Hyperbolic Geometry and Spaces of Riemann Surfaces

Linda Keen

Introduction

Classifying Riemann surfaces is a problem that has fascinated mathematicians for more than a century. Real analytic, complex analytic, and geometric solutions have been found using a variety of techniques. In this article I shall examine several approaches; I shall restrict myself to the situation where the surface is a torus or a punctured torus and make the description very explicit.

Moduli Spaces for Riemann Surfaces

A Riemann surface is a topological surface with a complex analytic structure on it; that is, the surface is covered by a set of charts so that the relation between the maps defined on overlapping neighborhoods is complex analytic. If S_1 and S_2 are two Riemann surfaces, it can happen that there exist homeomorphisms from S_1 to S_2, and yet none of these homeomorphisms is complex analytic. In other words, S_1 and S_2 have the same underlying topological surface but are distinct as Riemann surfaces. It turns out that, unless the underlying surface is the 2-sphere or the 2-sphere minus 1, 2, or 3 points, there is a continuum of distinct Riemann surfaces with the same underlying surface. How then might we characterize the set of all distinct Riemann surfaces for a given topological surface S? This set is known as the *moduli space* of the surface and is denoted Mod(S). To put the characterization problem more concretely:

- Can we realize Mod(S) as some natural geometric object (e.g., as a real analytic manifold, or perhaps even as a complex analytic manifold)?
- Can we find parameters (these are the "moduli"), at least for some large open subset of this manifold, so that as we vary the parameters there is some aspect of the complex structure of the corresponding Riemann surfaces that is visibly varying with the parameters?

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This article looks at several examples that illustrate what the problem is about and some of the methods that have been used to attack it. The geometric key is that a complex analytic homeomorphism is *conformal;* that is, it preserves angles locally. It is an easy exercise in calculus to show that if a map is complex analytic and invertible at a point, then the angle between two curves intersecting transversally at that point, measured as the angle between their tangents, is equal to the angle between the image curves at the image point. Maps that distort angles cannot be complex analytic.

First Simple Example Let λ be a real number in the unit interval $I = \{0 < \lambda < 1\}$, and consider the cyclic group

$$G_\lambda = \{g_n : z \mapsto \lambda^n z,\ n \in \mathbf{Z}\}$$

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of conformal homeomorphisms of the punctured plane, $\mathbf{C}^* = \mathbf{C} - \{0\}$, to itself. The natural map $\mathbf{C}^* \to \mathbf{C}^*/G_\lambda \cong S_\lambda$ maps the half-open annulus $A_\lambda = \{|\lambda| \le |z| < 1\}$ one-to-one onto the quotient S_λ. Because $g_1 : z \to \lambda z$ maps the unit circle one-to-one onto the inner boundary of A_λ, we see that S_λ is topologically a torus: The image α of the unit circle and the image β of the real axis are a pair of generators for its homology. Projecting the complex structure from $\mathbf{C}$ onto S_λ makes it a Riemann surface.

When λ is close to 1, β is very short, so we get a very "skinny" torus, and as λ decreases, the torus gets "fatter." Because complex analytic maps preserve conformal geometry, this distortion is reflected in the complex structure, so it is plausible that we get a whole continuum of different complex structures as λ varies in the interval.

Now suppose that λ is no longer real but is a complex number $re^{i\theta}$ in the punctured unit disk, $D^* = \{z : 0 < |z| < 1\}$. Define the group G_λ, the quotient $\mathbf{C}^*/G_\lambda \cong S_\lambda$, and the annulus A_λ as above. The element $g_1 \in G_\lambda$ still identifies the inner boundary of A_λ with the outer boundary, but now the inner circle is twisted by the angle θ before it is glued. This twisting distorts the complex structure of the quotient, so for fixed r and varying θ, there is another whole continuum of different structures. In fact, classical theorems from elliptic function theory tell us that *every* possible complex structure on the torus is obtained from some $\lambda \in D^*$. Thus, D^* is a good candidate for our natural realization of Mod(S) and λ is a natural parameter. However, D^* is *not* quite Mod(S) because many different λ's may give rise to the same structure. We shall return to this question after the next section. We shall see that the parameter space D^* is, in fact, a covering space of Mod(S). It is typical that the moduli space is difficult to find; one often has to settle for a covering space.

Second Simple Example A more usual representation of the torus is obtained by considering the group $G_\tau = \{g_{m,n}(z) = z+m+n\tau : m, n \in \mathbf{Z}\}$, where τ is in the upper half-plane U, and forming the quotient $\mathbf{C} \to \mathbf{C}/G_\tau \cong S_\tau$. The parallelogram P_τ spanned by 1 and τ, with its opposite sides glued, is the analogue of the annulus. The complex structure on the torus is inherited from $\mathbf{C}$ and depends on G_τ. As uniform stretching doesn't change the complex structure, the quotient of $\mathbf{C}$ by the group $\{z \to z + rm + rn\tau : m, n \in \mathbf{Z}\}$, where r is any positive scalar, determines a torus equivalent to S_τ. The space of moduli is the collection of groups G_τ, $\tau \in U$. The parameter space U is, therefore, another covering space of Mod(S) that is easy to find and to work with. To find Mod(S) one has to see how different choices of generators for the groups G_τ are related. For these groups one knows how to do this: The classical modular group $PSL(2, \mathbf{Z})$ relates pairs of generators.

The plane $\mathbf{C}$ is simply connected and is the universal cover of the torus. The exponential maps the parallelogram P_τ onto the annulus A_λ for $\lambda = \exp(2\pi i\tau)$. Because the parameters τ and $-1/\tau$ give rise to equivalent tori, so do the parameters $\lambda = \exp(2\pi i\tau)$ and $\lambda' = \exp(-2\pi i/\tau)$. If $\tau = it$, for $t > 0$ real and large, λ is real and very small, so the underlying torus is fat. On the other hand, λ' is real and close to 1, so the underlying torus is skinny. Thus, a torus that is fat from one perspective is skinny from another.

The relation between τ and λ also shows that the parameter space D^* is an intermediate covering space between U and Mod(S).

Boundary Behavior In our examples, the plane domains in which the parameter spaces are embedded have boundaries. This means that if our parameter reaches the boundary, something has happened—the construction of the torus no longer works.

Let us look at what happens when $\lambda = re^{i\theta}$ tends to the boundary of D^*. The absolute value of the parameter, $r = |\lambda|$, is measuring the size of the annulus. The open arc from 1 to λ projects to a closed curve β on the torus S_λ. If $\theta = 0$, so that $\lambda = r$, the length of β on S_λ is $|\log r|$; as $r \to 0$, it becomes infinite. On the other hand, suppose $\theta = 2\pi ip/q$ for p/q rational, and consider the collection of arcs

$$\{(r, 1), (re^{i\theta}, e^{i\theta}), (re^{2i\theta}, e^{2i\theta}), \ldots, (re^{(q-1)i\theta}, e^{(q-1)i\theta})\}.$$

They project to a closed curve on S_λ that I again call β. Now if $r \to 1$, the arcs get short, β becomes "pinched," and its length on S_λ goes to zero. In either case, there is no longer a torus; it has become a doubly infinite cylinder.

Now let us look at the parameter space U. What happens as τ approaches the rational points on the boundary of U? Suppose $\tau = p/q + it$, $p, q \in \mathbf{Z}$, $t \in \mathbf{R}^+$. Draw the parallelogram P_τ spanned by 1 and τ; it contains the vertical line joining the origin and $-p + q\tau = qit$, which projects to a closed curve β on S_τ. As $t \to 0$, β is "pinched," and when $t = 0$, S_τ has degenerated to a doubly infinite cylinder again.

It is much more difficult to describe what is degenerating on S_τ as we approach the irrational points on the boundary. If we write $\tau = r + it$, where r is irrational and $t > 0$, the projection β of the vertical line in the parallelogram joining 0 and $-r + \tau = it$ never closes up on S_τ and, hence, is an open curve of infinite length. If we call α the projection of the generator joining 0 and 1, we see that what is getting shorter is the length of the segment of β between its successive intersections with α.

Third Example: A First Taste of Hyperbolic Geometry In our first example, we can think of the annulus A_λ as the torus cut open along a curve. The domain $\mathbf{C}^*$ is "tiled" by more annuli $A_n = g_n(A_\lambda)$; the annuli A_n don't overlap and together fill out all of $\mathbf{C}^*$. The group G_λ is a discrete group of conformal self-maps of $\mathbf{C}^*$.

In our second example, we can think of the parallelogram P_τ as the torus cut along a pair of simple curves that intersect exactly once. The group of trans-

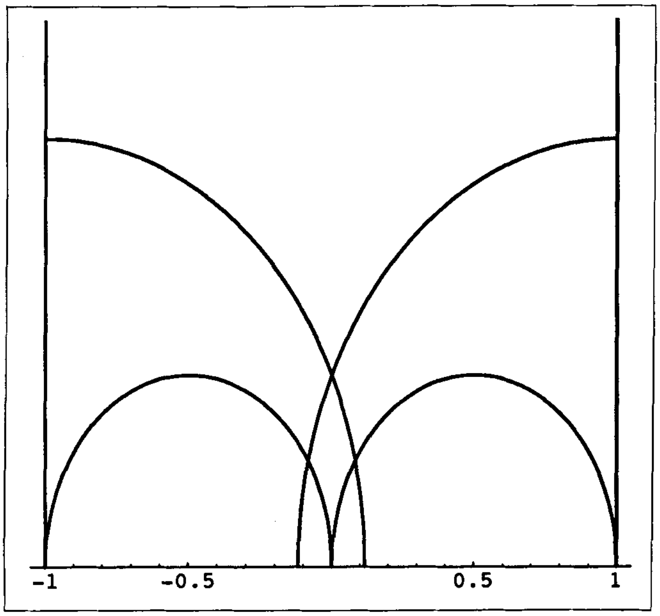

Figure 1. The hyperbolic quadrilateral.

lations G_τ determines a collection of parallelogram tiles, $P_{m,n} = g_{m,n}(P_\tau)$, that do not overlap and fill out all of **C**. Again G_τ is a discrete group of conformal self-maps of **C**.

We can apply this idea of "tiling" to obtain techniques that work not only on tori but also on more complicated Riemann surfaces. We cut the surface up to obtain a piece P of complex plane; we then try to find a group G of conformal maps to obtain a collection $\{gP\}$ of images of P that fill up some simply connected domain Ω in the plane without overlap; each element of G should be a conformal self-map of Ω.

I illustrate again with a simple example. Start with a torus, and take a pair of simple closed curves on it intersecting in the point η. Now remove the point η to obtain a *punctured* torus. Cutting along the curves gives us a quadrilateral with its corners removed. As we try to make our tiling, we see that because Ω is simply connected, the removed corners will have to be on the boundary of the domain Ω we are tiling. It follows that Ω will have to have at least four boundary points, and hence, by the Riemann mapping theorem, cannot be complex analytically equivalent to either **C** or $\mathbf{C}^*$.

Let us get very specific. Suppose that P (see Fig. 1) is the region inside the upper half-plane U bounded by the semicircles

$$C_1 = \{|z + 1/2| = 1/2\} \cap U \text{ and}$$
$$C_2 = \{|z - 1/2| = 1/2\} \cap U$$

and by the semi-infinite vertical lines

$$I_1 = \{\Re z = 1\} \cap U \quad \text{and} \quad I_2 = \{\Re z = -1\} \cap U.$$

Now consider the linear fractional transformations

$$g(z) = \frac{2z+1}{z+1}, \qquad h(z) = \frac{z+1}{z+2},$$

and let $G = \langle g, h \rangle$ be the group they generate.

We easily compute

$$g(-1) = \infty, \quad g(0) = 1 \quad \text{and} \quad g((-1+i)/2) = 1 + i,$$

so g maps the semicircle C_1 onto the vertical line I_1. Moreover, it maps P onto a quadrilateral gP adjacent to P along I_1. It does not overlap P, and its vertices are again on **R**. Similarly we see that h maps the vertical line I_2 to the semicircle C_2; and that hP is a quadrilateral with vertices on **R**, not overlapping P, and adjacent to P along the semicircle. The group G is a discrete group of conformal self-maps of U, and because P has zero angles, one may show that the images of P under G do, in fact, tile U.

This example gives us a once-punctured torus, and it has a complex structure inherited from U. Now we can puncture any torus (the torus being homogeneous, it doesn't matter *where* we puncture it), so there is again a whole family of possible complex structures for the punctured torus. How can we introduce parameters into the group we just constructed to vary it and obtain these other punctured tori?

In the late nineteenth century, Poincaré [1] discovered a technique which he, Fricke [2], and others used on this problem. It was used again by a number of people in the mid-1960s, including Ahlfors [3], Bers [4], Fenchel [5], Maskit [6], and the author [7–9]; in the 1970s, it was enlarged and developed further by Thurston, Sullivan, and Gromov [10]. What Poincaré remarked on was that the group of linear transformations

$$(az+b)/(cz+d), \quad a, b, c, d \in \mathbf{R},\ ad - bc > 0,$$

are not only conformal homeomorphisms of U but are also isometries with respect to the *hyperbolic metric* on U.

The hyperbolic metric is defined by $ds = |dz|/\Im z$. Geodesics are circles orthogonal to the real axis (and vertical lines). The distance from any point inside U to a point on $\mathbf{R} \cup \{\infty\}$ is infinite.

In our second example, where we tile the plane by parallelograms, we may convince ourselves that we can choose the basic parallelogram in any shape by choosing the lengths of the sides and the angle between them. These lengths and the angle determine generating translations for the group. Because rescaling doesn't change the complex structure of the quotient, we may always assume one of the lengths is 1. Then, as the angles of a parallelogram add up to 2π, four copies fit around each corner, and we can tile the plane.

In our punctured torus example, the quadrilateral P is bounded by hyperbolic geodesics, but they have infinite length. Moreover, they meet at 0 angles at the boundary. Are there hyperbolic geometric invariants sitting inside P somewhere? Does it have a "hyperbolic shape"? The answer is yes!

The hyperbolic isometry $g(z)$ fixes exactly two points on $\mathbf{R}$ and leaves the hyperbolic geodesic A_g joining them invariant. This geodesic is called the *axis* of g. Unlike the Euclidean case, the hyperbolic distance between z and $g(z)$, $d_U(z, g(z))$, is not the same for all $z \in U$. This distance is minimal for any $z \in A_g$; the minimum distance l_g is called the *translation length* of g. Similarly the isometry h has an axis A_h and a translation length l_h; one sees that A_g and A_h intersect in exactly one point.

Suppose now that we try to construct an arbitrary hyperbolic quadrilateral P with four infinite sides, meeting in vertices on the real axis, and such that there are hyperbolic isometries g and h identifying the pairs of opposite sides. It is a theorem, certainly known to Fricke and Fenchel, but first published by the author [8], that

- the "shape" of such a hyperbolic P is determined by the translation length of either isometry, l_g or l_h, and the angle θ between the axes A_g and A_h, and
- there is a P and a group for any given shape.

Only one length is necessary in this case because there are no isometries that change scale.

In sum, we have constructed a simply connected covering of the moduli space of a punctured torus parametrized by two real variables, $\{(l_g, \theta) \in \mathbf{R}^+ \times (0, \pi)\}$. These parameters have a geometric interpretation on the surface. There is also a simple way to write these parameters as real analytic functions of the coefficients of the generators of the group.

An important point here is that the methods of Examples 1 and 2 do *not* generalize to surfaces of higher genus, but these methods *do*.

Complex Moduli Spaces

The parameters for conformal structures on Riemann surfaces that we found above using hyperbolic geometric methods have many desirable properties. They are intrinsically defined; we can explicitly compute the polygonal tile and, hence, the group they determine; they work for arbitrary Riemann surfaces.

In the first two examples, we see how the complex structure on the torus depends on the parameter as a complex variable, so these parameter spaces have a rich structure. The methods, however, depend on elliptic function theory and work only for tori. In the third example, where the methods do generalize, the complex structure on the punctured torus depends on the parameters as independent real variables, so the parameter space has less structure. Ideally one would like to find a method for constructing parameter spaces for general Riemann surfaces so that the parameters are complex and the dependence of the geometry of the Riemann surface on these complex variables can be understood.

The Punctured Torus Revisited Here is another complex representation of the moduli space of a punctured torus that will generalize.

For $\mu \in U$, consider the group $G_\mu = \langle g, h_\mu \rangle$ where

$$g(z) = z + 2, \qquad h_\mu(z) = \frac{1}{z} + \mu.$$

Using techniques originated by Maskit (e.g., [6], VII), one can show that for appropriately chosen μ, there is a simply connected domain $\Omega(G_\mu)$ such that the group G_μ is a discrete group of conformal automorphisms and $\Omega(G_\mu)/G_\mu$ is a punctured torus.

To get an indication of how this works, choose $\mu = 3i$ and let P be the region (see Fig. 2)

- between the vertical lines $\Re z = -1$, $\Re z = 1$ and the circles $|z| = 1$ and $|z - 3i| = 1$,
- with vertices $-1, 1, 1 + 3i$ and $-1 + 3i$.

The map $g(z)$ takes the left side of P to the right side and maps P to a translate adjacent along the right side. The map $h(z)$ takes the lower semicircular boundary onto the upper one and maps P to a quadrilateral adjacent along the upper semicircle. This is the start of our tiling. It is not obvious, but it follows from Maskit's theory that the images of P under G_{3i} do not overlap. As we generate these images of P, they fill out some domain $\Omega(G_{3i})$ in $\mathbf{C}$, which, by construction, is invariant under G_{3i}.

Unlike Example 3, where the images of the tile P filled out the recognizable upper half-plane, the domain $\Omega(G_{3i})$ is not easily described; in fact, $\Omega(G_\mu)$ is different for different choices of μ. To get some idea of what the domains $\Omega(G_\mu)$ can look like, in Figures 3, 4, and 5 I show the computer pictures made by Ian Redfern at Warwick University for the groups G_μ with $\mu = 3i$, $\mu = 0.0533 + 1.9i$, and $\mu = 0.5001 + 1.667i$. The domain $\Omega(G_\mu)$ is the complement of the closed circles; its boundary is quite intricate.

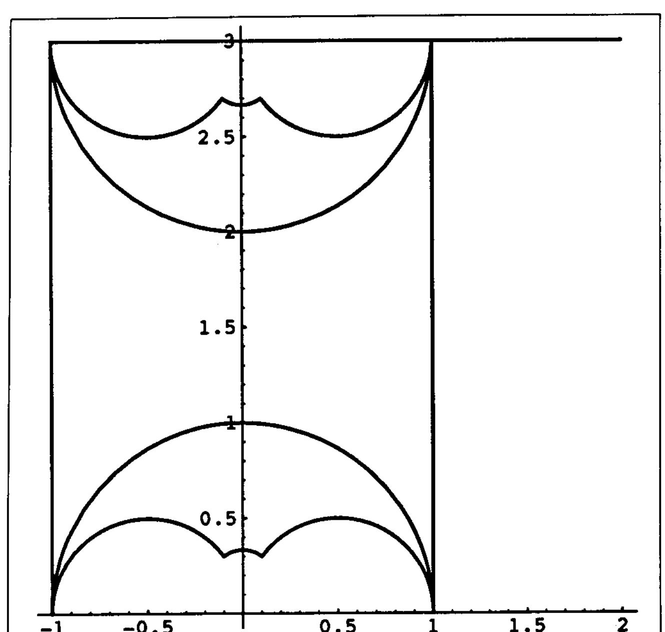

Figure 2. The tile P for G_{3i}.

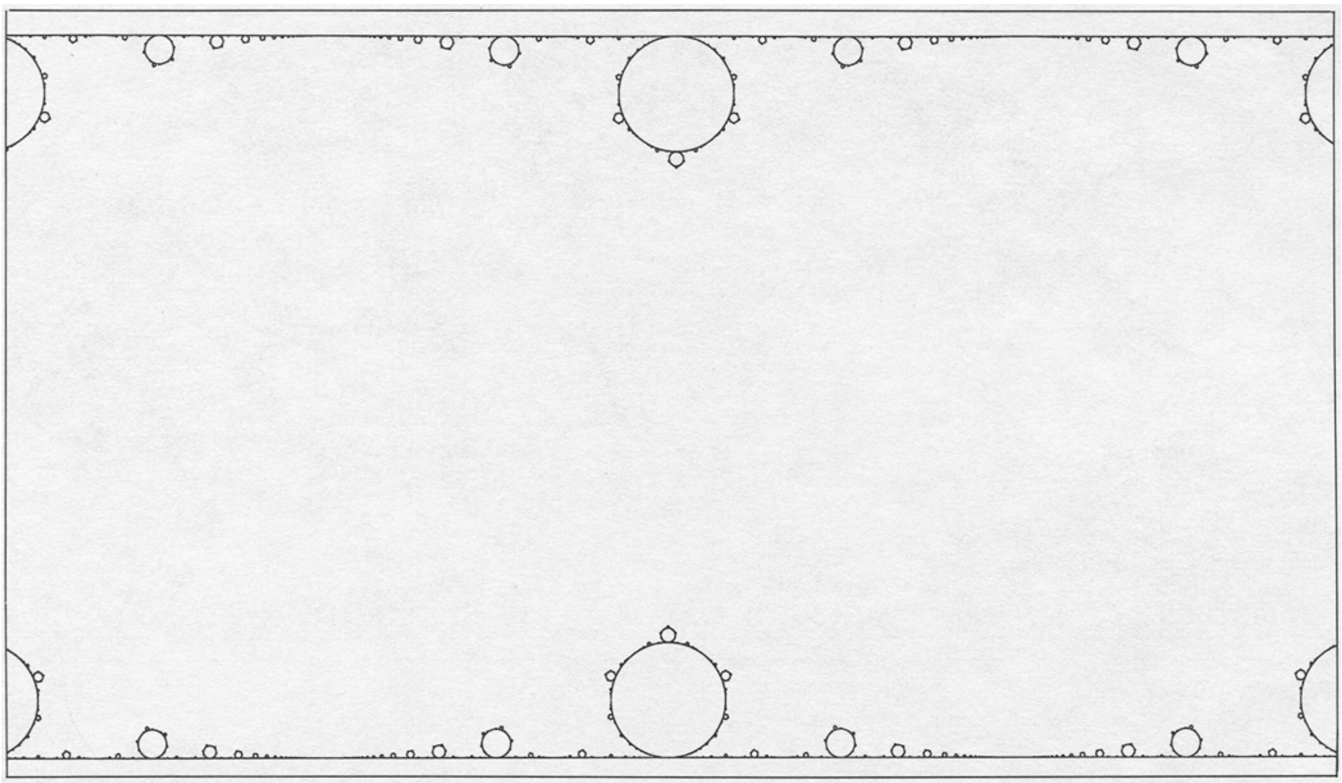

Figure 3. $\Omega(G_{3i})$.

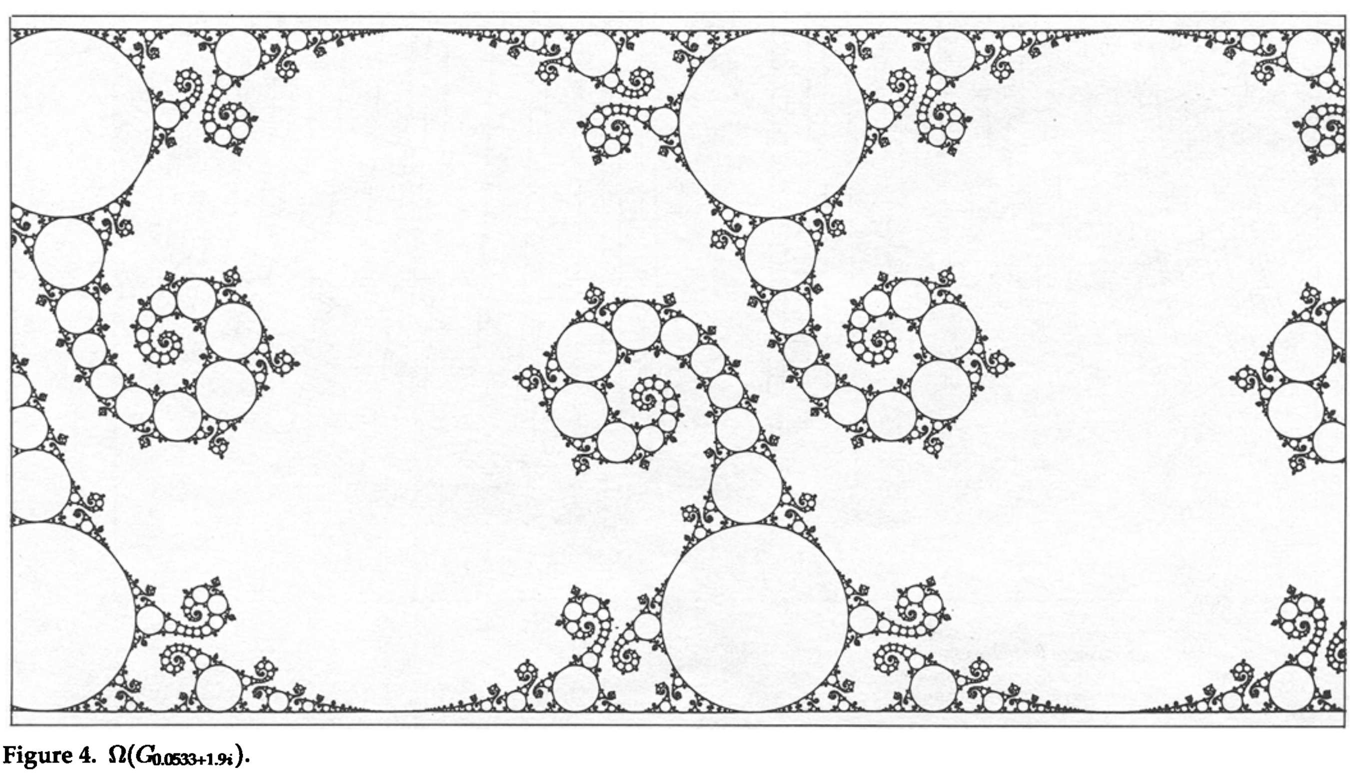

Figure 4. $\Omega(G_{0.0533+1.9i})$.

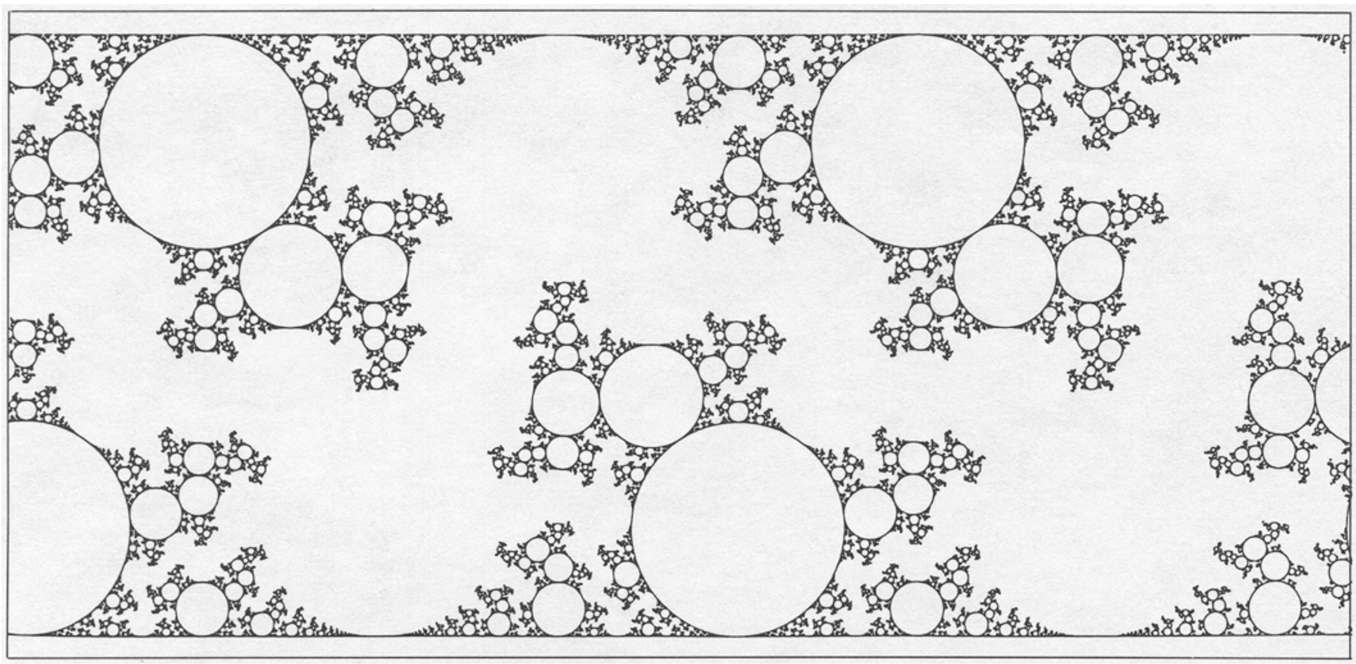

Figure 5. $\Omega(G_{0.5001+1.667i})$.

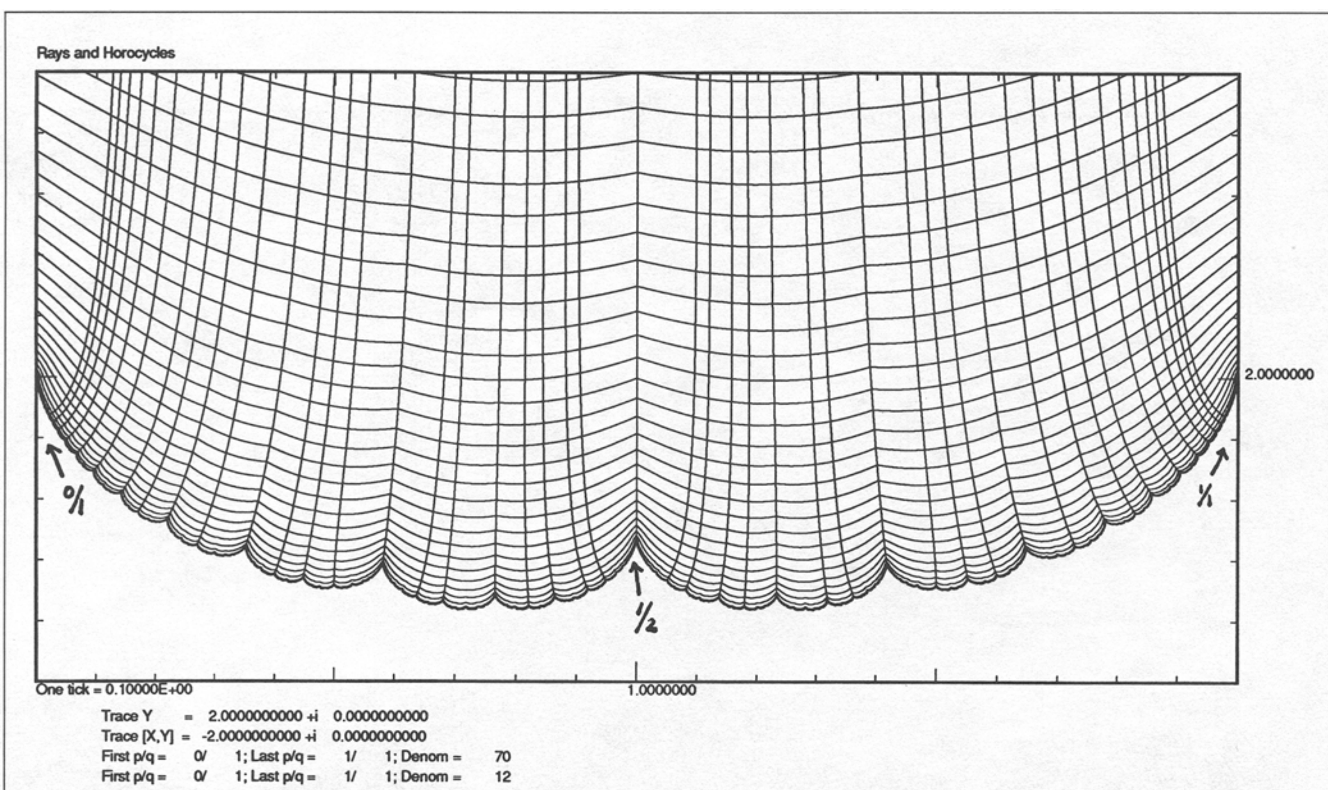

Figure 6. The Maskit embedding with pleating coordinates.

The Maskit Parameter Space. To show that the specific group G_{3i} gives us another way to represent a particular punctured torus by forming the quotient $\Omega(G_{3i})/G_{3i}$ requires Maskit's combinatorial theory for groups that represent Riemann surfaces. To show that every punctured torus can be represented by some group in the family $\{G_\mu\}$ takes a different set of techniques, from partial differential equations and the theory of quasi-conformal mappings developed by Ahlfors and Bers in the early 1960s (see, e.g., [3, 4, 11]). I will not explain these theories, but just report that for the punctured torus they tell us that:

- there is a simply connected domain $\mathcal{M} \subseteq U$ such that for $\mu \in \mathcal{M}$ there is a domain $\Omega(G_\mu)$ on which the group G_μ acts as a group of conformal homeomorphisms and such that $\Omega(G_\mu)/G_\mu$ represents a punctured torus, and
- every punctured torus is represented by some $\mu \in \mathcal{M}$.

In the first example, the cyclic group depending on a parameter $\lambda \in D^*$, we saw that the boundary of the disk was a natural boundary because the tori had degenerated. Similarly, the Maskit parameter space $\mathcal{M}$ is embedded as a domain inside the upper half-plane and the tori must degenerate as we approach its boundary. How can we find or describe this boundary? Using Maskit's techniques one can prove that the half-plane $\Im\mu > 2$ is contained in $\mathcal{M}$. Therefore, the boundary $\partial\mathcal{M}$ in $\mathbf{C}$ is in the horizontal strip $0 < \Im\mu \leq 2$. Wright [12] used experimental techniques to compute this boundary and came up with the picture in Figure 6. This picture was the jumping-off point for the author's ongoing collaboration with Caroline Series on complex moduli spaces [13–17].

Hyperbolic Geometry Again — This Time in Three Dimensions When Poincaré realized that linear fractional transformations with real coefficients were isometries of the hyperbolic plane, he also realized that linear fractional transformations with complex coefficients were isometries of hyperbolic 3-space. Hyperbolic 3-space can be modeled by the upper half-space $\mathbf{H}^3 = \{(z,t) : z \in \mathbf{C},\ t \in \mathbf{R}^+\}$. The hyperbolic metric there is given by $ds = \sqrt{|dz|^2 + dt^2}/t$. Geodesics are circles orthogonal to the base $\mathbf{C}$, and hyperbolic planes are hemispheres orthogonal to the base $\mathbf{C}$.

Linear fractional transformations map circles and straight lines in $\mathbf{C}$ onto circles and straight lines; in fact, each is a product of an even number of reflections in lines and inversions in circles. Given a circle in $\mathbf{C}$, we can view it as the equator of a sphere in $\mathbf{R}^3$. An inversion in the circle extends naturally to an inversion in the sphere. Similarly, reflections in lines in $\mathbf{C}$ extend to reflections in the planes through them orthogonal to $\mathbf{C}$. The isometry of $\mathbf{H}^3$ corresponding to a linear transformation is a product of these extended inversions and reflections. One checks that since there are an even number, the upper half-space is preserved. It is an exercise to check that the metric is preserved.

Fenchel [5] and Greenberg [18], in the early 1960s, began to use techniques of 3-dimensional hyperbolic geometry to study groups representing Riemann surfaces. The idea was that because the group was discrete, one could look at the quotient 3-manifold, $\mathbf{H}^3/G$. It is a manifold with boundary, and the Riemann surfaces represented by the group are the boundary components.

Let us see how this works in our third example, the group $G = \langle g, h\rangle$. The action of G on U can be extended to $\mathbf{H}^3$ and to the lower half-plane L. Reflect the circles C_i and the lines I_i, $i = 1, 2$, in the real axis. These reflections determine a region $\bar{P}$ in L, and L/G is again a punctured torus. Note that the surfaces U/G and L/G are *anti*holomorphically equivalent, for the maps on local neighborhoods are given by complex conjugation. The hemispheres over the circles C_i and the vertical half-planes over the lines I_i, $i = 1, 2$, bound a region $R \subset \mathbf{H}^3$; and g identifies the hemisphere over C_1 with the plane over I_1, while h identifies the plane over I_2 with the hemisphere over C_2. The polyhedron R is a tile for the group G acting on $\mathbf{H}^3$. The quotient $(U \cup L \cup \mathbf{H}^3)/G$ is a 3-manifold whose boundary consists of a pair of antiholomorphically equivalent punctured tori. As we shall see, not all groups of linear fractional transformations acting on $\mathbf{H}^3$ act so symmetrically with respect to the real line, nor are the relations among their boundary surfaces so easy to determine.

In the early 1970s, Marden [19] studied the relationship between groups of linear fractional transformations acting on $\mathbf{H}^3$ and the topological properties of their quotient 3-manifolds, and in the late 1970s, Thurston [10] introduced revolutionary new techniques involving this hyperbolic geometry to attack classification problems for both Riemann surfaces and 3-manifolds.

Convex Hulls and Pleated Surfaces Let us return to the family of groups $\{G_\mu\}$ for $\mu \in \mathcal{M}$. For each group we have an open plane domain $\Omega(G_\mu)$ invariant under G_μ. The boundary, $\Lambda(G_\mu) = \partial\Omega(G_\mu)$, is a closed G_μ-invariant set called the *limit set of* G_μ. Let us turn our attention to it.

One of Thurston's ideas was to consider the (hyperbolic) *convex hull* $\mathcal{C}$ in $\mathbf{H}^3$ of the set $\Lambda(G_\mu)$ and to study its boundary. This boundary is also G_μ-invariant, and one can prove that there is a G_μ-invariant component of this boundary, $\partial\mathcal{C}(G_\mu)$, that is homeomorphic to $\Omega(G_\mu)$. The quotient, $\partial\mathcal{C}(G_\mu)/G_\mu$, is, therefore, again a punctured torus.

Thurston saw that $\partial\mathcal{C}$ had certain geometric properties that were very useful. It is a surface in $\mathbf{H}^3$ made of pieces of hyperbolic plane joined along geodesic curves that, because of the convexity, can only meet on $\hat{\mathbf{C}} = \partial\mathbf{H}^3$ in points of $\Lambda(G_\mu)$. The quotient surface $S_\mu = \partial\mathcal{C}(G_\mu)/G_\mu$ is, therefore, also made up of pieces of hyperbolic plane joined along nonintersecting geodesics. Thurston called such surfaces *pleated*, and the geodesics along which they are pleated, the *pleating locus*.

Before we see how to extract information about the groups G_μ from these ideas, let us see why they don't give any new information about the groups of Example 3. There, the set Ω is always U and the limit set Λ is the real line. The convex hull of Λ is the vertical plane above Λ and so is equal to its boundary. There is only one hyperbolic plane and so no geodesic where two planes are joined; the pleating locus is, therefore, empty.

Figure 4 is a picture of $\Lambda(G_{0.0533+1.9i})$. This limit set is very intricate and its convex hull $\mathcal{C}$ has interior. To get a sense of what $\mathcal{C}$ looks like, note that by definition the hyperbolic geodesic joining any pair of points in Λ belongs to $\mathcal{C}$, as does the hyperbolic triangle spanned by any three points in Λ. If four or more points of Λ lie in a circle C, the convex polygon they span is in a plane and in $\mathcal{C}$. If, moreover, there are no points of Λ inside C, then the hyperbolic plane spanned by C intersects $\partial\mathcal{C}$ in a hyperbolic convex polygon.

If we look carefully at Figure 3, 4, or 5, we see a pattern of closed circles and other overlapping circles with missing boundary arcs. The interiors of these circles contain no points in Λ. The boundary of the convex hull in $\mathbf{H}^3$ consists of the intersection of the hyperbolic planes spanned by all the circles that we see. The full planes spanned by the closed circles belong to $\partial\mathcal{C}$. Over the other circles, the piece of the plane belonging to $\partial\mathcal{C}$ is an infinite-sided convex polygon. When a pair of circles intersect, the planes spanning them intersect in a circular arc that is a boundary curve of the polygon on each plane. It is a geodesic with its endpoints in the limit set; $\partial\mathcal{C}$ is "bent" along this geodesic at an angle equal to the angle between the circles. The set of geodesics formed by the intersecting planes is the pleating locus.

In computer pictures for various groups $G_\mu \in \mathcal{M}$ made by Wright and Redfern, and particularly for those groups near the boundary, one could see patterns of circles in the limit sets $\Lambda(G_\mu)$, and we thought that there should be meaning to the patterns. For example, note that the patterns in Figures 3, 4, and 5 are decidedly different. What Series and I realized is that whenever a pattern of circles appears in $\Lambda(G_\mu)$, the quotient S_μ is pleated along some simple closed curve and the curve is determined by the pattern!

Enumerating Simple Closed Curves on the Punctured Torus Consider the unpunctured torus with simple closed geodesics α and β intersecting once. We may assume α is the projection of the line in $\mathbf{C}$ joining 0 and 1, and β is the projection of the line joining 0 and τ. Every simple closed geodesic on the torus then has the form $p\alpha + q\beta$ and is the projection of a line joining 0 to $p + q\tau$ for relatively prime integers p and q. For each such pair, (p, q), we have a family of parallel lines projecting onto a family of parallel geodesics.

The fundamental group of the punctured torus, $\pi_1(S)$, is also generated by a pair of simple closed geodesics intersecting once, but it is a free group not an abelian group. We know from Maskit's theory that for $\mu \in \mathcal{M}$, the domain $\Omega(G_\mu)$ is simply connected; it follows that G_μ is isomorphic to $\pi_1(S)$. The "forgetful map" from S into the unpunctured torus, defined by forgetting the puncture, shows that each simple closed curve on S is also a simple closed curve on the unpunctured torus. It induces a projection on fundamental groups

$$\pi_1(S) \mapsto \mathbf{Z} + \mathbf{Z},$$

from which we see that there are many elements in $\pi_1(S)$ that project to $(p\alpha + q\beta)$.

It is an interesting fact, proved by Series [20], that there is a unique simple closed geodesic $\gamma_{p/q}$ in the inverse image of $(p\alpha + q\beta)$. Moreover, there is a unique conjugacy class in G_μ containing a shortest cyclically reduced representative for that geodesic. Hence, there is a canonical word $W_{p/q}$ in G_μ associated to each simple closed geodesic on the punctured torus. These words may be enumerated recursively using continued fractions.

Pleating Curves and Moduli Given a linear fractional transformation, $(az + b)/(cz + d)$, we may assume without loss of generality that $ad - bc = 1$. The *trace* of the transformation, $a + d$, is then well defined, and conjugate transformations have the same trace.

If we look at words in G_μ, they are compositions of the maps g and h_μ, so their coefficients and their traces are polynomials in μ with integral coefficients.

The crucial observation [13] that relates the complex parameter μ to the geometry of the hyperbolic 3-manifold $\mathbf{H}^3/G_\mu$ is

THEOREM 1. *Whenever the quotient of the convex hull boundary S_μ is pleated along the curve $\gamma_{p/q}$, the trace polynomial* $\operatorname{Tr} W_{p/q}(\mu)$ *is real-valued.*

We also prove

THEOREM 2. *For any pair of relatively prime integers, (p, q), there is some $\mu \in \mathcal{M}$ such that S_μ is pleated along $\gamma_{p/q}$.*

These theorems together give this picture of the parameter space:

THEOREM 3. *The space $\mathcal{M}$ is foliated by real analytic curves $\mathcal{P}_r$, $r \in \mathbf{R}$, a dense subfamily of which is defined by properly chosen branches of the curves defined by*

$$\Im \operatorname{Tr} W_{p/q}(\mu) = 0$$

(the "vertical" curves in Fig. 6). These curves meet the boundary of $\mathcal{M}$ at points where the torus has degenerated because the curve $\gamma_{p/q}$ has been pinched.

The final piece of the relationship between the complex and geometric parameters is given by

THEOREM 4. *There is a family, $\{l_r(\mu)\}_{r\in\mathbf{R}}$, of analytic maps from $\mathcal{M}$ to $\mathbf{C}$ that vary continuously with r. The value $l_r(\mu)$, for r rational and $\mu \in \mathcal{P}_r$, equals the appropriately normalized length of the pleating locus. The pairs $(r, l_r(\mu))$ define a new set of coordinates for $\mathcal{M}$.*

The level curves $l_r(\mu)$ = const are the "horizontal" curves in Figure 6.

We have generalized these techniques to *twice-punctured* tori and expect them to generalize to arbitrary surfaces [15, 17].

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