 0$.

Then the first condition implies the dilatation of ψ_t tends to one as t tends to infinity along the sequence c_n. Thus the conjugacy $\phi_{f,\,g}$ can be deformed to be arbitrarily close to conformal, and therefore the Teichmüller (pre)distance between $\mathscr{R}_f$ and $\mathscr{R}_g$ is zero. But condition (b) implies there is no similarity of the plane sending U_f to U_g, and therefore $\mathscr{R}_f$ and $\mathscr{R}_g$ represent distinct points in Teichmüller space. (Note that any such similarity must have positive real derivative since f and g are bounded.)

Thus the premetric is not a metric.

It only remains to construct f and g. This is easily carried out using almost periodic functions. Let $d(x)$ be the distance from x to the integers, and let h be the periodic Lipschitz function $h(x)=\max(0, 1/10-d(x))$. Define

$$f(x)=\sum_p 2^{-p}h\left(\frac{x}{p}\right) \qquad \text{and} \qquad g(x)=\sum_p 2^{-p}h\left(\frac{x+a_p}{p}\right),$$

where p runs over the primes and where $0<a_p<p/2$ is a sequence of integers tending to infinity. By the Chinese remainder theorem, we can find integers c_n such that $c_n=-a_p \mod p$ for the first n primes. Then the Lipschitz constant of $f(x)-g(x+c_n)$ is close to zero because after translating by c_n, the first n terms in the sum for g agree with those for f. This verifies condition (a).

By similar reasoning, f and g have the same infimum and supremum, namely 0 and $\sum 2^{-p}/10$. So if $f(x)=\alpha+\beta g(x+\gamma)$, we must have $\beta=1$ and

$\alpha = 0$. But the supremum is attained for f and not for g, so we cannot have $f(x) = g(x + \gamma)$.

5. FOLIATED RIEMANN SURFACES

In contrast to the case of Kleinian groups, *indiscrete* groups can arise in the analysis of iterated rational maps and other more general dynamical systems. Herman rings provide prototypical examples. A Herman ring (or any annulus equipped with an irrational rotation) carries a natural foliation by circles. The deformations of this dynamical system can be thought of as the Teichmüller space of a foliated annulus. Because an irrational rotation has indiscrete orbits, its deformations are not canonically reduced to those of a traditional Riemann surface—although we will see the Teichmüller space of a foliated annulus is naturally isomorphic to the upper halfplane.

More generally, we will analyze the Teichmüller space of a complex manifold equipped with an arbitrary set of *covering relations* (these include automorphisms and self-coverings).

5.1. *Foliated Annuli*

Let X be an annulus of finite modulus. The Teichmüller space of X is infinite-dimensional, due to the presence of ideal boundary. However X carries a natural foliation by circles, or more intrinsically by the orbits of $\mathrm{Aut}_0(X)$, the identity component of its automorphism group. The addition of this dynamical system to X greatly rigidifies its structure.

To describe $\mathrm{Teich}(X, \mathrm{Aut}_0(X))$ concretely, consider a specific annulus $A(R) = \{z: 1 < |z| < R\}$ whose universal cover is given by $\widetilde{A}(R) = \{z: 0 < \mathrm{Im}(z) < \log R\}$, with $\pi_1(A(R))$ acting by translation by multiples of 2π. Then $\mathrm{Aut}_0(A(R)) = S^1$ acting by rotations, which lifts to $\mathbb{R}$ acting by translations on $\widetilde{A}(R)$. Any point in $\mathrm{Def}(A(R), S^1)$ is represented by $(\phi, A(S), S^1)$ for some radius $S > 1$, and the conjugacy lifts to a map $\tilde{\phi}: \widetilde{A}(R) \to \widetilde{A}(S)$ which (by virtue of quasiconformality) extends to the boundary and can be normalized so that $\tilde{\phi}(0) = 0$. Since ϕ conjugates S^1 to S^1, $\tilde{\phi}$ commutes with real translations, and so $\tilde{\phi} \mid \mathbb{R} = \mathrm{id}$ and $\tilde{\phi} \mid (i + \mathbb{R}) = \tau + \mathrm{id}$ for some complex number τ with $\mathrm{Im}(\tau) = \log S$.

The group $\mathrm{QC}_0(A(R), S^1)$ consists exactly of those $\omega: A(R) \to A(R)$ commuting with rotation and inducing the identity on the ideal boundary of $A(R)$, because by Theorem 4.1 there is a canonical isotopy of such ω to the identity through maps commuting with S^1. It follows that τ uniquely determines the marking of $(A(S), S^1)$. Similarly elements of the modular group $\mathrm{Mod}(A(R), S^1)$ are uniquely determined by their ideal boundary values; the map $z \mapsto R/z$ interchanges the boundary components, and those automorphisms preserving the boundary components are represented by maps of the form $z \mapsto \exp(2\pi i(\theta_1 + \theta_2 \log|z|))\, z$. In summary:

THEOREM 5.1. *The Teichmüller space of the dynamical system* $(A(R), S^1)$ *is naturally isomorphic to the upper half-plane* $\mathbb{H}$. *The modular group* $\mathrm{Mod}(A(R), S^1)$ *is isomorphic to a semidirect product of* $\mathbb{Z}/2$ *and* $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$.

The map $x + iy \mapsto x + \tau y$ from $\tilde{A}(R)$ to $\tilde{A}(S)$ has dilatation $(i-\tau)/(i+\tau)$ $d\bar{z}/dz$, which varies holomorphically with τ. This shows:

THEOREM 5.2. *The map* $\mathrm{Def}(A(R), S^1) \to \mathrm{Teich}(A(R), S^1)$ *is a holomorphic submersion with a global holomorphic cross-section.*

5.2. *Covering Relations*

More generally, we can analyze the Teichmüller space of a set of relations defined by covering mappings.

DEFINITION. Let X be a Riemann surface. A relation $R \subset X \times X$ is a *covering relation* if $R = (\pi \times \pi)(S)$ where $\pi: \tilde{X} \to X$ is the universal covering of X and S is the graph of an automorphism γ of $\tilde{X}$.

EXAMPLE. Let $f: X \to Y$ be a covering map. Then the relation

$$R = \{(x, x'): f(x) = f(x')\} \subset X \times X\}$$

is a disjoint union of covering relations. To visualize these, fix $x \in X$; then for each x' such that $f(x) = f(x')$, the maximal analytic continuation of the branch of $f^{-1} \circ f$ sending x to x' is a covering relation.

It is easy to check:

THEOREM 5.3. *If* $\mathcal{R}$ *is a collection of covering relations on* X, *then*

$$\mathrm{Teich}(X, \mathcal{R}) \cong \mathrm{Teich}(\tilde{X}, \Gamma),$$

where $\Gamma \subset \mathrm{Aut}(\tilde{X})$ *is the closure of the subgroup generated by* $\pi_1(X)$ *and by automorphisms corresponding to lifts of elements of* $\mathcal{R}$.

The case where X is hyperbolic is completed by:

THEOREM 5.4. *Let* Γ *be a closed subgroup of* $\mathrm{Aut}(\mathbb{H})$.

1. *If* Γ *is discrete, then*

$$\mathrm{Teich}(\mathbb{H}, \Gamma) \cong \mathrm{Teich}(\mathbb{H}/\Gamma)$$

where $\mathbb{H}/\Gamma$ is a Riemann surface if Γ is torsion free, and a complex orbifold otherwise.

2. *If Γ is one-dimensional and stabilizes a geodesic, then* $\mathrm{Teich}(\mathbb{H}, \Gamma)$ *is naturally isomorphic to* $\mathbb{H}$.

3. *Otherwise,* $\mathrm{Teich}(\mathbb{H}, \Gamma)$ *is trivial* (*a single point*).

Proof. Let Γ_0 be the identity component of Γ. The case where Γ is discrete is immediate, using Theorem 4.1.

Next assume Γ_0 is one-dimensional; then Γ_0 is a one-parameter group of hyperbolic, parabolic or elliptic transformations. Up to conformal conjugacy, there is a unique one-parameter group of each type, and no pair are quasiconformally conjugate. Thus any point in $\mathrm{Teich}(\mathbb{H}, \Gamma)$ has a representative $([\phi], \mathbb{H}, \Gamma')$ where $\Gamma'_0 = \Gamma_0$. In particular we can assume $\phi \in \mathrm{QC}(\mathbb{H}, \Gamma_0)$.

The group Γ_0 is hyperbolic if and only if it stabilizes a geodesic. In this case Γ/Γ_0 is either trivial or $\mathbb{Z}/2$, depending on whether Γ preserves the orientation of the geodesic or not. The trivial case arises for the dynamical system $(A(R), S^1)$, which we have already analyzed; its Teichmüller space is isomorphic to $\mathbb{H}$. The $\mathbb{Z}/2$ case is similar.

The group Γ_0 is elliptic if and only if it is equal to the stabilizer of a point $p \in \mathbb{H}$; in this case $\Gamma_0 = \Gamma$. Since an orientation-preserving homeomorphism of the circle conjugating rotations to rotations is itself a rotation, $\phi \mid S^1_\infty = \gamma \mid S^1_\infty$ for some $\gamma \in \Gamma$. By Theorem 4.1, ϕ is Γ-equivariantly isotopic to γ rel S^1_∞, and so $[\phi]$ is represented by a conformal map. Therefore the Teichmüller space is a point.

The group Γ_0 is parabolic if and only if it has a unique fixed point p on S^1_∞. We may assume $p = \infty$; then Γ_0 acts by translations. Since a homeomorphism of $\mathbb{R}$ to itself normalizing the group of translations is a similarity, we have $\phi \mid S^1_\infty = \gamma \mid S^1_\infty$ where $\gamma(z) = az + b$. Again, Theorem 4.1 provides an equivariant isotopy from ϕ to γ rel S^1_∞, so the Teichmüller space is a point in this case as well.

Now suppose Γ_0 is two-dimensional. Then its commutator subgroup is a 1-dimensional parabolic subgroup; as before, this implies ϕ agrees with a conformal map on S^1_∞ and the Teichmüller space is trivial.

Finally, if Γ_0 is three-dimensional then $\Gamma = \mathrm{Aut}(\mathbb{H})$ and even the deformation space of Γ is trivial.

5.3. *Disjoint Unions*

Let $(X_\alpha, \mathcal{R}_\alpha)$ be a collection of holomorphic dynamical systems, where $\mathcal{R}_\alpha$ is a collection of covering relations on a complex manifolds X_α. Let

$$\prod{}' \mathrm{Teich}(X_\alpha, \mathcal{R}_\alpha)$$

be the restricted product of Teichmüller spaces; we admit only sequences $([\phi_\alpha], Y_\alpha, S_\alpha)$ whose Teichmüller distances from $([\mathrm{id}], X_\alpha, \mathscr{R}_\alpha)$ are uniformly bounded above.

THEOREM 5. *Let $(X, \mathscr{R})$ be the disjoint union of the dynamical systems $(X_\alpha, \mathscr{R}_\alpha)$. Then*

$$\mathrm{Teich}(X, \mathscr{R}) \cong \prod{}' \mathrm{Teich}(X_\alpha, \mathscr{R}_\alpha).$$

Proof. The product above is easily seen to be the quotient of the restricted product of deformation spaces by the restricted product of groups $\mathrm{QC}_0(X_\alpha, \mathscr{R}_\alpha)$.

There is one nontrivial point to verify: that $\prod' \mathrm{QC}_0(X_\alpha, \mathscr{R}_\alpha) \subset \mathrm{QC}_0(X, \mathscr{R})$. The potential difficulty is that a collection of uniformly quasiconformal isotopies, one for each X_α, need not piece together to form a uniformly quasiconformal isotopy on X. This point is handled by Theorem 4.1, which provides an isotopy $\phi_{\alpha, t}$ whose dilatation is bounded in terms of that of ϕ_α. The isotopy respects the dynamics by naturality: every covering relation lifts to an automorphism of the universal cover commuting with ϕ_α. (An easy variant of Theorem 4.1 applies when some component Riemann surfaces are not hyperbolic.)

COROLLARY 5.6. *The Teichmüller space of a complex manifold is the restricted product of the Teichmüller spaces of its components.*

6. THE TEICHMÜLLER SPACE OF A RATIONAL MAP

Let $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d > 1$. By the main result of [44], every component of the Fatou set Ω of f is preperiodic, so it eventually lands on a cyclic component of one of the five types classified in Theorem 2.1. In this section we describe the Teichmüller space of $(\hat{\mathbb{C}}, f)$ as a product of factors from these cyclic components and from invariant complex structures on the Julia set. We will also describe a variant of the proof of the no wandering domains theorem.

6.1. *Self-Covering Maps*

Let $f: X \to X$ be a holomorphic endomorphism of a complex 1-manifold. To simplify notation, we will use f both to denote both the map and the dynamical system consisting of a single relation, the graph of f.

DEFINITIONS. The *grand orbit* of a point $x \in X$ is the set of y such that $f^n(x) = f^m(y)$ for some $n, m \geqslant 0$. We denote the grand orbit equivalence relation by $x \sim y$. If $f^n(x) = f^n(y)$ for some $n \geqslant 0$ then we write $x \approx y$ and say x and y belong to the same *small orbit*.

The space X/f is the quotient of X by the grand orbit equivalence relation. This relation is *discrete* if all orbits are discrete; otherwise it is *indiscrete*.

In preparation for the analysis of rational maps, we first describe the case of a self-covering.

THEOREM 6.1. *Assume every component of X is hyperbolic, f is a covering map and X/f is connected.*

1. *If the grand orbit equivalence relation of f is discrete, then X/f is a Riemann surface or orbifold, and*

$$\mathrm{Teich}(X, f) \cong \mathrm{Teich}(X/f).$$

2. *If the grand orbit equivalence relation is indiscrete, and some component A of X is an annulus of finite modulus, then*

$$\mathrm{Teich}(X, f) \cong \mathrm{Teich}(A, \mathrm{Aut}_0(A)) \cong \mathbb{H}.$$

3. *Otherwise, the Teichmüller space of (X, f) is trivial.*

Proof. First suppose X has a component A which is periodic of period p. Then $\mathrm{Teich}(X, f) \cong \mathrm{Teich}(A, f^p)$. Identify the universal cover of A with $\mathbb{H}$, and let Γ be the closure of the group generated by $\pi_1(A)$ and a lift of f^p. It is easy to see that Γ is discrete if and only if the grand orbits of f are discrete, which implies $X/f \cong \mathbb{H}/\Gamma$ and case 1 of the Theorem holds. If Γ is indiscrete, then A is a disk, a punctured disk or an annulus and f^p is an irrational rotation, by Theorem 3.2; then cases 2 and 3 follow by Theorem 5.4.

Now suppose X has no periodic component. If the grand orbit relation is discrete, case 1 is again easily established. So assume the grand orbit relation is indiscrete; then X has a component A with nontrivial fundamental group. We have $\mathrm{Teich}(X, f) \cong \mathrm{Teich}(A, \mathscr{R})$ where $\mathscr{R}$ is the set of covering relations arising as branches of $f^{-n} \circ f^n$ for all n. The corresponding group Γ is the closure of the union of discrete groups $\Gamma_n \cong \pi_1(f^n(A))$. Since Γ is indiscrete, Γ is abelian and therefore 1-dimensional, by a variant of Lemma 3.3.

Thus Γ stabilizes a geodesic if and only if $\pi_1(A)$ stabilizes a geodesic, giving case 2; otherwise, by Theorem 5.4, the Teichmüller space is trivial. ∎

6.2. *Rational Maps*

Now let $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d > 1$.

The superattracting basins, Siegel disks and Herman rings of a rational map are canonical foliated by the components of the *closures* of the grand orbits. In the Siegel disks, Herman rings, and near the superattracting cycles, the leaves of this foliation are real-analytic circles. In general countably many leaves may be singular.

Let $\hat{J}$ be closure of the grand orbits of all periodic points and all critical points of f. By Theorem 2.1, it is easy to see that $\hat{J}$ is the union of:

1. The Julia set of f;

2. The grand orbits of the attracting and superattracting cycles and the centers of Siegel disks (a countable set); and

3. The leaves of the canonical foliations which meet the grand orbit of the critical points (a countable union of one-dimensional sets).

Let $\hat{\Omega} = \hat{\mathbb{C}} - \hat{J} \subset \Omega$. Then $f: \hat{\Omega} \to \hat{\Omega}$ is a covering map without periodic points.

Let $\hat{\Omega} = \Omega^{\mathrm{dis}} \sqcup \Omega^{\mathrm{fol}}$ denote the partition into open sets where the grand orbit equivalence relation is discrete and where it is indiscrete. In other words, Ω^{dis} contains the points which eventually land in attracting or parabolic basins, while Ω^{fol} contains those which land in Siegel disks, Herman rings and superattracting basins.

Parallel to Theorem 4.7 for Kleinian groups, we have the following result for rational maps:

THEOREM 6.2. *The Teichmüller space of a rational map f of degree d is naturally isomorphic to*

$$M_1(J, f) \times \mathrm{Teich}(\Omega^{\mathrm{fol}}, f) \times \mathrm{Teich}(\Omega^{\mathrm{dis}}/f),$$

where Ω^{dis}/f is a complex manifold.

Here $M_1(J, f)$ denotes the restriction of $M_1(\hat{\mathbb{C}}, f)$ to J.

Proof. Since $\hat{J}$ contains a dense countable dynamically distinguished subset, $\omega/\hat{J} = \mathrm{id}$ for all $\omega \in \mathrm{QC}_0(\hat{\mathbb{C}}, f)$, and so

$$\mathrm{QC}_0(\hat{\mathbb{C}}, f) = \mathrm{QC}_0(\Omega^{\mathrm{fol}}, f) \times \mathrm{QC}_0(\Omega^{\mathrm{dis}}, f)$$

by Theorem 4.3. This implies the theorem with the last factor replaced by $\mathrm{Teich}(\Omega^{\mathrm{dis}}, f)$, using the fact that J and $\hat{J}$ differ by a set of measure zero. To complete the proof, write Ω^{dis} as $\bigcup \Omega_i^{\mathrm{dis}}$, a disjoint union of totally invariant open sets such that each quotient $\Omega_i^{\mathrm{dis}}/f$ is connected. Then by Theorems 5.5 and 6.1, we have

$$\mathrm{Teich}(\Omega^{\mathrm{dis}}, f) = \prod{}' \mathrm{Teich}(\Omega_i^{\mathrm{dis}}, f) = \prod{}' \mathrm{Teich}(\Omega_i^{\mathrm{dis}}/f) = \mathrm{Teich}(\Omega^{\mathrm{dis}}/f).$$

Note that Ω^{dis}/f is a complex manifold (rather than an orbifold), because $f \mid \Omega^{\mathrm{dis}}$ has no periodic points. ∎

Next we give a more concrete description of the factors above.

THEOREM 6.3. *The quotient space Ω^{dis}/f is a finite union of Riemann surfaces, one for each cycle of attractive or parabolic components of the Fatou set of f.*

An attractive basin contributes an n-times punctured torus to Ω^{dis}/f, while a parabolic basin contributes an $n+2$-times punctured sphere, where $n \geqslant 1$ is the number of grand orbits of critical points landing in the corresponding basin.

Proof. Every component of Ω^{dis} is preperiodic, so every component X of Ω^{dis}/f can be represented as the quotient Y/f^p, where Y is obtained from a parabolic or attractive basin U of period p be removing the grand orbits of critical points and periodic points.

First suppose U is attractive. Let x be the attracting fixed point of f^p in U, and let $\lambda = (f^p)'(x)$. After a conformal conjugacy if necessary, we can assume $x \in \mathbb{C}$. Then by classical results, there is a holomorphic linearizing map

$$\psi(z) = \lim \lambda^{-n}(f^{pn}(z) - x)$$

mapping U onto $\mathbb{C}$, injective near x and satisfying $\psi(f^p(z)) = \lambda(\psi(z))$.

Let U' be the complement in U of the grand orbit of x. Then the space of grand orbits in U' is isomorphic to $\mathbb{C}^*/\langle z \mapsto \lambda z \rangle$, a complex torus. Deleting the points corresponding to critical orbits in U', we obtain Y/f^p.

Now suppose U is parabolic. Then there is a similar map $\psi\colon U \to \mathbb{C}$ such that $\psi(f^p(z)) = z+1$, exhibiting $\mathbb{C}$ as the small orbit quotient of U. Thus the space of grand orbits in U is the infinite cylinder $\mathbb{C}/\langle z \mapsto z+1 \rangle \cong \mathbb{C}^*$. Again deleting the points corresponding to critical orbits in U, we obtain Y/f^p.

In both cases, the number n of critical orbits to be deleted is at least one, since the immediate basin of an attracting or parabolic cycle always contains a critical point. Thus the number of components of Ω^{dis}/f is bounded by the number of critical points, namely $2d-2$. ∎

For a detailed development of attracting and parabolic fixed points, see, e.g., [11, Chapter II].

LEMMA 6.4. *Let $\Delta^n \subset \mathrm{Def}(\hat{\mathbb{C}}, f)$ be a polydisk mapping injectively to the Teichmüller space of f. Then $n \leqslant 2d-2 = \dim \mathrm{Rat}_d$, where d is the degree of f.*

Proof. Let

$$\eta\colon \mathrm{Def}(\hat{\mathbb{C}}, f) \to V = \mathrm{Rat}_d/\mathrm{Aut}(\hat{\mathbb{C}})$$

be the holomorphic map which sends $(\phi, \hat{\mathbb{C}}, g)$ to g, a rational map determined up to conformal conjugacy. The space of such rational maps is a complex orbifold V of dimension $2d-2$. If $n > 2d-2$, then the fibers of the map $\Delta^n \to V$ are analytic subsets of positive dimension; hence there is an arc (ϕ_t, g_0) in Δ^n lying over a single map g_0. But as t varies, the marking of g_0 changes only by a uniformly quasiconformal isotopy, contradicting the assumption that Δ^n maps injectively to $\mathrm{Teich}(\hat{\mathbb{C}}, \Gamma)$. ∎

DEFINITION. An *invariant line field* on a positive-measure totally invariant subset E of the Julia set is the choice of a real 1-dimensional subspace $L_e \subset L_e\hat{\mathbb{C}}$, varying measurably with respect to $e \in E$, such that f' transforms L_e to $L_{f(e)}$ for almost every $e \in E$.

Equivalently, an invariant line field is given by a measurable Beltrami differential μ supported on E with $|\mu| = 1$ such that $f^*\mu = \mu$. The correspondence is given by $L_e = \{v \in T_e\hat{\mathbb{C}} : \mu(v) = 1 \text{ or } v = 0\}$.

THEOREM 6.5. *The space $M_1(J, f)$ is a finite-dimensional polydisk, whose dimension is equal to the number of ergodic components of the maximal measurable subset of J carrying an invariant line field.*

Proof. The space $M_1(J, f)$ injects into $\mathrm{Def}(\hat{\mathbb{C}}, f)$ (by extending μ to be zero on the Fatou set), so $M_1(J, f)$ is finite-dimensional by the preceding Lemma. The line fields with ergodic support form a basis for $M(J, f)$. ∎

COROLLARY 6.6. *On the Julia set there are finitely many positive measure ergodic components outside of which the action of the tangent map of f is irreducible.*

DEFINITIONS. A critical point is *acyclic* if its forward orbit is infinite. Two points x and y in the Fatou set are in the same *foliated equivalence class* if the closures of their grand orbits agree. For example, if x and y are on the same leaf of the canonical foliation of a Siegel disk, then they lie in a single foliated equivalence class. On the other hand, if x and y belong to an attracting or parabolic basin, then to lie in the same foliated equivalence class they must have the same grand orbit.

THEOREM 6.7. *The space* $\mathrm{Teich}(\Omega^{\mathrm{fol}}, f)$ *is a finite-dimensional polydisk, whose dimension is given by the number of cycles of Herman rings plus the number of foliated equivalence classes of acyclic critical points landing in Siegel disks, Herman rings or superattracting basins.*

Proof. As for Ω^{dis}, we can write $\Omega^{\mathrm{fol}} = \bigcup \Omega_i^{\mathrm{fol}}$, a disjoint union of totally invariant open sets such that $\Omega_i^{\mathrm{fol}}/f$ is connected for each i. Then

$$\mathrm{Teich}(\Omega^{\mathrm{fol}}, f) = \prod{}' \mathrm{Teich}(\Omega_i^{\mathrm{fol}}, f),$$

and by Theorem 6.1 each factor on the right is either a complex disk or trivial. By Theorem 5.2, each disk factor can be lifted to $\mathrm{Def}(\hat{\mathbb{C}}, f)$, so by Lemma 6.4 the number of disk factors is finite. A disk factor arises whenever Ω_i^{fol} has an annular component. A cycle of foliated regions with n critical leaves gives n periodic annuli in the Siegel disk case, $n+1$ in the case of a Herman ring, and n wandering annuli in the superattracting case. If two critical points account for the same leaf, then they lie in the same foliated equivalence class. ∎

THEOREM 6.8 (Number of Moduli). *The dimension of the Teichmüller space of a rational map is given by* $n = n_{AC} + n_{HR} + n_{LF} - n_P$, *where*

- n_{AC} *is the number of foliated equivalence classes of acyclic critical points in the Fatou set,*
- n_{HR} *is the number of cycles of Herman rings,*
- n_{LF} *is the number of ergodic line fields on the Julia set, and*
- n_P *is the number of parabolic cycles.*

Proof. The Teichmüller space of an n-times punctured torus has dimension n, while that of an $n+2$-times punctured sphere has dimension $n-1$. Thus the dimension of $\mathrm{Teich}(\Omega^{\mathrm{dis}}/f)$ is equal to the number of grand orbits of acyclic critical points in Ω^{dis}, minus n_P. We have just seen the number of remaining acyclic critical orbits (up to foliated equivalence), plus n_{HR}, gives the dimension of $\mathrm{Teich}(\Omega^{\mathrm{fol}}, f)$. Finally n_{LF} is the dimension of $M_1(J, f)$. ∎

6.3. *No Wandering Domains*

A *wandering domain* is a component Ω_0 of the Fatou set Ω such that the forward iterates $f^i(\Omega_0)$; $i > 0$ are disjoint. The main result of [44] states:

THEOREM 6.9. *The Fatou set of a rational map has no wandering domain.*

Here is another proof, using the results above.

LEMMA 6.10 (Baker). *If f has a wandering domain Ω_0, then $f^n(\Omega_0)$ is a disk for all n sufficiently large.*

A simple geometric proof of this Lemma was found by Baker in 1983 (as described in the last paragraph of [6]). The result also appears in [4, 11, and 3].

Proof of Theorem 6.9. By the Lemma, if f has a wandering domain then it has a wandering disk Ω_0. For all n large enough, $f^n(\Omega_0)$ contains no critical points, so it maps bijectively to $f^{n+1}(\Omega_0)$. Thus the Riemann surface Ω^{dis}/f has a component X which is a finitely punctured disk. (One puncture appears for each grand orbit containing a critical point and passing through Ω_0.) But the Teichmüller space of X is infinite dimensional, contradicting the finite-dimensionality of the Teichmüller space of f. ▮

6.4. *Discreteness of the Modular Group*

Proof of Theorem 2.3 (*Discreteness of the Modular group*). The group $\mathrm{Mod}(\hat{\mathbb{C}}, f)$ acts isometrically on the finite-dimensional complex manifold $\mathrm{Teich}(\hat{\mathbb{C}}, f)$ with respect to the Teichmüller metric. The stabilizer of a point $([\phi], \hat{\mathbb{C}}, g)$ is isomorphic to $\mathrm{Aut}(g)$ and hence finite; thus the quotient of $\mathrm{Mod}(\hat{\mathbb{C}}, f)$ by a finite group acts faithfully. By compactness of quasiconformal maps with bounded dilatation, $\mathrm{Mod}(\hat{\mathbb{C}}, f)$ maps to a closed subgroup of the isometry group; thus $\mathrm{Mod}(\hat{\mathbb{C}}, f)$ is a Lie group. If $\mathrm{Mod}(\hat{\mathbb{C}}, f)$ has positive dimension, then there is an arc $(\phi_t, \hat{\mathbb{C}}, f)$ of inequivalent markings of f in Teichmüller space; but such an arc can be lifted to the deformation space, which implies each ϕ_t is in $\mathrm{QC}_0(f)$ (just as in the proof of Lemma 6.4), a contradiction.

Therefore $\mathrm{Mod}(\hat{\mathbb{C}}, f)$ is discrete. ▮

Remark. Equivalently, we have shown that for any $K > 1$, there are only a finite number of nonisotopic quasiconformal automorphisms of f with dilatation less than K.

7. HOLOMORPHIC MOTIONS AND QUASICONFORMAL CONJUGACIES

DEFINITIONS. Let X be a complex manifold. A *holomorphic family* of rational maps $f_\lambda(z)$ over X is a holomorphic map $X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, given by $(\lambda, z) \mapsto f_\lambda(z)$.

Let $X^{\mathrm{top}} \subset X$ be the set of *topologically stable* parameters. That is, $\alpha \in X^{\mathrm{top}}$ if and only if there is a neighborhood U of α such that f_α and f_β are topologically conjugate for all $\beta \in U$.[3]

[3] In the terminology of smooth dynamics, these are the *structurally stable* parameters relative to the family X.

The space $X^{\mathrm{qc}} \subset X^{\mathrm{top}}$ of *quasiconformally stable* parameters is defined similarly, except the conjugacy is required to be quasiconformal.

Let $X_0 \subset X$ be the set of parameters such that the number of critical points of f_λ (counted *without* multiplicity) is locally constant. For $\lambda \in X_0$ the critical points can be locally labeled by holomorphic functions $c_1(\lambda), ..., c_n(\lambda)$. Indeed, X_0 is the maximal open set over which the projection $C \to X$ is a covering space, where C is variety of critical points

$$\{(\lambda, c) \in X \times \hat{\mathbb{C}} : f'_\lambda(c) = 0\}.$$

A *critical orbit relation* of f_λ is a set of integers (i, j, a, b) such that $f^a(c_i(\lambda)) = f^b(c_j(\lambda))$; here $a, b \geqslant 0$.

The set $X^{\mathrm{post}} \subset X_0$ of *postcritically stable* parameters consists of those λ such that the set of critical orbit relations is locally constant. That is, $\lambda \in X^{\mathrm{post}}$ if any coincidence between the forward orbits of two critical points persists under a small change in λ. This is clearly necessary for topologically stability, so $X^{\mathrm{top}} \subset X^{\mathrm{post}}$.

The main result of this section is the following.

THEOREM 7.1. *In any holomorphic family of rational maps, the topologically stable parameters are open and dense.*

Moreover the structurally stable, quasiconformally stable and postcritically stable parameters coincide ($X^{\mathrm{top}} = X^{\mathrm{qc}} = X^{\mathrm{post}}$).

This result was anticipated and nearly established in [30, Theorem D]. The proof is completed and streamlined here using the Harmonic λ-Lemma of Bers and Royden.

7.1. *Holomorphic Motions*

DEFINITION. A *holomorphic motion* of a set $A \subset \hat{\mathbb{C}}$ over a connected complex manifold with basepoint (X, x) is a mapping $\phi : X \times A \to \hat{\mathbb{C}}$, given by $(\lambda, a) \mapsto \phi_\lambda(a)$, such that:

1. For each fixed $a \in A$, $\phi_\lambda(a)$ is a holomorphic function of λ;
2. For each fixed $\lambda \in X$, $\phi_\lambda(a)$ is an injective function of a; and
3. The injection is the identity at the basepoint (that is, $\phi_x(a) = a$).

We will use two fundamental results about holomorphic motions:

THEOREM 7.2 (The λ-Lemma). *A holomorphic motion of A has a unique extension to a holomorphic motion of $\overline{A}$. The extended motion gives a continuous map $\phi : X \times \overline{A} \to \hat{\mathbb{C}}$. For each λ, the map $\phi_\lambda : A \to \hat{\mathbb{C}}$ extends to a quasiconformal map of the sphere to itself.*

See [30]; the final statement appears in [7].

DEFINITION. Let $U \subset \hat{\mathbb{C}}$ be an open set with $|\hat{\mathbb{C}} - U| > 2$. A Beltrami coefficient μ on U is *harmonic* if locally

$$\mu = \mu(z) \frac{d\bar{z}}{dz} = \frac{\bar{\Phi}}{\rho^2} = \frac{\bar{\Phi}(z)\, d\bar{z}^2}{\rho(z)^2\, dz\, d\bar{z}},$$

where $\Phi(z)\, dz^2$ is a holomorphic quadratic differential on U and ρ^2 is the area element of the hyperbolic (or Poincaré) metric on U.

THEOREM 7.3 (The Harmonic λ-Lemma). *Let*

$$\phi\colon \Delta \times A \to \hat{\mathbb{C}}$$

be a holomorphic motion over the unit disk, with $|A| > 2$. *Then there is a* unique *holomorphic motion of the whole sphere*

$$\bar{\phi}\colon \Delta(1/3) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}},$$

defined over the disk of radius $1/3$ *and agreeing with* ϕ *on their common domain of definition, such that the Beltrami coefficient of* $\bar{\phi}_\lambda(z)$ *is harmonic on* $\hat{\mathbb{C}} - \bar{A}$ *for each* λ.

Remarks. The Harmonic λ-Lemma is due to Bers–Royden [7]. The extension it provides has the advantage of uniqueness, which will be used below to guarantee compatibility with dynamics. For other approaches to extending holomorphic motions, see [47] and [40]. These motions are the main tool in our construction of conjugacies.

DEFINITION. Let $f_\lambda(z)$ be a holomorphic family of rational maps over (X, x). A holomorphic motion *respects the dynamics* if it is a conjugacy: that is, if

$$\phi_\lambda(f_x(a)) = f_\lambda(\phi_\lambda(a))$$

whenever a and $f_x(a)$ both belong to A.

THEOREM 7.4. *For any* $x \in X^{\mathrm{post}}$, *there is a neighborhood* U *of* x *and a holomorphic motion of the sphere over* (U, x) *respecting the dynamics.*

Proof. Choose a polydisk neighborhood V of x in X^{post}. Then over V the critical points of f_λ can be labeled by distinct holomorphic functions $c_i(\lambda)$. In other words,

$$\{c_1(\lambda), ..., c_n(\lambda)\}$$

defines a holomorphic motion of the critical points of f_x over V.

The condition of constant critical orbit relations is exactly what we need to extend this motion to the forward orbits of the critical points. That is, if we specify a correspondence between the forward orbits of the critical points of f_x and f_λ by

$$f_x^a(c_i(x)) \mapsto f_\lambda^a(c_i(\lambda)),$$

the mapping we obtain is well-defined, injective, and depends holomorphically on λ.

Next, we extend this motion to the *grand* orbits of the critical points. There is a unique extension compatible with the dynamics. Indeed, consider a typical point where the motion has already been defined, say by $q(\lambda)$. Let

$$Z = \{(\lambda, p) : f_\lambda(p) = q(\lambda)\} \subset V \times \hat{\mathbb{C}} \xrightarrow{\pi} V$$

be the graph of the multivalued function $f_\lambda^{-1}(q(\lambda))$. If $q(\lambda)$ has a preimage under f_λ which is a critical point $c_i(\lambda)$, then this critical point has constant multiplicity and the graph of c_i forms one component of Z. The remaining preimages of $q(\lambda)$ have multiplicity one, and therefore $\pi^{-1}(\lambda)$ has constant cardinality as λ varies in V. Consequently Z is a union of graphs of single-valued functions giving a holomorphic motion of $f_x^{-1}(q(x))$.

The preimages of $q(\lambda)$ under f_x^n are treated similarly, by induction on n, giving a holomorphic motion of the grand orbits compatible with the dynamics.

By the λ-lemma, this motion extends to one sending $\hat{P}(x)$ to $\hat{P}(\lambda)$, where $\hat{P}(\lambda)$ denotes the closure of the grand orbits of the critical points of f_λ.

If $|\hat{P}(x)| \leqslant 2$, then f_λ is conjugate to $z \mapsto z^n$ for all $\lambda \in V$; the theorem is easy in this special case. Otherwise $\hat{P}(x)$ contains the Julia set of f_x, and its complement is a union of hyperbolic Riemann surfaces.

To conclude the proof, we apply the Bers–Royden Harmonic λ-lemma to extend the motion of $\hat{P}(x)$ to an *unique* motion $\phi_\lambda(z)$ of the whole sphere, such that the Beltrami coefficient $\mu_\lambda(z)$ of $\phi_\lambda(z)$ is harmonic on $\hat{\mathbb{C}} - \hat{P}(x)$. This motion is defined on a polydisk neighborhood U of x of one-third the size of V.

For each λ in U the map

$$f_\lambda : (\hat{\mathbb{C}} - \hat{P}(\lambda)) \to (\hat{\mathbb{C}} - \hat{P}(\lambda))$$

is a covering map. Define another extension of the motion to the whole sphere by

$$\psi_\lambda(z) = f_\lambda^{-1} \circ \phi_\lambda \circ f_x(z),$$

where $z \in \widehat{\mathbb{C}} - \hat{P}(x)$ and the branch of the inverse is chosen continuously so that $\psi_x(z) = z$. (On $\hat{P}(x)$ we set leave the motion the same, since it already respects the dynamics).

The rational maps f_λ and f_x are conformal, so the Beltrami coefficient of ψ_λ is simply $f_x^*(\mu_\lambda)$. But f_x is a holomorphic local isometry for the hyperbolic metric on $\widehat{\mathbb{C}} - \hat{P}(x)$, so it pulls back harmonic Beltrami differentials to harmonic Beltrami differentials. By uniqueness of the Bers–Royden extension, we have $\psi_\lambda = \phi_\lambda$, and consequently the motion ϕ respects the dynamics. ∎

COROLLARY 7.5. *The postcritically stable, quasiconformally stable and topologically stable parameters coincide.*

Proof. It is clear that $X^{\mathrm{qc}} \subset X^{\mathrm{top}} \subset X^{\mathrm{post}}$. By the preceding theorem, if $x \in X^{\mathrm{post}}$ then for all λ in a neighborhood of x we have $\phi_\lambda \circ f_\lambda = f_x \circ \phi_\lambda$, where $\phi_\lambda(z)$ is a holomorphic motion of the sphere. By the λ-lemma, $\phi_\lambda(z)$ is quasiconformal, so $X^{\mathrm{post}} \subset X^{\mathrm{qc}}$. ∎

Remark. The monodromy of holomorphic motions in non-simply connected families of rational maps can be quite interesting; see [31, 32, 19, 8, 9].

7.2. *Density of Structural Stability*

The completion of the proof of density of topological stability will use the notion of J-stability developed by Mañé, Sad, and Sullivan. Theorems 7.6 and 7.7 below are from [30]; see also [33, Section 4; 27].

THEOREM 7.6. *Let $f_\lambda(z)$ be a holomorphic family of rational maps over X. Then the following conditions on a parameter $x \in X$ are equivalent.*

1. *The total number of attracting and superattracting cycles of f_λ is constant on a neighborhood of x.*

2. *Every periodic point of f_x is attracting, repelling or persistently indifferent.*

3. *The is a neighborhood (U, x) over which the Julia set J_x admits a holomorphic motion compatible with the dynamics.*

Here a periodic point z of f_x of period p is *persistently indifferent* if there is a holomorphic map $w: U \to \widehat{\mathbb{C}}$ defined on a neighborhood of x such that $f_\lambda^p(w(\lambda)) = w(\lambda)$ and $|(f_\lambda^p)'(w(\lambda))| = 1$ for all $\lambda \in U$.

DEFINITION. The $x \in X$ such that the above conditions hold form the *J-stable* parameters of the family $f_\lambda(z)$.

THEOREM 7.7. *The J-stable parameters are open and dense.*

Proof. The local maxima of $N(\lambda)$ = (the number of attracting cycles of f_λ) are open and dense (since $N(\lambda) \leqslant 2d-2$) and equal to X^{stable} by the preceding result. ∎

The next result completes the proof of Theorem 7.1.

THEOREM 7.8. *The postcritically stable parameters are open and dense in the set of J-stable parameters.*

Remark. A similar result was established in [30].

Proof. Using Corollary 7.5, we have $X^{\text{post}} \subset X^{\text{stable}}$ because $X^{\text{post}} = X^{\text{top}}$ and topological conjugacy preserves the number of attracting cycles. By definition X^{post} is open, so it only remains to prove it is dense in X^{stable}.

For convenience, we first pass to the subset $X_0 \subset X^{\text{stable}}$ where two additional conditions hold:

(a) every superattracting cycle is persistently superattracting, and

(b) the number of critical points of f_λ is locally maximized.

Then X_0 is the complement of a proper complex analytic subvariety of X^{stable}, so it is open and dense.

Let U be an arbitrary polydisk in X_0, and label the critical points of f_λ by $c_i(\lambda)$, $i = 1, ..., n$ where $c_i : U \to \hat{\mathbb{C}}$ are holomorphic maps.

We will show that for each i and j, there is an open dense subset of U on which the critical orbit relations between c_i and c_j are locally constant. Since the intersection of a finite number of open and dense sets is open and dense, this will complete the proof.

Because $U \subset X^{\text{stable}}$, any critical point in the Julia set remains there as the parameter varies, and the holomorphic motion of the Julia respects the labeling of the critical points. (Indeed $f: J \to J$ is locally m-to-1 at a point $z \in J$ if and only if z is a critical point of multiplicity $m-1$; compare [33, Section 4]). Thus the critical orbit relations are constant for critical points in the Julia set; so we will assume c_i and c_j both lie in the Fatou set.

For $a, b \geqslant 0$, with $a \neq b$ if $i = j$, consider the critical orbit relation:

$$f_\lambda^a(c_i(\lambda)) = f_\lambda^b(c_j(\lambda)). \tag{$*$}$$

Let $U_{ij} \subset U$ be the set of parameters where no relation of the form $(*)$ holds for any a, b. If $U_{ij} = U$ we are done. If U_{ij} is empty, then (by the Baire category theorem), some relation of the form $(*)$ holds on an open subset of U, and hence throughout U. It is easy to verify that all relations between c_i and c_j are constant in this case as well.

In the remaining case, there are $a, b \geqslant 0$ and a $\lambda_0 \in U$ such that $(*)$ holds, but $(*)$ fails outside a proper complex analytic subvariety of U. To complete the proof, we will show λ_0 is in the closure of the interior of U_{ij}. In other words, we will find an open set of small perturbations of λ_0 where there are no relations between the forward orbits of c_i and c_j.

First consider the case where $i = j$. Since the Julia set moves continuously over U and periodic cycles do not change type, there is natural correspondence between components of the Fatou set of f_λ as λ varies. This correspondence preserves the types of periodic components as classified by Theorem 2.1 (using the fact that superattracting cycles are persistently superattracting over X_0).

We may assume c_i lands in an attracting or superattracting basin, or in a Siegel disk, since $(*)$ cannot hold in a Herman ring or in a parabolic basin. Thus $f^a_{\lambda_0}(c_i(\lambda_0))$ lands on the corresponding attracting, superattracting or indifferent cycle. In the attracting and superattracting cases, the critical point is asymptotic to this cycle (but not equal to it) for all λ sufficiently close to λ_0 but outside the variety where $(*)$ holds. In the Siegel disk case, $f^a_\lambda(c_i(\lambda))$ lies in the Siegel disk but misses its center when $(*)$ fails, implying its forward orbit is infinite in this case as well. Thus λ_0 is in the closure of the interior of U_{ii}, as desired.

Now we treat the case $i \neq j$. Since we have already shown that the self-relations of a critical point are constant on an open dense set, we may assume c_i and c_j have constant self-relations on U. Then if c_i has a finite forward orbit for one (and hence all) parameters in U, so does c_j (by $(*)$), and they both land in the same periodic cycle when $\lambda = \lambda_0$. Since each periodic cycle moves holomorphically over U, $(*)$ holds on all of U and we are done.

So henceforth we assume c_i and c_j have infinite forward orbits for all parameters in U. Increasing a and b if necessary, we can assume the point

$$p = f^a_{\lambda_0}(c_i(\lambda_0)) = f^b_{\lambda_0}(c_j(\lambda_0))$$

lies in a periodic component of the Fatou set. If this periodic component is an attracting, superattracting or parabolic basin, we may assume that p lies close to the corresponding periodic cycle.

In the attracting and parabolic cases, we claim there is a ball B containing p such that for all λ sufficiently close to λ_0, the sets

$$\langle f^n_\lambda(B) : n = 0, 1, 2, ... \rangle$$

are disjoint and $f^n_\lambda \mid B$ is injective for all $n > 0$. This can be verified using the local models of attracting and parabolic cycles. Now for λ near λ_0 but outside the subvariety where $(*)$ holds, $f^a_\lambda(c_i(\lambda))$ and $f^b_\lambda(c_j(\lambda))$ are distinct points in B. Thus $c_i(\lambda)$ and $c_j(\lambda)$ have disjoint forward orbits and we have shown λ_0 is in the closure of the interior of U_{ij}.

In the superattracting case, this argument breaks down because we cannot obtain injectivity of all iterates of f_λ on a neighborhood of p. Instead, we will show that near λ_0, c_i and c_j lie on different leaves of the canonical foliation of the superattracting basin, and therefore have distinct grand orbits.

To make this precise, choose a local coordinate with respect to which the dynamics takes the form $Z \mapsto Z^d$. More precisely, if the period of the superattracting cycle is k, let $Z_\lambda(z)$ be a holomorphic function of (λ, z) in a neighborhood of (λ_0, p), which is a homeomorphism for each fixed λ and which satisfies

$$Z_\lambda(f_\lambda^k(z)) = Z_\lambda^d(z).$$

(The existence of Z follows from classical results on superattracting cycles; cf. [11, Section II.4].) Since $(*)$ does not hold throughout U, there is a neighborhood V of λ_0 on which

$$Z_\lambda(f_\lambda^a(c_i(\lambda))) \neq Z_\lambda(f_\lambda^b(c_j(\lambda)))$$

unless $\lambda = \lambda_0$. Note too that neither quantity vanishes since each critical point has an infinite forward orbit. Shrinking V if necessary we can assume

$$|Z_\lambda(f_\lambda^a(c_i(\lambda)))|^d < |Z_\lambda(f_\lambda^b(c_j(\lambda)))| < |Z_\lambda(f_\lambda^a(c_i(\lambda)))|^{1/d}$$

for $\lambda \in V$. If we remove from V the proper real-analytic subset where

$$|Z_\lambda(f_\lambda^a(c_i(\lambda)))| = |Z_\lambda(f_\lambda^b(c_j(\lambda)))|,$$

we obtain an open subset of U_{ij} with λ_0 in its closure. (Here we use the fact that two points where $\log\log(1/|Z|)$ differs by more than zero and less than $\log d$ cannot be in the same grand orbit.)

Finally we consider the case of a Siegel disk or Herman ring of period k. In this case, for all $\lambda \in U$, the forward orbit of $c_i(\lambda)$ determines a dense subset of $C_i(\lambda)$, a union of k invariant real-analytic circles. This dynamically labeled subset moves injectively as λ varies, so the λ-lemma gives a holomorphic motion

$$\phi\colon U \times C_i(\lambda_0) \to \widehat{\mathbb{C}},$$

which respects the dynamics. For each fixed λ, $\phi_\lambda(z)$ is a holomorphic function of z as well, since f_λ is holomorphically conjugate to a linear rotation in domain and range.

By assumption, $f_\lambda^b(c_j(\lambda) \in C_i(\lambda)$ when $\lambda = \lambda_0$. If this relation holds on an open subset V of U, then

$$g(\lambda) = \phi_\lambda^{-1}(f_\lambda^b(c_j(\lambda)))$$

is a holomorphic function on V with values in $C_i(\lambda_0)$, hence constant. It follows that $(*)$ holds on V, and hence on all of U and we are done. Otherwise, $c_i(\lambda)$ and $c_j(\lambda)$ have disjoint forward orbits for all λ outside the proper real-analytic subset of U where $f_\lambda^b(c_j(\lambda)) \in C_i(\lambda)$. ∎

From the proof we have a good qualitative description of the set of points that must be removed to obtain X^{top} from X^{stable}.

Corollary 7.9. *The set X^{stable} is the union of the open dense subset X^{top} and a countable collection of proper complex and real-analytic subvarieties. Thus X^{top} has full measure in X^{stable}.*

The real-analytic part occurs only when f_λ has a foliated region (a Siegel disk, Herman ring, or a persistent superattracting basin) for some $\lambda \in X^{\text{stable}}$.

Remark. On the other hand, Rees has shown that X^{stable} need not have full measure in X [37].

Corollary 7.10. *If X is a connected J-stable family of rational maps with no foliated regions, then X^{post} is connected and $\pi_1(X^{\text{post}})$ maps surjectively to $\pi_1(X)$.*

Proof. The complement $X - X^{\text{post}}$ is a countable union of proper complex subvarieties, which have codimension two. ∎

8. HYPERBOLIC RATIONAL MAPS

In this section we show that two topologically conjugate hyperbolic rational maps are quasiconformally conjugate.

Proof of Theorem 2.9 (Topological conjugacy). The stable regions of an expanding (alias hyperbolic) rational map are either attracting or superattracting basins. Let $\phi: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a topological conjugacy between two expanding maps f_1 and f_2. We will deform ϕ to a quasiconformal conjugacy.

The map ϕ preserves the extended Julia set $\hat{J}$ and the regions Ω^{dis} and Ω^{fol} introduced in Section 6 because these are determined by the topological dynamics. Thus ϕ descends to a homeomorphism $\bar{\phi}$ between the Riemann surfaces $\Omega_1^{\text{dis}}/f_1$ and $\Omega_2^{\text{dis}}/f_2$. These surfaces are finite unions of punctured tori. By standard surface topology we may deform $\bar{\phi}$ through an isotopy $\bar{\phi}_t$ to a quasiconformal map between these surfaces. (Indeed, after deformation we can arrange that the mapping is the restriction of a smooth map between closed tori.) Lifting this deformation, we obtain an isotopy of $\phi_t : \Omega_1^{\text{dis}} \to \Omega_2^{\text{dis}}$, starting at ϕ and respecting the dynamics for all t.

It is easy to see this isotopy converges to the trivial deformation along the grand orbits of the periodic points and critical points in the attracting regions. We claim it also converges to the trivial deformation along the Julia set. Indeed, the hyperbolic property implies each map is expanding with respect to some conformal metric near the Julia set. Thus pulling back the deformation by f_i^{-n} contracts it exponentially, so the limiting deformation is trivial.

We have arrived at a deformation of our conjugacy which is trivial on the Julia set and which renders the conjugacy quasiconformal on Ω^{dis}. A similar argument may be employed to deform the conjugacy on the superattractive regions to get quasiconformality on Ω^{fol}. (By taking care to preserve circular symmetry, an isotopy defined on a single annulus or punctured disk component of Ω^{fol} extends equivariantly to the entire grand orbit.)

By well-known removability results, a homeomorphism which is quasiconformal except on a closed countable union of isolated points and real-analytic arcs is globally quasiconformal [26, Section I.8.3]. Thus at the conclusion of the isotopy we obtain a conjugacy ψ which is quasiconformal on the entire Fatou set Ω.

Finally we claim ψ is quasiconformal on the Julia set. Consider a small circle C centered at a point $x \in J(f_1)$ (Fig. 3). By the distortion lemma for expanding conformal maps [43], as long as an iterate $f_1^n(C)$ remains close to the Julia set, f_1^n distorts ratios of distances between points in C by a bounded amount (it is a quasi-similarity). In particular, the ratio between the inradius and outradius of $f_1^n(C)$ about $f_1^n(x)$ is bounded independent of C.

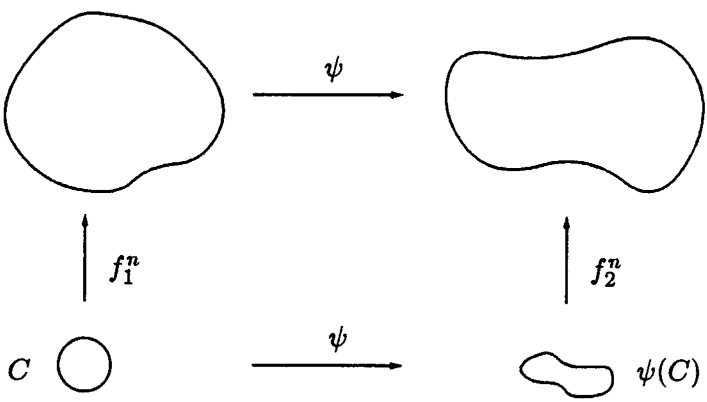

Fig. 3. Proof of quasiconformality of ψ.

By the expanding property, we can also choose n large enough that $f_1^n(C)$ has definite diameter. Then its image under ψ is still has a bounded ratio of inradius to outradius. Since ψ is a conjugacy,

$$\psi \circ f_1^n(C) = f_2^n \circ \psi(C).$$

Applying the distortion lemma once more, we conclude that $\psi(C)$ has a bounded ratio of inradius to outradius, because it maps with bounded distortion under f_2^n to $\psi(f_1^n(C))$. Thus the circular dilatation of ψ is bounded on the Julia set, and therefore ψ is quasiconformal on the whole sphere. ∎

Remark. In the proof above, the Koebe distortion theorem for univalent functions can be used in place of the distortion theorem for expanding maps. (One then appeals to the property $J(f) \cap P(f) = \emptyset$ to construct univalent inverses near the Julia set.)

9. COMPLEX TORI AND INVARIANT LINE FIELDS ON THE JULIA SET

A central problem in conformal dynamics is to settle:

CONJECTURE 9.1 (Density of Hyperbolicity). *The hyperbolic rational maps are open and dense among all rational maps of a given degree.*

In this section we formulate a conjecture about the ergodic theory of a *single* rational map which implies the density of hyperbolicity.

Let $X = \mathbb{C}/\Lambda$ be a complex torus; then X also has a group structure coming from addition on $\mathbb{C}$. Let $\wp\colon X \to \hat{\mathbb{C}}$ be a degree two holomorphic map to the Riemann sphere such that $\wp(-z) = \wp(z)$; such a map is unique up to automorphisms of $\hat{\mathbb{C}}$ and can be given by the Weierstrass $\wp$-function.

Let $F\colon X \to X$ be the endomorphism $F(z) = nz$ for some integer $n > 1$. Since $n(-z) = -(nz)$, there is a unique rational map f on the sphere such that the diagram

$$\begin{tikzcd} \mathbb{C}/\Lambda \arrow[r, "F"] \arrow[d, "\wp"'] & \mathbb{C}/\Lambda \arrow[d, "\wp"'] \\ \hat{\mathbb{C}} \arrow[r, "f"] & \hat{\mathbb{C}} \end{tikzcd}$$

commutes. (Compare [25].)

DEFINITION. A rational map f is *double covered by an integral torus endomorphism* if it arises by the above construction.

It is easy to see that repelling periodic points of F are dense on the torus, and therefore the Julia set of f is equal to the whole sphere. Moreover, F and $z \mapsto -z$ preserve the line field tangent to geodesics of a constant slope on X (or more formally, the Beltrami differential $\mu = d\bar{z}/dz$), and therefore f has an invariant line field on $\hat{\mathbb{C}}$.

CONJECTURE 9.2 (No Invariant Line Fields). *A rational map f carries no invariant line field on its Julia set, except when f is double covered by an integral torus endomorphism.*

THEOREM 9.3. *The no invariant line fields conjecture implies the density of hyperbolic dynamics in the space of all rational maps.*

Proof. Let $X = \mathrm{Rat}_d$ be the space of all rational maps of a fixed degree $d > 1$, and let X^{qc} be the open dense set of quasiconformally stable maps in this universal family. Let $f \in X^{\mathrm{qc}}$. Then $\mathrm{Teich}(\hat{\mathbb{C}}, f)/\mathrm{Mod}(\hat{\mathbb{C}}, f)$ parameterizes the component of $X^{\mathrm{qc}}/\mathrm{Aut}(\hat{\mathbb{C}})$ containing f. Since the modular group is discrete and $\mathrm{Aut}(\hat{\mathbb{C}})$ acts with finite stabilizers, we have

$$\dim \mathrm{Teich}(\hat{\mathbb{C}}, f) = \dim \mathrm{Rat}_d - \dim \mathrm{Aut}(\hat{\mathbb{C}}) = 2d - 2.$$

Clearly f has no indifferent cycles, since by J-stability these would have to persist on an open neighborhood of f in Rat_d and then on all of Rat_d. Therefore f has no Siegel disks or parabolic basins. Similarly f has no periodic critical points, and therefore no superattracting basins. By a Theorem of Mañé, f has no Herman rings [29]. Thus all stable regions are attracting basins. Finally f is not covered by an integral torus endomorphism because such rational maps form a proper subvariety of Rat_d.

By Theorem 6.8, the dimension of $\mathrm{Teich}(\hat{\mathbb{C}}, f)$ is given by $n_{AC} + n_{LF}$, the number of grand orbits of acyclic critical points in the Fatou set plus the number of independent line fields on the Julia set. The no invariant line fields conjecture then implies $n_{LF} = 0$, so $n_{AC} = 2d - 2$. Thus all critical points of f lie in the Fatou set and converge to attracting periodic cycles, and therefore f is hyperbolic. ∎

By a similar argument one may establish:

THEOREM 9.4. *The no invariant line fields conjecture implies the density of hyperbolic maps in the space of polynomials of any degree.*

Remarks. If f is covered by an integral torus endomorphism, then $\mathrm{Mod}(\hat{\mathbb{C}}, f)$ contains $\mathrm{PSL}_2(\mathbb{Z})$ with finite index (compare [22]). It seems likely that the modular group is finite for any other rational map whose Julia set is the sphere. This finiteness would follow from the no invariant line fields conjecture as well, since then $\mathrm{Mod}(\hat{\mathbb{C}}, f) = \mathrm{Aut}(f)$.

10. EXAMPLE: QUADRATIC POLYNOMIALS

To illustrate the general theory, we will describe the topological conjugacy classes of hyperbolic maps in the family of quadratic polynomials. Every quadratic polynomial is conformally conjugate to one of the form

$$f_c(z) = z^2 + c,$$

for a unique $c \in \mathbb{C}$. Let $H \subset \mathbb{C}$ be the open set of hyperbolic parameters. From the definition $c \in H$ if and only if the critical point $z = 0$ tends to an attracting or superattracting cycle under iteration. Thus H is the disjoint union of the following sets:

$H(p)$: the parameters such that $z = 0$ tends to an attracting cycle of period $p \geqslant 1$;

$H_0(p)$: those where the critical point itself has order p; and

$H(\infty)$: those where the critical point tends to ∞.

THEOREM 10.1. *The hyperbolic quadratic polynomials are classified up to conjugacy as follows.*

1. *Each component of $H(p)$ is isomorphic to a punctured disk and represents a single quasiconformal conjugacy class. The multiplier of the attracting cycle gives a natural isomorphism to Δ^*.*

2. *The set $H_0(p)$ is a finite set of points, corresponding to the punctures of $H(p)$.*

3. *The set $H(\infty)$ is isomorphic to a punctured disk and consists of a single quasiconformal conjugacy class.*

Hyperbolic maps belonging to different components of $H(p)$, $H_0(p)$ and $H(\infty)$ represent different topological conjugacy classes.

Proof. Any component U of $H(p)$ or of $H(\infty)$ is open, and the forward orbit of $z = 0$ under f_c is infinite for c in U. Thus the critical orbit relations are constant on U, so by Theorem 2.7 U represents a single quasiconformal conjugacy class.

Pick a basepoint $c \in U$; then

$$U \cong \mathrm{Teich}(\hat{\mathbb{C}}, f_c)/\mathrm{Mod}(\hat{\mathbb{C}}, f_c)$$

because any deformation of f_c is conformally conjugate to a quadratic polynomial, and two polynomials are identified by the modular group if and only if they are conformally conjugate.

If $c \in H(p)$, then $z=0$ is contained in an attracting basin Ω_0 and $f_c^p : \Omega_0 \to \Omega_0$ is conformally conjugate to the Blaschke product $B_\lambda : \Delta \to \Delta$, where

$$B_\lambda(z) = \frac{z(z+\lambda)}{1+\bar{\lambda}z},$$

and λ is the multiplier of the attracting cycle of f_c. The Julia set of a hyperbolic map has measure zero, and therefore carries no invariant line field. The superattracting basin of ∞ is conjugate to (Δ, z^2) and thus has no moduli. Therefore the Teichmüller space of f_c is isomorphic $\mathbb{H}$, the Teichmüller space of the punctured torus

$$Y = \Omega^{\mathrm{dis}}/f = (\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau) - \{0\}, \qquad \tau = \frac{\log \lambda}{2\pi i}.$$

On this torus there is a dynamically distinguished isotopy class of oriented simple closed curve γ, which lifts to a closed loop around the attracting periodic point. The modular group of f_c is generated by a Dehn twist about this loop, which acts by $\tau \mapsto \tau + 1$. Indeed, the modular group can be no larger because γ must be fixed. On the other hand, a Dehn twist about γ gives an element of $\mathrm{Mod}(\Delta, B_\lambda)$ fixing the ideal boundary of the disk, and so it can be extended from an automorphism of f_c^p on Ω_0 to an automorphism of $(\hat{\mathbb{C}}, f_c)$, fixing every point outside the grand orbit of Ω_0. (This Dehn twist can realized as monodromy in the family B_λ as λ loops once around the parameter $\lambda = 0$.) Since $\lambda = \exp(2\pi i \tau) \in \Delta^*$ is a complete invariant of a torus Y with a distinguished curve γ, U is a punctured disk parameterized by the multiplier.

As $\lambda \to 0$ at the puncture, c remains bounded, so there is a limiting polynomial which belongs to $H_0(p)$. Conversely a punctured neighborhood of $c \in H_0(p)$ belongs to $H(p)$, because the multiplier is continuous. The set $H_0(p)$ is contained in the roots of $f_c^p(0) = 0$ and is therefore finite.

We now turn to the open set $H(\infty)$. It is easy to see that $H(\infty)$ contains a neighborhood of ∞ and it has no bounded component (by the maximum principle), so it is connected. For $c \in H(\infty)$, the canonical foliation of the basin of infinity has a countable set of distinguished leaves meeting the grand orbit of $z=0$. Thus Ω^{fol} has a wandering annulus, and therefore $\mathrm{Teich}(\hat{\mathbb{C}}, f_c) \cong \mathbb{H}$. The modular group of f_c is at most $\mathbb{Z}$, since the critical point $z=0$ is distinguished and so any quasiconformal automorphism must fix the boundary of the wandering annulus pointwise. Since $\pi_1(H(\infty)) \neq \{1\}$, the modular group is nontrivial and $H(\infty)$ is isomorphic to a punctured disk.

Any two topologically conjugate hyperbolic maps are quasiconformally conjugate, and hence belong to the same component of $H(p)$, $H_0(p)$ or $H(\infty)$, by Theorem 2.9. ∎

COROLLARY 10.2 (Douady–Hubbard). *The Mandelbrot set*

$$M = \{c\colon f_c^n(0) \text{ does not tend to infinity as } n \to \infty\}$$

is connected.

Proof. If M were disconnected, then $H(\infty) = \mathbb{C} - M$ would have fundamental group larger than $\mathbb{Z}$, contradicting the previous theorem. ∎

THEOREM 10.3. *Let $h(p)$ denote the number of connected component of $H(p)$. Then*:

$$\sum_{p \mid n} h(p) = 2^{n-1}.$$

Proof. Let $Q(c) = f_c^n(0)$. By induction on n, $Q(c)$ is a monic polynomial with integral coefficients such that $Q'(c) = 1 \bmod 2$. Thus the resultant of $Q(c)$ and $Q'(c)$ is odd, and therefore $Q(c)$ has distinct roots. But these roots correspond exactly to the parameters c such that f_c is of type (ii) and $p \mid n$. Thus

$$\sum_{p \mid n} h(p) = \deg Q = 2^{n-1}$$

(using Theorem 10.1). ∎

This proof that $Q(c)$ has simple roots was pointed out by Allan Adler and by Andrew Gleason.

Remarks. It is not hard to show that $\mathbb{C} - \partial M$ consists exactly of the J-stable parameters in the family f_c, and that

$$\mathbb{C} = H \sqcup \partial M \sqcup Q,$$

where Q is the open set of c such that the Julia set of f_c has positive measure and carries an invariant line field [33, Section 4]. Thus hyperbolic dynamics is dense in the quadratic family if and only if Q is empty.

Douady and Hubbard formulated these theorems somewhat differently (without quasiconformal conjugacy) but proved them using our quasiconformal deformation technique introduced in [44] together with their explicit description of quadratic maps which amplifies greatly on the count of Theorem 10.3 [13, 14].

The papers of this series probably would not have been written without the inspiration and insight of these "bonshommes."

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