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ON THE QUASICONFORMAL SURGERY OF RATIONAL FUNCTIONS

BY MITSUHIRO SHISHIKURA

ABSTRACT. — The method of quasiconformal surgery for rational functions, considered as complex analytic dynamical systems, is developed. This applied to give the sharpest estimates for the numbers of cycles of stable regions corresponding to D. Sullivan's classification. As another application, rational functions having Herman rings are constructed.

Introduction

Consider a complex analytic dynamical system on the Riemann sphere, which is defined by a rational function of degree greater than one. Each connected component of the complement of its Julia set is called a *stable region*. D. Sullivan [17] proved that every stable region is eventually cyclic, and that cyclic stable regions can be classified into five types—attractive basin, superattractive basin, parabolic basin, Siegel disk and Herman ring.

One of the aims of this paper is to give the sharpest estimates for the numbers of such cycles (Corollary 2, Theorem 3 and 4). As a consequence, we shall show that a rational function of degree d cannot have more than $2(d-1)$ cycles of stable regions. This answers a question in [17]. Moreover, it has at most $d-2$ Herman rings, hence if $d=2$, there is no Herman ring.

These results are obtained by means of surgeries based on the theory of quasiconformal mappings, which we call the *quasiconformal surgeries* (or *qc-surgeries*). Such surgery technique was first introduced by A. Douady and J. H. Hubbard for polynomial-like mappings (*see* [7] and [8]). We will formulate it and apply it in several cases. In this paper, we treat mainly three kinds of surgeries:

(1) To perturb a rational function so that all of its indifferent periodic points become attractive (Theorem 1). (Such a perturbation was expected by P. Fatou [9] in 1920);

(2) To decompose a rational function which has Herman rings into ones having Siegel disks;

(3) To construct a rational function having Hermann rings from ones having Siegel disks. [This is the counter procedure of (2).]

These three are combined to prove the estimates. Also, the third yields, for any p, a rational function of degree 3 with Herman rings of order p (Theorem 5).

In paragraph 1, we review the theory of complex analytic dynamical systems on the Rieman sphere, and prepare some terms and notations. Main theorems are stated precisely in paragraph 2. In paragraph 3, we provide our fundamental lemma for qc-surgery, which is applied in following sections.

The surgery (1) and its applications (Theorem 1 and Proposition 1) are given in paragraphs 4 and 5. The surgery (2), together with the dispositions of Herman rings and its inverse images, is discussed in paragraph 6. Combining these results, we give the estimates (Theorem 2 and Theorem 3) in paragraphs 7 and 8. In paragraph 9, we demonstrate the surgery (3) and prove Theorem 5 and 6. Finally, in paragraph 10, we show that our estimates are optimum, by constructing examples.

This paper is based on the author's Master's Thesis (in Japanese, 1985) at Kyoto University. *See* also [16].

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1. Preliminaries

1.0. Let $f(z)$ be a rational function of z with complex coefficients. We consider the dynamical system $f: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, where $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. The *degree* of f, $\deg f$, is the maximum of degrees of its denominator and of its numerator, provided they are relatively prime. Assume $d = \deg f \geqq 2$. We write $f^n = \underbrace{f \circ f \circ \ldots \circ f}_{n}$ (n-th iteration).

1.1. Let $z \in \bar{\mathbb{C}}$ be a periodic point of f of period p, i.e. $f^p(z) = z$ and $f^j(z) \neq z$ $(0 < j < p)$. The *multiplicator* of z is

$$\lambda = \begin{cases} (f^p)'(z) & (\text{if } z \neq \infty) \\ (A \circ f^p \circ A)'(0) & (\text{if } z = \infty, \text{ where } A(z) = 1/z). \end{cases}$$

We say that z is *attractive* (resp. *indifferent*, *repulsive*, *non-repulsive*), if $|\lambda| < 1$ (resp. $= 1$, > 1, $\leqq 1$). Moreover z is *rationally indifferent* (resp. *irrationally indifferent*), if $\lambda = e^{2\pi i \theta}$ where $\theta \in \mathbb{R}$ is rational (resp. irrational). (*See* 1.5 for further definitions.)

We call $\{z, f(z), \ldots, f^{p-1}(z)\}$ a *cycle*, and use the terms attractive, indifferent, etc. also for cycles.

1.2. A point z is called a *critical point* of f, if f is not one to one on any neighborhood of z. If $z \neq \infty$ and $f(z) \neq \infty$, it is equivalent to $f'(z) = 0$. A rational function of degree d has $2(d-1)$ critical points (counted with multiplicities).

1.3. A point $z \in \bar{\mathbb{C}}$ is *normal* (with respect to f), if $\{f^n : n \geqq 0\}$ is equicontinuous on some neighborhood of z. The set of all normal points is called the *stable set* of f, denoted by D_f, and each of its connected components a *stable region*. The complement $J_f = \bar{\mathbb{C}} - D_f$ is the *Julia set* of f. These sets have the following properties (*see* [3], [4], [9] and [12]):

(*a*) Both J_f and D_f are completely invariant, i. e. $f(J_f) = J_f = f^{-1}(J_f)$, etc.

(*b*) The Julia set coincides with the closure of the set of repulsive periodic points.

(*c*) Each stable region is mapped by f onto some stable region.

We say that a stable region D is *cyclic*, if $f^p(D) = D$ for some $p \geqq 1$. The least such p is called the *order* of D.

1.4. THEOREM (D. Sullivan [17]). — Each stable region is eventually cyclic, i. e. if D is a stable region of f, $f^N(D)$ is cyclic for some $N \geqq 0$.

Moreover, let D be a cyclic stable region of order p, then $(D, f^p|_D)$ is of one of the following types:

(AB) *attractive basin:* there exists an attractive periodic point z_0 of period p in D. When $n \to \infty$, $f^{np}(z) \to z_0$ uniformly on every compact set in D.

(PB) *parabolic basin:* there exists a rationally indifferent periodic point z_0 on the boundary ∂D, such that $f^p(z_0) = z_0$, $(f^p)'(z_0) = 1$. When $n \to \infty$, $f^{np}(z) \to z_0$ uniformly on every compact set in D.

(SD) *Siegel disk:* $f^p|_D$ is conformally conjugate to an irrational rotation on the unit disk $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, i. e. there exist a conformal mapping $\varphi : D \to \Delta$ and an irrational number θ such that $\varphi \circ f^p = e^{2\pi i \theta} . \varphi$ on D. We call θ (mod 1) the *rotation number* and $z_0 = \varphi^{-1}(0)$ the *center*.

(HR) *Herman ring:* $f^p|_D$ is conformally conjugate to an irrational rotation on an annulus $\{\zeta : r < |\zeta| < 1\}$, for some $0 < r < 1$. The rotation number is defined as in (SD), in addition, up to sign.

Remark. — In (AB), if the multiplicator of z_0 is 0, D is called a *superattractive basin*, (SAB). We have (SAB) included into (AB), although D. Sullivan did not.

If D is attractive basin (resp. a parabolic basin, etc.), we call D, $f(D), \ldots, f^{p-1}(D)$ an *AB-cycle* (resp. a *PB-cycle*, etc.).

1.5. RELATION TO PERIODIC POINTS. — In each of (AB), (PB) and (SD), there is an associated periodic point z_0, which is attractive, rationally indifferent, irrationally indifferent, respectively. Conversely if z_0 is an attractive periodic point (resp. a rationally indifferent periodic point), there is an AB-cycle (resp. a finite number of PB-cycles) which has z_0 as the limit point. However, if z_0 is an irrationally indifferent periodic point, there is not always an SD-cycle containing z_0.

For example, let θ be an irrational number satisfying the following Diophantine condition:

there exist positive constants C and σ such that

$$|\theta - p/q| > C/q^{\sigma}, \qquad \text{for} \quad p, q \in \mathbb{Z}, \quad q \geqq 1. \tag{1.1}$$

If f has a periodic point z with multiplicator $e^{2\pi i \theta}$, then f has a Siegel disk whose center is z (*cf.* Siegel [15]).

Irrational numbers satisfying the Diophantine condition form a full-measure set of $\mathbb{R}$.

On the contrary, there is a dense set of irrational numbers such that if θ belongs to it, a periodic point with multiplicator $e^{2\pi i\theta}$ cannot be the center of a Siegel disk, for any rational function (*cf.* Cremer [6], and also [3]). Let us call an irrationally indifferent periodic point a *Siegel point* if it is the center of a Siegel disk, and a *Cremer point* otherwise. Siegel-cycles and Cremer-cycles are similarly defined.

Note that Herman rings have nothing to do with periodic points.

1.6. RELATION TO CRITICAL POINTS. — It is classically known that each AB-cycle or PB-cycle contains at least one critical point (*see* [9], [17] and also Lemma 5). Hence the number of AB-cycles and PB-cycles is at most $2(d-1)$.

It is also known [9] that the boundary of a Siegel disk or a Herman ring is contained in the closure of the forward orbits of critical points. Moreover it is conjectured that every SD-cycle has at least one critical point on its boundary (and as for an HR-cycle at least two corresponding to its boundary components) (*see* Herman [11]).

1.7. NOTATIONS. — Let D be a subset of $\bar{\mathbb{C}}$. Define:

$n_{\text{attr}}(f, \mathrm{D})=$ the number of attractive cycles of f, entirely contained in D.

If $\mathrm{D}=\bar{\mathbb{C}}$, we omit D. If there is no confusion about f, we omit f.

Similarly, define n_{indiff}, n_{rat}, n_{irr}, n_{Cremer}, n_{AB}, n_{PB}, n_{SD} and n_{HR} for indifferent cycles, rationally indifferent cycles, irrationally indifferent cycles, Cremer-cycles, AB-cycles, PB-cycles, SD-cycles and HR-cycles, respectively.

Also we define

$n_c(f, \mathrm{D})=$ the number of the critical points of f contained in D, where critical points are counted with multiplicities.

Remark. — The arguments from now on goes, even if critical points are counted without multiplicities.

1.8. Let γ be an oriented Jordan curve in $\bar{\mathbb{C}}$. Then $\bar{\mathbb{C}}-\gamma$ is divided into two connected components. We call the component which lies on the left-hand side of γ the *interior* of γ, denoted by Int γ, and the other the *exterior* of γ, denoted by Ext γ.

Let A be an annulus i.e. a doubly connected region. Fix an orientation of A, by choosing a generator of its fundamental group. We can define similarly its interior and exterior as the components of $\bar{\mathbb{C}}-\mathrm{A}$.

2. Main theorems

We prove the following theorems.

THEOREM 1. — *Let f be a rational function of degree d. Denote by $z_0, \ldots, z_{\mathrm{N}}$ all non-repulsive periodic points of f. There exist, for $0<\varepsilon<\varepsilon_0$, rational functions f_ε of degree d*

and points $z_0^\varepsilon, \ldots, z_N^\varepsilon$ *of* $\bar{\mathbb{C}}$ *such that:*

(i) *When* $\varepsilon \to 0$, $f_\varepsilon \to f$ *uniformly and* $z_i^\varepsilon \to z_i$ *with respect to the metric of* $\bar{\mathbb{C}}$;

(ii) *If* $f(z_i) = z_j$, $f_\varepsilon(z_i^\varepsilon) = z_j^\varepsilon$. *Each* z_i^ε *is an attractive periodic point of* f_ε *with the same period as* z_i.

Therefore,

$$n_{\text{attr}}(f_\varepsilon) \geqq n_{\text{attr}}(f) + n_{\text{indiff}}(f).$$

COROLLARY 1:

$$n_{\text{atrr}}(f) + n_{\text{indiff}}(f) \leqq 2(d-1). \tag{2.1}$$

Remark. — As mentioned in paragraphs 1.5 and 1.6, P. Fatou [9] proved that

$$n_{\text{attr}} + n_{\text{rat}} \leqq n_{\text{AB}} + n_{\text{PB}} \leqq n_c(\text{D}_f) \leqq 2(d-1).$$

After that, he surmised (2.1) (*see* [9, 2ᵉ mémoire], p. 66), but he succeeded only in showing that one can perturb f so that at least half of indifferent cycles become attractive. So it has been known that

$$n_{\text{attr}} + \frac{1}{2} n_{\text{indiff}} \leqq 2(d-1).$$

Concerning polynomials, A. Douady and J. H. Hubbard have obtained a result similar to Theorem 1 and the estimate

$$n_{\text{attr}}(p, \mathbb{C}) + n_{\text{indiff}}(p, \mathbb{C}) \leqq d-1$$

for a polynomial p of degree d, instead of Corollary 1. (*See* [3] and Example in paragraph 3, Remark in paragraph 4.) Their method, however, does not work for rational functions having no attractive cycle.

THEOREM 2. — *For any rational function f of degree d,*

$$n_c(\text{D}_f) + n_{\text{irr}} + 2\, n_{\text{HR}} \leqq 2(d-1). \tag{2.2}$$

As noted above, $n_{\text{AB}} + n_{\text{PB}} \leqq n_c(\text{D}_f)$, and by the definition, $n_{\text{irr}} = n_{\text{SD}} + n_{\text{Cremer}}$. Therefore, we have

COROLLARY 2:

$$n_{\text{AB}} + n_{\text{PB}} + n_{\text{SD}} + 2\, n_{\text{HR}} + n_{\text{Cremer}} \leqq 2(d-1). \tag{2.3}$$

Remark. — D. Sullivan [17] has already shown, combining diverse estimates, that $n_{\text{AB}} + n_{\text{PB}} + n_{\text{SD}} + n_{\text{HR}} \leqq 8(d-1)$. Then, he asked whether, in this estimate, $8(d-1)$ can be replaced by $2(d-1)$. Corollary 2 solves this problem affirmatively (or equally Problem 7.8 of [3]).

Remark. — If the conjecture in paragraph 1.6 was true, one could get directly (2.3) with n_{Cremer} omitted.

Theorem 2 implies $n_{HR} \leqq d-1$. But we know more precisely:

THEOREM 3:

$$n_{HR} \leqq d-2. \tag{2.4}$$

In particular, a rational function of degree 2 has no Herman ring.

Concerning the number of cycles of the respective types, the estimates (2.3) and (2.4) are best possible. In fact, we have:

THEOREM 4. — *Suppose that m_{AB}, m_{PB}, etc. and d are nonnegative integers satisfying* (2.3) *and* (2.4), *with n_{AB}, etc. replaced by m_{AB}, etc. Then there exists a rational function of degree d satisfying $m_{AB} = n_{AB}(f)$, etc.*

Remark. – *See* Herman [10], Question VII.7.(2). Theorem 3 and 4 give an answer to his question.

In the proof of Theorem 4, we construct a rational function which has Herman rings from those having Siegel disks, using the qc-surgery. The same technique applies to prove:

THEOREM 5. — *Let $p \geqq 1$. There exist rational functions f_A, f_B satisfying* (A) *and* (B), *respectively, where we give all the Herman rings suitable orientations, which are respected by f_A or f_B.*

(A) *f_A has a cycle of Herman rings $A_1, \ldots, A_p$ of order p such that $A_j \subset \operatorname{Ext} A_i$, for $i \neq j$.*

(B) *f_B has a cycle of Herman rings A_1 and A_2 of order 2 such that $A_2 \subset \operatorname{Ext} A_1$ and $A_1 \subset \operatorname{Int} A_2$.*

Moreover, f_A (resp. f_B) can be chosen to be of degree 3 (resp. degree 4).

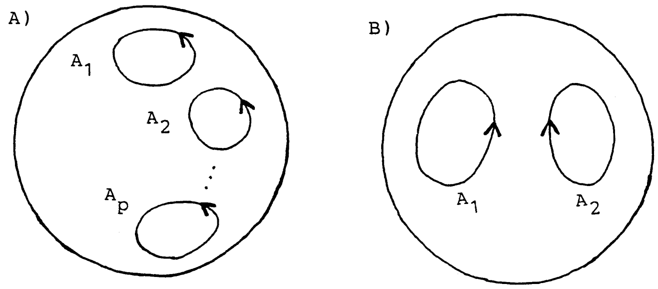

Fig. 1. — Herman rings A_i yielded by Theorem 5. The rings are indicated only by invariant curves in them. The arrows signify their orientations.

See Figure 1 for the disposition of A_i. A numerical experiment related to (A) with $p=2$ is reported in [16].

Remark. – M. R. Herman [10] also constructed a rational function with Herman rings by a different method, but without determining its degree which is probably higher. Our method enables us to interpret the dynamics of a function with Herman rings in terms of that of functions with Siegel disks. For example, we obtain:

THEOREM 6. — *Let θ be an irrational number. The following conditions are equivalent:*

(i) *there exists a rational function which has a Herman ring of rotation number* θ;

(ii) *there exists a rational function which has a Siegel disk of rotation number* θ.

3. Fundamental lemma for QC-surgery

The surgery means a method to create from given rational functions a new one preserving their dynamics (in some sense). Unfortunately, we cannot glue different analytic functions directly, because of the theorem of identity. However, if one abandons the analyticity, in other words, if one considers their conjugations by certain homeomorphisms, glueing can be possible. It comes into question, in turn, whether the resulting map is conjugate to a rational function. In order to reproduce a rational function, we make use of the theory of quasiconformal mappings.

DEFINITIONS. — Let Ω, Ω' be domains of $\mathbb{C}$. A homeomorphism $\varphi : \Omega \to \Omega'$ is a *quasiconformal mapping* (*qc-mapping*) if φ is absolutely continuous on almost all lines parallel to real-axis and almost all lines parallel to imaginary-axis, and if $|\mu_\varphi| \leqq k$ a. e. (almost everywhere with respect to the Lebesgue measure), for some $k<1$, where $\mu_\varphi = \varphi_{\bar{z}}/\varphi_z$. (*See* Ahlfors [1].) Quasiconformal mappings on Riemannian surfaces are defined by means of local charts.

A *quasi-regular mapping* is a composite of a qc-mapping and an analytic function. (*Cf.* [13] in which this is called a quasiconformal function.)

Here is our formulation of the qc-surgery.

LEMMA 1 (Fundamental lemma for qc-surgery). — *Let* $g : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ *be a quasi-regular mapping. Suppose that there are disjoint open sets* E_i *of* $\bar{\mathbb{C}}$, *qc-mappings* $\Phi_i : E_i \to E'_i \subset \bar{\mathbb{C}}$ $(i=1, \ldots, m)$ *and integer* $N \geqq 0$, *satisfying the following conditions:*

(i) $g(E) \subset E$, *where* $E = E_1 \cup \ldots \cup E_m$;

(ii) $\Phi \circ g \circ \Phi_i^{-1}$ *is analytic in* $E'_i = \Phi_i(E_i)$, *where* $\Phi : E \to \bar{\mathbb{C}}$ *is defined by* $\Phi|_{E_i} = \Phi_i$;

(iii) $g_{\bar{z}} = 0$ *a. e. on* $\bar{\mathbb{C}} - g^{-N}(E)$.

Then there exists a qc-mapping φ *of* $\bar{\mathbb{C}}$ *such that* $\varphi \circ g \circ \varphi^{-1}$ *is a rational function.*

Moreover, $\varphi \circ \Phi_i^{-1}$ *is conformal in* E'_i *and* $\varphi_{\bar{z}} = 0$ *a. e. on* $\bar{\mathbb{C}} - \bigcup_{n \geqq 0} g^{-n}(E)$.

Proof (*see* [7], [8]). — Define a measurable conformal structure σ on $\bar{\mathbb{C}}$ as follows. Let σ_0 be the conformal structure defined by $|dz|$. Set $\sigma = \Phi^* \sigma_0$ on E, where $\Phi^* \sigma_0$ means the pull-back of σ_0 by Φ, defined except on a null set. By (ii), $\sigma|_E$ is invariant for g, in the sense that $g^* \sigma = \sigma$ a. e. on E. Pulling back σ by g, define σ on $\bigcup_{n \geqq 0} g^{-n}(E)$.

Finally, set $\sigma = \sigma_0$ on the remaining part of $\bar{\mathbb{C}}$.

The g-invariance (a. e.) of σ with respect to g follows from the definition and (iii). Moreover, the distortion of σ with respect to σ_0 is uniformly bounded. In fact, if Φ is K_1-qc and g is K_2-quasi-regular, and if σ is represented as $|dz+\mu.d\bar{z}|$, then $\|\mu\|_\infty \leqq k=(K-1)/(K+1)$ a. e., where $K=K_1.K_2^N$. By the measurable mapping theorem (*cf.* [1]), there exists a K-qc-mapping φ of $\bar{\mathbb{C}}$ such that $\varphi^*\sigma_0=\sigma$ a. e. Then, $f=\varphi\circ g\circ\varphi^{-1}$ respects a. e. the standard conformal structure σ_0. Hence f is locally 1-qc, i. e. conformal, except at a finite number of its critical points. By the removable singularity theorem, f is analytic on $\bar{\mathbb{C}}$, therefore, a rational function. □

This lemma means glueing of $g|_{\bar{\mathbb{C}}-E}$ and $\Phi\circ g\circ\Phi_i^{-1}$. Note that, to get a qc-mapping of $\bar{\mathbb{C}}$, it is enough to construct a C^1-diffeomorphism of $\bar{\mathbb{C}}$. This makes our surgeries easier.

Example. — We exercise this surgery technique here for Douady-Hubbard's polynomial like mappings. (*See* Douady [7] and Douady-Hubbard [8].)

Let U_1, U_2 be simply connected domains in $\mathbb{C}$, whose boundaries consist of analytic Jordan curves, and satisfying $\bar{U}_1\subset U_2$. Suppose that $f: U_1\to U_2$ is holomorphic, proper of degree d and then extends continuously to ∂U_1. Then $(U_1, U_2; f)$ is called a *polynomial-like function.*

Fix $R>1$, and construct a qc-mapping

$$\Phi:\quad \bar{\mathbb{C}}-\bar{U}_1\to\{z\in\bar{\mathbb{C}}: |z|>R\}$$

such that:

$\Phi(\infty)=\infty$; Φ is conformal in $\bar{\mathbb{C}}-\bar{U}_2$;

Φ extends to ∂U_1, and satisfies $(\Phi(z))^d=\Phi(f(z))$ on ∂U_1.

Define

$$g=\begin{cases} f & \text{on } U_1 \\ \Phi^{-1}(\{\Phi(z)\}^d) & \text{on } \bar{\mathbb{C}}-U_1. \end{cases}$$

Applying Lemma 1 to g, $E=\bar{\mathbb{C}}-\bar{U}_1$, Φ and $N=1$, we obtain a qc-mapping φ and a rational function $p(z)=\varphi\circ g\circ\varphi^{-1}$. It is easy to see that p is a polynomial of degree d, provided that $\varphi(\infty)=\infty$.

4. Perturbations

In this section, we perturb a rational function f, in order to make its non-repulsive periodic points attractive.

Before doing this, we state some Lemmas. An easy calculation shows:

LEMMA 2. — *Let $h(z)$ be a polynomial of degree k. Define $H_\varepsilon:\bar{\mathbb{C}}\to\bar{\mathbb{C}}$ for $\varepsilon\in\mathbb{C}$, by*

$$H_\varepsilon(z)=z+\varepsilon.h(z).\rho(|\varepsilon|^{1/k}.|z|) \qquad \text{for} \quad z\in\mathbb{C},$$

$$H_\varepsilon(\infty)=\infty,$$

where ρ *is a* C^∞*-function on* $\mathbb{R}$ *such that* $0 \leqq \rho \leqq 1$, $\rho = 1$ *on* $[0, 1]$ *and* $\rho = 0$ *on* $[2, \infty)$. *Then, for small* ε, H_ε *is qc. Furthermore,* $H_\varepsilon \to \mathrm{id}_{\bar{\mathbb{C}}}$ *uniformly* (*w.r.t. the metric of* $\bar{\mathbb{C}}$) *and* $\| \mu_{H_\varepsilon} \|_\infty \to 0$, *when* $\varepsilon \to 0$.

LEMMA 3. — *Suppose that a polynomial* $h(z)$ *and open sets* E_ε *of* $\bar{\mathbb{C}}$ $(\varepsilon_0 > \varepsilon \geqq 0)$ *satisfy*:

(4.1) $$E_0 \subset E_\varepsilon \text{ and } E_\varepsilon \text{ are uniformly bounded in } \mathbb{C};$$

(4.2) $$f(\infty) \in E_0;$$

(4.3) $$f \circ (\mathrm{id} + \varepsilon . h)(E_\varepsilon) \subset E_\varepsilon.$$

Set $g_\varepsilon = f \circ H_\varepsilon$, *where* H_ε *is defined in Lemma 2. Then, for small* $\varepsilon > 0$, *there exist qc-mappings* φ_ε *of* $\bar{\mathbb{C}}$ *such that* $f_\varepsilon = \varphi_\varepsilon \circ g_\varepsilon \circ \varphi_\varepsilon^{-1}$ *are rational functions and that* $\varphi_\varepsilon \to \mathrm{id}_{\bar{\mathbb{C}}}, f_\varepsilon \to f$ *uniformly, when* $\varepsilon \to 0$.

Proof. — Let $V_\varepsilon = \{ z \in \bar{\mathbb{C}} : |z| > (1/|\varepsilon|)^{1/k} \}$. For small $\varepsilon > 0$, $E_\varepsilon \cap \bar{V}_\varepsilon = \varnothing$ and $g_\varepsilon(\bar{V}_\varepsilon) \subset E_\varepsilon$. By (4.3), $g_\varepsilon(E_\varepsilon) \subset E_\varepsilon$. Moreover, if ε is small enough, g_ε is quasi-regular by Lemma 2 and $(g_\varepsilon)_{\bar{z}} = 0$ on $E_\varepsilon \cup (\bar{\mathbb{C}} - g^{-1}(E_\varepsilon)) \subset \bar{\mathbb{C}} - \bar{V}_\varepsilon$.

Hence Lemma 1 can be applied to g_ε, E_ε, $\Phi = \mathrm{id}_{E_\varepsilon}$ and $N = 1$. Thus qc-mappings φ_ε are obtained. The second assertion follows from the parametrized measurable mapping theorem (*cf.* [1]), since $\| \mu_{g_\varepsilon} \|_\infty = \| \mu_{H_\varepsilon} \|_\infty \to 0 \, (\varepsilon \to 0)$. (*See* the proof of Lemma 1.)

Note that $\varphi_\varepsilon |_{E_\varepsilon}$ is conformal. □

LEMMA 4. — *Let* $\zeta_1, \ldots, \zeta_m$ *be distinct points of* $\mathbb{C}$, *and* $B_1, \ldots, B_n$ *pairwise disjoint closed sets of* $\mathbb{C}$ *homeomorphic to a closed disk. And for each* j, *let* h_j *be a holomorphic function in a neighborhood of* B_j. *Suppose that if* $\zeta_i \in B_j$, $h_j(\zeta_i) = 0$ *and* $h'_j(\zeta_i) = -1$.

Then, for any $\delta > 0$, *there is a polynomial* $h(z)$ *such that*

$$h(\zeta_i) = 0, \qquad h'(\zeta_i) = -1 \qquad (i = 1, \ldots, m)$$

and $|h - h_j| < \delta$ *on* B_j $(j = 1, \ldots, n)$.

Proof. — Take a polynomial p_1 satisfying:

$$p_1(\zeta_i) = 0, \qquad p'_1(\zeta_i) = -1 \qquad (i = 1, \ldots, m)$$

and let $p_2(z) = \prod_i (z - \zeta_i)^2$. Then $(h_j - p_1)/p_2$ is holomorphic in a neighborhood of B_j $(j = 1, \ldots, n)$. By Runge's theorem, there is a polynomial $q(z)$ such that

$$|(h_j - p_1)/p_2 - q| < \delta / \sup_{\zeta \in B_j} |p_2(\zeta)| \quad \text{on } B_j.$$

Clearly, $h = p_1 + p_2 . q$ verifies the conditions. □

Let $\{ z_0, z_1, \ldots, z_{p-1} \}$ be one of non-repulsive cycles of f. First, pay attention only to this cycle. We are going to construct the perturbations, according as this cycle is attractive, rationally indifferent, Siegel-cycle or Cremer-cycle.

Case 1. — z_i are attractive. — Let $E_\varepsilon \equiv E_0$ be the union of small disks centered at z_i, such that $f(\bar{E}_0) \subset E_0$. By a coordinate transformation, we may assume that

$\infty \in f^{-1}(\mathrm{E}_0) - \bar{\mathrm{E}}_0$. Let h be an arbitrary polynomial such that $h(z_i) = 0$ $(i = 0, \ldots, p-1)$. It is easily checked that (4.1)-(4.3) hold, for small ε. Therefore Lemma 3 can be applied.

Case 2. — z_i are rationally indifferent. — It follows from the theory of normal forms (*cf.* [2]), that there exists an analytic local diffeomorphism ψ at 0, such that $\psi(0) = z_0$, and

$$\psi^{-1} \circ f^p \circ \psi(z) = \lambda z (1 - z^m + O(z^{m+1})),$$

where $\lambda = (f^p)'(z_0)$ is a root of unity; $\lambda^m = 1$. Let

$$\mathrm{E}'_0 = \{\zeta : 0 < |\zeta| < r_0, \ |\arg \zeta^m| < \pi/3\}.$$

Check that if r_0 is sufficiently small, $\bar{\mathrm{E}}'_0$ is contained in the domain of ψ and satisfies:

$$f^p(\psi(\bar{\mathrm{E}}'_0)) \subset \psi(\mathrm{E}'_0 \cup \{0\}). \tag{4.4}$$

(*See* the flower theorem in [3], [5] and (4.5) below.) By a coordinate transformation, we may assume that

$$\infty \in f^{-1}(\psi(\mathrm{E}'_0)) - f^{p-1}(\psi(\bar{\mathrm{E}}'_0)).$$

Let h be a polynomial such that $h(z_i) = 0$ and $h'(z_i) = -1$. Consider $\mathrm{G}_\varepsilon(z) = \psi^{-1} \circ g_\varepsilon^p \circ \psi(z)$, for small ε, where g_ε is as in Lemma 3. It is easily seen that

$$\begin{cases} \mathrm{G}_\varepsilon(z) = \lambda z [(1-\varepsilon)^p - z^m + O(\varepsilon z) + O(z^{m+1})] & (\text{as } \varepsilon, z \to 0), \\ |\mathrm{G}_\varepsilon(z)| = |z| \,.\, [(1-\varepsilon)^p - \mathrm{Re}\, z^m + O(\varepsilon z) + O(z^{m+1})], & \\ \arg \mathrm{G}_\varepsilon(z) = \arg \lambda z - \mathrm{Im}\, z^m + O(\varepsilon z) + O(z^{m+1}) & (\mathrm{mod}\ 2\pi). \end{cases} \tag{4.5}$$

For $\varepsilon > 0$, define $\mathrm{E}'_\varepsilon = \mathrm{E}'_0 \cup \{|\zeta| < \varepsilon^{z/(2m-1)}\}$. We show that:

$$\text{if } \varepsilon \text{ is sufficiently small,} \quad \mathrm{G}_\varepsilon(\bar{\mathrm{E}}'_\varepsilon) \subset \mathrm{E}'_\varepsilon. \tag{4.6}$$

First, if $\varepsilon^{z/(2m-1)} < |z| = r < r_1$ and $\arg z^m = \pm \pi/3$, then

$$\arg \mathrm{G}_\varepsilon(z) - \arg \lambda z = \mp \frac{\sqrt{3}}{2} r^m (1 + O(r_1^{1/2})).$$

If we take a sufficiently small r_1,

$$\mathrm{G}_\varepsilon(\partial \mathrm{E}'_\varepsilon \cap \{\varepsilon^{z/(2m-1)} < |z| < r_1\}) \subset \mathrm{E}'_\varepsilon.$$

Fix this r_1. Secondly, if $|z| = \varepsilon^{z/(2m-1)}$,

$$|\mathrm{G}_\varepsilon(z)| = |z| (1 - p\varepsilon + o(\varepsilon)).$$

Hence $\mathrm{G}_\varepsilon(\partial \mathrm{E}'_\varepsilon \cap \{|z| \leqq \varepsilon^{z/(2m-1)}\}) \subset \mathrm{E}'_\varepsilon$, for small ε. Finally, for small ε, $\mathrm{G}_\varepsilon(\partial \mathrm{E}'_\varepsilon \cap \{|z| \geqq r_1\}) \subset \mathrm{E}'_\varepsilon$, since $\mathrm{G}_0(\partial \mathrm{E}'_0 \cap \{|z| \geqq r_1\}) \subset \mathrm{E}'_0$. [*See* (4.4).]

Thus (4.6) is proved. Set $E_\varepsilon = \psi(E'_\varepsilon) \cup g_\varepsilon \circ \psi(E'_\varepsilon) \cup \ldots \cup g_\varepsilon^{p-1} \circ \psi(E'_\varepsilon)$. Obviously, E_ε satisfies (4.1)-(4.3).

Case 3. — z_i are Siegel points. — Let S_i be the Siegel disks containing z_i and $\psi_i : S_i \to \Delta = \{ |z| < 1 \}$ conformal mappings such that $\psi_i(z_i) = 0$ $(i = 0, \ldots, p-1)$. Set $B_i = \psi_i^{-1}(\{ |\zeta| \leqq r \})$, for $0 < r < 1$. Then $f(B_i) = B_{i+1}$, $i = 0, \ldots, p-1$, where $B_p = B_0$. Fix r so that

(4.7) $S_i - \mathring{B}_i$ do not intersect with the forward orbits of critical points.

By a coordinate transformation, we may assume $\infty \in f^{-1}(\mathring{B}_0) - \mathring{B}_{p-1}$. Let $h_i(z) = -\psi_i(z)/\psi'_i(z)$ on S_i. By Lemma 4, there is a polynomial h such that

$$(4.8) \qquad h(z_i) = 0, \qquad h'(z_i) = -1 \qquad \text{and} \qquad |h - h_i| < \delta \quad \text{on } B_i.$$

Let H_ε, g_ε be as in Lemma 2 and Lemma 3. It is easy to see that if δ is sufficiently small, $H_\varepsilon(B_i) \subset B_i$, hence $g_\varepsilon(B_i) \subset B_{i+1}$. If we set $E_\varepsilon = \bigcup_i B_i$, (4.1)-(4.3) are satisfied.

Case 4. — z_i are Cremer points. — We may assume $f(\infty) = z_0$ and $\infty \neq z_{p-1}$. Let h be a polynomial satisfying $h(z_i) = 0$ and $h'(z_i) = -1$. We shall construct open sets E_ε satisfying (4.3).

It follows from the theory of normal forms (*cf.* [2]), that there exists an analytic local diffeomorphism ψ at 0 such that $\psi(0) = z_0$ and

$$\psi^{-1} \circ f^p \circ \psi(z) = \lambda z + O(z^{k+3}),$$

where $k = \deg h$. Define $E'_\varepsilon = \psi(\{ \zeta : |\zeta| < |\varepsilon|^{1/(k+1)} \})$ and $E_\varepsilon = E'_\varepsilon \cup \ldots \cup g_\varepsilon^{p-1}(E'_\varepsilon)$, where g_ε is as in Lemma 3. Since

$$\psi^{-1} \circ g_\varepsilon^p \circ \psi(z) = \lambda z [(1 - \varepsilon)^p + O(\varepsilon z) + O(z^{k+2})],$$

a simple estimate shows that if $\varepsilon > 0$ is sufficiently small, $g_\varepsilon^p(\bar{E}'_\varepsilon) \subset E'_\varepsilon$, hence $g_\varepsilon(E_\varepsilon) \subset E_\varepsilon$. (This argument implies the fact that the distance from z_0 to the boundary of its basin tends to zero slower than ε^β when $\varepsilon \to 0$, for any $\beta > 0$.)

We cannot use Lemma 3 in this case, since there is no non-empty open set E_0 satisfying (4.1). Its conclusion, however, holds by the following. Let V_ε be as before. It is easily verified that $g_\varepsilon(\bar{V}_\varepsilon) \subset E_\varepsilon$ and $E_\varepsilon \cap V_\varepsilon = \varnothing$, for small ε. Hence, as in the proof of Lemma 3, φ_ε and f_ε are obtained.

Thus, in each case, applying Lemma 3 or its variant, we have obtained φ_ε and f_ε. Consider them for $\varepsilon > 0$ small enough. Clearly, $z_i \in E_\varepsilon$ and $\{ z_i \}$ is an attractive cycle of g_ε. Define $z_i^\varepsilon = \varphi_\varepsilon(z_i)$ and $\tilde{E}_\varepsilon = \varphi_\varepsilon(E_\varepsilon)$. Then, $z_0^\varepsilon, \ldots, z_{p-1}^\varepsilon$ form an attractive cycle of f_ε, since $\varphi_\varepsilon|_{E_\varepsilon}$ is conformal. As $f_\varepsilon(\tilde{E}_\varepsilon) \subset \tilde{E}_\varepsilon$, it follows from Montel's theorem that $\tilde{E}_\varepsilon \subset D_{f_\varepsilon}$. Note that

$$\tilde{E}_\varepsilon = \tilde{E}_{0,\varepsilon} \cup \tilde{E}_{1,\varepsilon} \cup \ldots \cup \tilde{E}_{p-1,\varepsilon};$$

where $\tilde{E}_{i,\varepsilon}$ are the connected components of $\tilde{E}_\varepsilon$ and satisfy

$$z_i^\varepsilon \in \tilde{E}_{i,\varepsilon}, \qquad f_\varepsilon(\tilde{E}_{i,\varepsilon}) \subset \tilde{E}_{i+1,\varepsilon} \qquad (\tilde{E}_{p,\varepsilon} = \tilde{E}_{0,\varepsilon}).$$

Hence each $\tilde{E}_{i,\varepsilon}$ is contained in the attractive basin of z_i^ε.

Suppose that f has non-repulsive cycles other than $\{z_0, \ldots, z_{p-1}\}$. Again using Lemma 4, we can take the polynomial h so that it also satisfies the conditions as in Case 1-4 above, corresponding to each of these cycles. Then the arguments there are valid for these cycles, and the obtained perturbation makes all the non-repulsive periodic points attractive. Strictly speaking, let $z_0, \ldots, z_N$ be all of the non-repulsive periodic points of f, and define $z_i^\varepsilon = \varphi_\varepsilon(z_i)$ $(i=0, \ldots, N)$ as before, then $z_0^\varepsilon, \ldots, z_N^\varepsilon$ are attractive periodic points of f_ε.

Therefore, $n_{\mathrm{attr}}(f_\varepsilon) \geqq n_{\mathrm{attr}}(f) + n_{\mathrm{indiff}}(f)$. Finally, $\deg f = \deg f_\varepsilon$, since their topological degrees coincide. Thus Theorem 1 is proved.

Remark. — If f is a polynomial, one can perturbe it as a polynomial-like function, and obtain a perturbed polynomial by the surgery in Example in paragraph 3. (*See* Corollary 11.12 of [3].) In Case 1, *we can use a similar perturbation.*

Also in Case 3, we may use the same perturbation as Case 4. But we prefer that method for the sake of the proof of Theorem 2.

Remark. — It is also possible to perturb f so that some of indifferent cycles other than $\{z_0, \ldots, z_{p-1}\}$ become repulsive or indifferent.

5. Proof of theorem 2. Part I

Let $\tilde{D}_f$ be the D_f minus all inverse images of Herman rings.

PROPOSITION 1. — *For the* f_ε *constructed in paragraph* 4,

$$n_c(f_\varepsilon, \tilde{D}_{f_\varepsilon}) \geqq n_c(f, \tilde{D}_f) + n_{\mathrm{irr}}(f).$$

Therefore,

$$n_c(f, \tilde{D}_f) + n_{\mathrm{irr}}(f) \leqq n_c(f). \tag{5.1}$$

LEMMA 5. — *Let* f *be a rational function with* $\deg f \geqq 2$, *and* B *a simply connected domain of* $\bar{\mathbb{C}}$. *Suppose that* $f(B) \subset B$, $f|_B$ *is one to one, and* f *has an attractive fixed point* z_0 *in* B.

Then there exists a critical point c *of* f *such that*

$$f^N(c) \in B - f(B) \quad \textit{for some} \quad N \geqq 1.$$

Moreover, c *and* B *are contained in the same connected component of* $\bigcup_{n \geqq 0} f^{-n}(B)$.

Proof (*see* Theorem 5.8 of [3]). — If such c does not exist, one can define inductively analytic functions g_n on B such that

$$f^n \circ g_n = \mathrm{id}_B,$$
$$g_n(z_0) = z_0.$$

It follows from Montel's theorem that the family $\{g_n\}$ is normal, since it omits at least three values (for example points of J_f). This contradicts with the fact that $g'_n(z_0) = 1/(f'(z_0))^n \to \infty$, as $n \to \infty$. □

Proof of Proposition 1. — Let $\{z_0, z_1, \ldots, z_{p-1}\}$ be a non-repulsive cycle of f. We use the notations φ_ε, z_i^ε, $\tilde{E}_\varepsilon$ in paragraph 4.

If c is a critical point of f, then $c^\varepsilon = \varphi_\varepsilon(H_\varepsilon^{-1}(c))$ is a critical point of f_ε. This gives an 1 to 1 correspondance between critical points of f and f_ε, preserving their multiplicities. Hence $n_c(f) = n_c(f_\varepsilon)$, even if we adopted the convention that critical points are counted without multiplicities.

We make considerations according to the cases in paragraph 4.

Case 1 (resp. *Case* 2). — z_i are attractive (resp. rationally indifferent). — Let $c_1, \ldots, c_m$ be all of the critical points of f, which are eventually mapped in to the AB-cycle (resp. the PB-cycles) associated to z_i. Then, for some N, $f^N(c_j) \in E_0$. If ε is sufficiently small, $f_\varepsilon^N(c_j^\varepsilon) \in \tilde{E}_\varepsilon$. Hence c_j^ε are eventually mapped by f_ε into the AB-cycle associated to z_i^ε. (*See* paragraph 4.)

Case 3. — z_i are Siegel points. — Let $c_1, \ldots, c_m$ be all of the critical points of f eventually mapped into the SD-cycle associated to z_i. By (4.7), $f^N(c_i) \in E_0 = B_0 \cup \ldots \cup B_{p-1}$, for some N. As above, c_j^ε are eventually mapped by f_ε into the AB-cycle associated to z_i^ε. Besides, for small $\varepsilon > 0$, there exists a critical point c of f_ε other than c_j^ε such that c itself is contained in the AB-cycle assiated to z_i^ε. In fact, f_ε^p and $\tilde{B}_0 = \varphi_\varepsilon(B_0)$ satisfy the conditions of Lemma 5, and $\{f_\varepsilon^n(c_j^\varepsilon) : n \geqq 0, j = 1, \ldots, m\}$ does not intersect with $\tilde{B}_0 - f_\varepsilon^p(\tilde{B}_0)$, if $\varepsilon > 0$ is small.

Case 4. — z_i are Cremer points. — As mentioned in paragraph 1.6, the AB-cycle of f_ε associated to z_i^ε contains at least one critical point of f_ε.

Hence, corresponding to each of irrationally indifferent cycle of f, at least one critical point will newly fall into the stable region (into the AB-cycles) by the perturbation. We thus conclude that $\tilde{D}_{f_\varepsilon}$ contains at least $n_{irr}(f)$ critical points more than $\tilde{D}_f$. This implies Proposition 1.

6. The case where Herman rings exist

Suppose a rational function f has Herman rings. Let $\mathscr{A}_0$ be the collection of all Herman rings of f. (By Sullivan's result, $\mathscr{A}_0$ is finite. See Remark after Corollary 2, and also Remark after the proof of Proposition 2.) For each $A \in \mathscr{A}_0$, we associate an oriented analytic Jordan curve γ_A so that:

$$f(\gamma_A) = \gamma_{f(A)} \qquad (A \in \mathscr{A}_0, \text{ hence } f(A) \in \mathscr{A}_0);$$

γ_A does not intersect with the orbits of critical points.

Hence, if $f^p(A)=A$, $f^p(\gamma_A)=\gamma_A$.

Set

$$\mathscr{A}=\{\text{connected components of } A-\gamma_A : A\in\mathscr{A}_0\},$$
$$\Gamma_n=\{\text{connected components of } f^{-n}(\gamma_A) : A\in\mathscr{A}_0\}\ (n\geqq 0).$$

Each Γ_n consists of analytic Jordan curves. Assign them orientations so that f respects these orientations.

Let $\mathscr{X}=\mathscr{A}\cup\{\{x\} : x \text{ is a non-repulsive periodic point}\}$. Then every $X\in\mathscr{X}$ is contained in a connected component of $\bar{\mathbb{C}}-\cup\Gamma_n$, where $\cup\Gamma_n$ means $\bigcup_{\gamma\in\Gamma_n}\gamma$. We call a connected component of $\bar{\mathbb{C}}-\cup\Gamma_n$ an *n-piece*.

LEMMA 6. — *There exists an integer* $N\geqq 0$ *such that: if* X_1, X_2 $(\in\mathscr{X})$ *are in the same* N-*pièce, then* $f(X_1)$ *and* $f(X_2)$ *are contained in the same* N-*piece.*

Proof. — For each pair (X_1, X_2), define $n(X_1, X_2)$: If X_1 and X_2 are not in the same n-piece for some n, then $n(X_1, X_2)=$ the least such n; otherwise, $n(X_1, X_2)=0$.

The assertion holds for $N=\max\{n(X_1, X_2) : X_1, X_2\in\mathscr{X}\}$. (Note that $\mathscr{X}$ is finite.) □

Fix this N. Let

$$\mathscr{D}=\{\text{N-pieces}\};$$
$$\mathscr{D}_I=\{D\in\mathscr{D} : \text{for some } X\in\mathscr{X}, X\subset D\};$$
$$\mathscr{D}_{II}=\mathscr{D}-\mathscr{D}_I=\{D\in\mathscr{D} : \text{for all } X\in\mathscr{X},\ X\cap D=\varnothing\}.$$

Since each $X\in\mathscr{X}$ is periodic (as a set) with respect to f, we obtain immediately from Lemma 6:

LEMMA 6′. — *$\mathscr{D}_I$ is decomposed into disjoint cycles*

$$D_{i,0}, \ldots, D_{i,m_i-1} \qquad (i=1, \ldots, L;\ m_i\geqq 1),$$

(hence, $\mathscr{D}_I=\{D_{i,j}\}$) such that: if $X\subset D_{i,j}$, $X\in\mathscr{X}$, then

$$f(X)\subset D_{i,j+1},$$

where we write $D_{i,m_i}=D_{i,0}$.

For simplicity, let us fix i and write $D_j=D_{i,j}$, $m=m_i$.

LEMMA 7. — (i) $f(D_j)\supset D_{j+1}$.

(ii) *Let $\gamma\in\Gamma_N$ such that $\gamma\subset\partial D_j$. If $D_j\subset \operatorname{Int}\gamma$ (resp. $D_j\subset\operatorname{Ext}\gamma$), then $D_{j+1}\subset\operatorname{Int} f(\gamma)$ (resp. $D_{j+1}\subset\operatorname{Ext} f(\gamma)$).*

Proof. — (i) is trivial. (ii) Assume $D_j\subset\operatorname{Int}\gamma$. If $N\geqq 1$, $f(\gamma)\cap f(D_j)=\varnothing$. If $z\in D_j$ is sufficiently near to γ, $f(z)\in\operatorname{Int} f(\gamma)$. So $f(D_j)\cap\operatorname{Int} f(\gamma)\neq\varnothing$, hence

$$D_{j+1}\subset f(D_j)\subset\operatorname{Int} f(\gamma).$$

If $N=0$, there is $A \in \mathscr{A}$ such that $A \subset D_j$ and $\gamma \subset \partial A$. As above, $f(A) \cap \operatorname{Int} f(\gamma) \neq \varnothing$. Thus, $D_{j+1} \subset \operatorname{Int} f(\gamma)$, since

$$f(A) \subset D_{j+1} \quad \text{and} \quad D_{j+1} \cap f(\gamma) = \varnothing. \quad \square$$

DEFINITION. — Let $m\overline{\mathbb{C}} = (\mathbb{Z}/m\mathbb{Z}) \times \overline{\mathbb{C}} = \overline{\mathbb{C}}_0 \cup \ldots \cup \overline{\mathbb{C}}_{m-1}$, where $\overline{\mathbb{C}}_j = \{j\} \times \overline{\mathbb{C}}$. Define $\iota_j : \overline{\mathbb{C}} \to \overline{\mathbb{C}}_j$ by $\iota_j(z) = (j, z)$.

We call a map $g : m\overline{\mathbb{C}} \to m\overline{\mathbb{C}}$ a *cyclic map*, if $g(\overline{\mathbb{C}}_j) \subset \overline{\mathbb{C}}_{j+1}$ for all $j \in \mathbb{Z}/m\mathbb{Z}$. Moreover, g is a *cyclic rational map* if all $g|_{\overline{\mathbb{C}}_j}$ are rational functions. The notations and the results in paragraph 1 are naturally extended to cyclic rational maps.

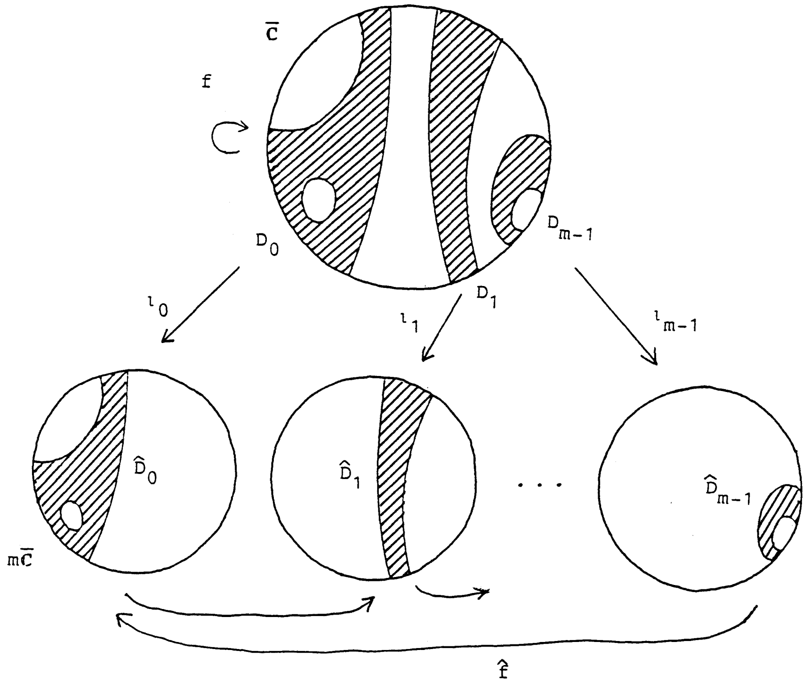

Fig. 2. – Definition of $\hat{D}$ and cyclic map $\hat{f}$.

Set :

$$\hat{D}_j = \iota_j(D_j); \qquad \hat{D} = \hat{D}_0 \cup \ldots \cup \hat{D}_{m-1};$$
$$\hat{\Gamma}_n = \{\iota_j(\gamma) \mid \gamma \in \Gamma_n,\ \gamma \subset \bar{D}_j\};$$
$$\hat{\mathscr{A}} = \{\iota_j(A) \mid A \in \mathscr{A},\ A \subset D_j\}.$$

Define a cyclic map $\hat{f} : \hat{D} \to m\overline{\mathbb{C}}$ by $\hat{f}(j, z) = (j+1, f(z))$. *See* Figure 2.

PROPOSITION 2. — *There exist rational functions* $f_0, \ldots, f_{m-1}$ *and qc-mappings* $\varphi_0, \ldots, \varphi_{m-1}$ *of* $\overline{\mathbb{C}}$ *satisfying* (i)-(v) *below.*

Define F, $\varphi : m\bar{\mathbb{C}} \to m\bar{\mathbb{C}}$ *by*

$$F(j, z) = (j+1, f_j(z)) \quad \textit{and} \quad \varphi(j, z) = (j, \varphi_j(z)).$$

(i) *The following diagram is commutative:*

$$\begin{array}{ccc} \hat{D} & \xrightarrow{\hat{f}} & m\bar{\mathbb{C}} \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ m\bar{\mathbb{C}} & \xrightarrow{F} & m\bar{\mathbb{C}} \end{array}$$

(ii) $\mu_\varphi = 0$ *a. e. on* $K_{\hat{f}} = \bigcup_{n \geqq 0} \hat{f}^{-n}(\hat{D})$.

(iii) $F(m\bar{\mathbb{C}} - \varphi(\hat{D})) \subset m\bar{\mathbb{C}} - \varphi(\hat{D})$.

(iv) *For each* $A \in \hat{\mathscr{A}}$, $\varphi(A)$ *is contained in a Siegel disk of* F.

(v) *There exists an integer* $M > 0$ *such that* $F^M(m\bar{\mathbb{C}} - \varphi(\hat{D}))$ *is contained in the union of attractive basins and Siegel disks of* F. *Therefore* $m\bar{\mathbb{C}} - \varphi(\hat{D}) \subset D_F$.

Proof. — Observe that Lemma 1 holds for a cyclic map g, with $\bar{\mathbb{C}}$ replaced by $m\bar{\mathbb{C}}$, and the resulting qc-mapping φ is chosen to be component-wise, i. e. $\varphi(\bar{\mathbb{C}}_j) = \bar{\mathbb{C}}_j$. So, we construct a cyclic quasi-regular mapping g extending $\hat{f}$, for which this version applies.

Step 0. — First, f is extended continuously to the boundary of D, i. e. to $\gamma \in \hat{\Gamma}_N$. Consider

$$\hat{\Gamma}'_N = \hat{\Gamma}_N \cup \{ \iota_{j+1}(f(\gamma)) \mid \gamma \in \Gamma_{N'}, \gamma \subset \partial D_j \}.$$

Each element has an orientation induced by ι_j such that if $\gamma \in \hat{\Gamma}_N$, then $\hat{f}(\gamma) \in \hat{\Gamma}'_N$ and $\hat{f}|_\gamma$ respects the orientation. By Lemma 7 (ii), alternating some of these orientations if necessary, we may assume that:

$$\hat{f} \text{ still respects the orientations;}$$

$$D_j \cap \operatorname{Ext}\gamma = \varnothing \quad \text{for} \quad \gamma \in \hat{\Gamma}'_N;$$

where Ext γ means the exterior of γ in $\bar{\mathbb{C}}_j$ if $\gamma \subset \bar{\mathbb{C}}_j$.

Let $E_\gamma = \overline{\operatorname{Ext}\gamma}$ for $\gamma \in \Gamma'_N$ and $\mathscr{E} = \{ E_\gamma \mid \gamma \in \hat{\Gamma}_N \}$. Note that $m\bar{\mathbb{C}} = D \cup \bigcup_{E \in \mathscr{E}} E$ (disjoint union). Define a relation « $\to$ » in $\mathscr{E}$ as follows: If $E \in \mathscr{E}$, then $\partial E \in \hat{\Gamma}_N$ and $\hat{f}(\partial E)$ is contained in $m\bar{\mathbb{C}} - \hat{D}$. Hence there exists a unique $E' \in \mathscr{E}$ such that $\hat{f}(\partial E) \subset E'$. Then we write $E \to E'$. We are going to define g on E so that $g(E) \subset E'$ and $g = \hat{f}$ on ∂E.

Step 1. — Define $\mathscr{E}_1 = \{ E_\gamma \mid \gamma \in \hat{\Gamma}_0 \}$. Let $E = E_\gamma \in \mathscr{E}_1$. Obviously, if $E \to E'$, then $E' \in \mathscr{E}_1$ and $\hat{f}(\partial E) = \partial E'$. Recall that $\hat{f}^p(\gamma) = \gamma$ for some q, since Γ_0 is a collection of inveriant curves in Herman rings. Hence there exist $E_k \in \mathscr{E}_1$ ($k = 0, 1, \ldots, q$) such that

$$E = E_0 \to E_1 \to \ldots \to E_q = E. \tag{6.1}$$

Moreover, $\mathscr{E}_1$ consists of cycles of the form (6.1).

Consider a cycle (6.1), where $E_0, \ldots, E_{q-1}$ are assumed to be distinct. Let $\gamma_k = \partial E_k$ and $S^1 = \{z \in C : |z| = 1\}$. Here, we choose the orientation of S^1 so that $\operatorname{Ext} S^1 = \Delta = \{|z| < 1\}$. By the definition of the Herman ring, there is a real analytic diffeomorphism $\psi_0 : \gamma_0 \to S^1$ respecting the orientation, such that $\psi_0 \circ \hat{f}^q \circ \psi_0^{-1}(z) = \lambda . z$, where $|\lambda| = 1$. Define $\psi_k : \gamma_k \to S^1$ by $\psi_k = \psi_0 \circ \hat{f}^{q-k}$ for $k = 1, \ldots, q$. Then the following diagram is commutative:

$$\begin{array}{ccccccc} \gamma_0 & \xrightarrow{\hat{f}} & \gamma_1 & \xrightarrow{\hat{f}} \ldots \xrightarrow{\hat{f}} & \gamma_q = \gamma_0 \\ \downarrow \psi_0 & & \downarrow \psi_1 & & \downarrow \psi_q = \psi_0 \\ S^1 & \xrightarrow{\lambda.} & S^1 & \xrightarrow{\text{id}} \ldots \xrightarrow{\text{id}} & S^1 \end{array}$$

As $\psi_k|_{\gamma_k}$ are real analytic and orientation preserving, there exist qc-mappings $\psi_k : E_k \to \bar{\Delta}$ extending $\psi_k|_{\gamma_k}$, where $\psi_q = \psi_0$. Define g on E_k so that the following diagram is commutative:

$$\begin{array}{ccccccc} E_0 & \xrightarrow{g} & E_1 & \xrightarrow{g} \ldots \xrightarrow{g} & E_q = E_0 \\ \downarrow \psi_0 & & \downarrow \psi_1 & & \downarrow \psi_q = \psi_0 \\ \bar{\Delta} & \xrightarrow{\lambda.} & \bar{\Delta} & \xrightarrow{\text{id}} \ldots \xrightarrow{\text{id}} & \bar{\Delta} \end{array}$$

Define Φ by $\Phi|_{E_k} = \psi_k$.

STEP 2. — Set

$$\mathscr{E}_2 = \{ E \in \mathscr{E} - \mathscr{E}_1 \mid \text{there exist } E_k \in \mathscr{E} \text{ satisfying } (6.1) \}.$$

Consider a cycle (6.1) for $E \in \mathscr{E}_2$, and assume $E_0, \ldots, E_{q-1}$ are distinct. Then $E_k \in \mathscr{E}_2$.- (*See* Step 1.)

Moreover, $\hat{f}(\gamma_{k_0}) \neq \gamma_{k_0+1}$, for some $k_0 (0 \leqq k_0 < q)$, where $\gamma_k = \partial E_k$. Indeed, if $\hat{f}(\gamma_k) = \gamma_{k+1}$ $(k = 0, \ldots, q-1)$, then $\hat{f}^q(\gamma_0) = \gamma_0$, hence $\gamma_0 \in \hat{\Gamma}_0$ by the definition of Γ_n. This contradicts with $E = E_{\gamma_0} \notin \mathscr{E}_1$.

Take smaller open disks E'_k such that $\bar{E}'_k \subset \mathring{E}_k$ $(k = 0, \ldots, q)$, $\hat{f}(\gamma_{k_0}) \subset E'_{k_0+1}$ and $E'_q = E'_0$. We can easily construct quasi-regular mappings g on E_k satisfying:

$$g = \hat{f} \quad \text{on } \gamma_k = \partial E_k;$$

$$g \text{ is analytic in } E'_k;$$

$$g(E_k) \subset E_{\hat{f}(\gamma_k)} \subset E_{k+1} \quad \text{and} \quad g(E'_k) \subset E'_{k+1}.$$

It follows immediately that $g^q(E_k) \subset E'_k$, since $g(E_{k_0}) \subset E'_{k_0+1}$.

Let $\Phi|_{E'_k} = \text{id}$.

STEP 3. — On $E \in \mathscr{E} - \mathscr{E}_1 \cup \mathscr{E}_2$, define g as a quasi-regular mapping so that $g = \hat{f}$ on ∂E and $g(E) \subset E'$, where $E \to E'$.

As $\mathscr{E}$ is finite, there èxist $E_k \in \mathscr{E}$ $(k = 0, \ldots, r)$ such that

$$E = E_0 \to E_1 \to \ldots \to E_r \quad \text{and} \quad E_r \in \mathscr{E}_1 \cup \mathscr{E}_2.$$

STEP 4. — Finally, let $g|_{\hat{D}}=\hat{f}$. We have thus defined g on the whole $m\bar{\mathbb{C}}$, which is continuous and quasi-regular.

Let

$$E^{(1)}=\bigcup_{E\in\mathscr{E}_1} E,\ E^{(2)}=\bigcup_{E\in\mathscr{E}_2} E' \quad \text{and} \quad E^*=E^{(1)}\cup E^{(2)},$$

where E′ denotes the open disk in Step 2 associated with $E\in\mathscr{E}_2$. By the construction, $g(m\bar{\mathbb{C}}-\hat{D})\subset m\bar{\mathbb{C}}-\hat{D}$, $g(E^*)\subset E^*$ and there exists an integer $M>0$ such that $g^M(m\bar{\mathbb{C}}-\hat{D})\subset E^*$. Hence, on $m\bar{\mathbb{C}}-g^{-M}(E^*)\subset\hat{D}$, $g_{\bar{z}}=\hat{f}_{\bar{z}}=0$. Moreover, the condition (ii) of Lemma 1 is verified for g, E^*, M and Φ defined in Step 1 and 2.

STEP 5. — The version of Lemma 1 for cyclic maps applies and then yields a qc-mapping φ of $m\bar{\mathbb{C}}$, such that $F=\varphi\circ g\circ\varphi^{-1}$ is analytic. We can choose φ of the form $\varphi(j, z)=(j, \varphi_j(z))$, where $\varphi_j(z)$ are qc-mappings of $\bar{\mathbb{C}}$, then $F(j, z)=(j+1, f_j(z))$, where $f_j(z)$ are rational functions.

The assertions (i)-(iii) of the proposition are easily verified. Consider $A\in\hat{\mathscr{A}}$ and $\gamma\in\hat{\Gamma}_0$ such that $\gamma\subset\partial A$. Let $S=\varphi(E_\gamma\cup A)$, which is a connected open set. For some $q\geqq 1$, $F^q(S)=S$ and $F^q|_S$ is conjugate to an irrational rotation on a disk. It is easily seen that S is a Siegel disk of F. So (iv) holds. On the other hand, it follows from the Schwarz's lemma that for $E\in\mathscr{E}_2$, there exists an attractive periodic point in φ(E′), whose basin contains φ(E). Thus (v) follows, and the proof of Proposition 2 is completed.

Remark. — It is possible to do this surgery with respect to subfamilies $\mathscr{A}_0^*\subset\mathscr{A}_0$, $\Gamma^*\subset\Gamma_n$ and $\mathscr{X}^*\subset\mathscr{X}$, provided that $f(\mathscr{A}_0^*)\subset\mathscr{A}_0^*$, etc.

In particular, if we take $\mathscr{A}_0^*=\{$ Herman rings intersecting with orbits of critical points $\}$, which is finite, we do not need to use the finiteness of Herman rings, in order to get the results in paragraph 7. (Because $n_c(D_F-\tilde{D}_F)=0$.)

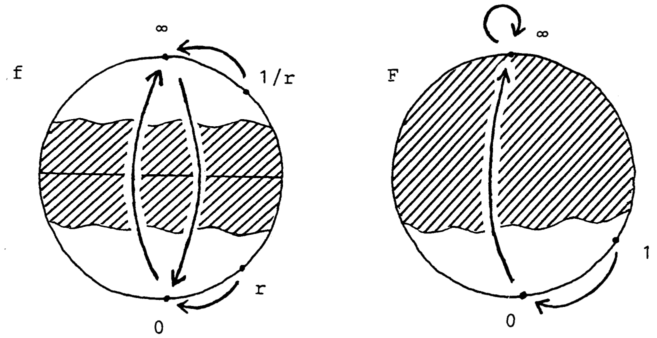

Fig. 3. — Example of the surgery in paragraph 6.

Example. — Let

$$f(z)=\frac{e^{i\alpha}}{z}\left(\frac{z-r}{1-rz}\right)^2,$$

where $\alpha \in \mathbb{R}$, $0<r<1/5$. Suppose that f has a Herman ring of rotation number θ containing $S^1=\{|z|=1\}$. Note that r and $1/r$ are critical points of f and are eventually periodic, in fact $f(r)=f^3(r)=0$, $f^2(r)=\infty$. Take $\Gamma=\Gamma_N=\{S^1\}$. Our surgery yields

$$F(z)=e^{2\pi i\theta}\frac{(z-1)^2}{z},$$

provided that the critical point corresponding to r is mapped by F^2 on the center of the Siegel disk. Check that F has ∞ as an irrational fixed point with multiplicator $e^{2\pi i\theta}$, and $F'(1)=0$, $F^2(1)=\infty$ (see *Fig.* 3).

7. Proof of theorem 2. Part II

Proposition 1 in paragraph yields the inequality (2.2) for a rational function without Herman ring. Now, assume f has Herman rings. We use the notations in paragraph 6.

Let

$$D_I=\bigcup_{D\in\mathscr{D}_I} D,\qquad D_{II}=\bigcup_{D\in\mathscr{D}_{II}} D\qquad\text{and}\qquad D^{(i)}=D_{i,\,0}\cup\ldots\cup D_{i,\,m_i-1}.$$

As before, we fix and omit i, except for $D^{(i)}$. Take φ and F in Proposition 2. It can be easily checked that Proposition 1 holds for cyclic rational maps, by a similar argument.

Consider the inequality (5.1) for F. It is also easy to see that F has no Herman ring and $\tilde{D}_F=D_F$. By (v) of Proposition 2, all the critical points of F in $m\bar{\mathbb{C}}-\varphi(\hat{D})$ are contained in the stable set D_F. Each irrationally indifferent cycle of F is either entirely contained in $\varphi(\hat{D})$ or the cycle of centers of an SD-cycle containing some $\varphi(A)$, where $A\in\hat{\mathscr{A}}$. Combined with these facts, that inequality yields:

$$(7.1)\quad n_c(\varphi(\hat{D})\cap D_F,\, F)+n_{irr}(\varphi(\hat{D}),\, F)+(\text{the number of cycles of }\hat{\mathscr{A}})\leqq n_c(\varphi(\hat{D}),\, F).$$

To express this inequality in terms of f, we need the following two lemmas.

LEMMA 8. —

$$\bigcup_j \varphi(\iota_j(D_f\cap D_j))\subset D_F.$$

Proof. — Let $z\in D_f\cap D_j$ and put $\tilde{z}=\varphi\circ\iota_j(z)$. Fix a metric on $m\bar{\mathbb{C}}$, and let $d_0=\text{dist}(J_F,\, m\bar{\mathbb{C}}-\varphi(\hat{D}))$. If $\text{dist}(F^n(\tilde{z}),\, m\bar{\mathbb{C}}-\varphi(\hat{D}))<d_0$ for some $n\geqq 0$, then $\tilde{z}\in D_F$. Alternatively, suppose that $\text{dist}(F^n(\tilde{z}),\, m\bar{\mathbb{C}}-\varphi(\hat{D}))\geqq d_0$ for all $n\geqq 0$. As f^n are equicontinuous in a neighborhood of z, there is a smaller neighborhood U such that $f^n(U)\subset D_{n+j}$ $(n\geqq 0)$, where the subscript is to be considered modulo m. Therefore, it follows from Proposition 2 (i) that $F^n\circ\varphi\circ\iota_j=\varphi\circ\iota_{n+j}\circ f^n$ on U. Thus $\tilde{z}$ is normal with respect to F, *i. e.* $\tilde{z}\in D_F$. □

LEMMA 9. — *Let z be a non-repulsive periodic point of f in D_j. Then $\tilde{z}=\varphi\circ\iota_j(z)$ is a periodic point of* F. *Moreover, $\tilde{z}$ is attractive (resp. rationally indifferent, Siegel point, Cremer point), if and only if z is so.*

Proof. — By the choice of N in paragraph 6, $f^n(z) \in D_{n+j}$ and $F^n(\tilde{z}) = \varphi \circ \iota_{n+j}(z)$, for all $n \geqq 0$. So $\tilde{z}$ becomes a periodic point of F. The second assertion follows from the following topological characterizations:

A fixed point z of a rational function f is; attractive (resp. repulsive): there exist an arbitrarily small neighborhood U of z such that $f(\bar{U}) \subset U$ (resp. $f(U) \supset \bar{U}$); indifferent: neither attractive nor repulsive; rationally indifferent: not attractive and there are an integer $k \geqq 1$ and an arbitrary small connected open set U such that $z \in \partial U$, $f^k(U) \subset U$ and $f^{nk}(\zeta) \to z\,(n \to \infty)$ for $\zeta \in U$; Siegel point : topologically conjugate to an irrational rotation in a neighborhood of z. □

Lemma 8 and Lemma 9 yield

$$n_c(D^{(i)} \cap D_f, f) \leqq n_c(\varphi(\hat{D}) \cap D_F, F),$$

and

$$n_{irr}(D^{(i)}, f) \leqq n_{irr}(\varphi(\hat{D}), F).$$

Let $n_{\mathscr{A}}(D^{(i)}, f)$ denote the number of cycles of $\mathscr{A}$ contained in $D^{(i)}$, *which is now equal to the number of cycles of* $\hat{\mathscr{A}}$.

Thus we have

$$n_c(D^{(i)} \cap D_f, f) + n_{irr}(D^{(i)}, f) + n_{\mathscr{A}}(D^{(i)}, f) \leqq n_c(D^{(i)}, f). \tag{7.2}$$

Note that $n_{irr}(f) = \sum_i n_{irr}(D^{(i)}, f)$, by the definition of $\mathscr{D}_I$. Since each Herman ring is divided into two components which are in $\mathscr{A}$, two cycles of $\mathscr{A}$ correspond to each HR-cycle. Hence

$$\sum_i n_{\mathscr{A}}(D^{(i)}, f) = 2\, n_{HR}(f).$$

Summing up (7.2) for i, we obtain

$$n_c(D_I \cap D_f, f) + n_{irr}(f) + 2\, n_{HR}(f) \leqq n_c(D_I, f). \tag{7.3}$$

On the other hand, we have

$$n_c(D_{II} \cap D_f) \leqq n_c(D_{II}, f). \tag{7.4}$$

Summing up (7.3) and (7.4), we obtain the desired inequality (2.2). So our proof of Theorem 2 is completed.

8. Proof of theorem 3

If $n_c(D_f) \neq 0$ or $n_{irr}(f) \neq 0$, the assertion immediately follows from Theorem 2. Now, assume that $n_c(D_f) = 0$ and $n_{irr}(f) = 0$. We need only to show that the equality in (2.2) does not hold. We continue to use the notations in paragraph 6.

We say that $A \in \mathscr{A}_0$ is *innermost* if $\operatorname{Int} \gamma_A \cap A' = \varnothing$ for $A' \in \mathscr{A}_0$, $A' \neq A$. Reversing the orientations if necessary, one can find at least one innermost ring A_0. Define $A_j = f^j(A_0)$, for $j \geqq 0$, and let p be the order of A, then $A_p = A_0$. Set $A'_j = \operatorname{Int} \gamma_{A_j} \cap A_j$.

LEMMA 10. — *There exists* k *such that: the component* C_k *of* $\bar{\mathbb{C}} - f^{-1}(\gamma_{A_{k+1}})$ *containing* A'_k *is not simply connected, and* A_{k+1} *is innermost.*

Proof. — There are two possibilities.

CASE 1. — All the A_j are innermost: $f(\operatorname{Int} \gamma_{A_k}) \not\subset \operatorname{Int} \gamma_{A_{k+1}}$, for some k, since A_0 is a Herman ring. Then C_k *is not* simply connected.

CASE 2. — One of A_j is not innermost: Choose k such that A_{k+1} is innermost and A_k is not. Then C_k is not simply connected, since $\operatorname{Int} \gamma_{A_k}$ contains an element of $\mathscr{A}_0$, which is mapped by f to $\operatorname{Ext} \gamma_{A_{k+1}}$. □

LEMMA 11. — *The* C_k *in Lemma* 10 *contains at least two critical points.*

Proof. — $f|_{C_k} : C_k \to \operatorname{Int} \gamma_{A_{k+1}}$ is proper, hence a branched covering. If C_k contains at most one critical point, it must be simply connected, and this contradicts with the choice of k in Lemma 10. □

Let $D_{i,j}$ be the element of $\mathscr{D}_I$ containing A'_k. Note that $\bar{C}_k$ is the union of the closures of some elements of $\mathscr{D}$. Since A_{k+1} is innermost, $f(C_k) = \operatorname{Int} \gamma_{A_{k+1}}$ contains no element of $\mathscr{A}$ but A'_{k+1}. So C_k contains no element of $\mathscr{A}$ but A'_k and no element of $\mathscr{D}_I$ but $D_{i,j}$. Therefore,

$$D_{i,j} \subset C_k \subset \overline{D_{i,j} \cup D_{II}}.$$

From Lemma 11, $n_c(D_{i,j}) + n_c(D_{II}) \geqq 2$. It also follows that $n_{\mathscr{A}}(D^{(i)}, f) = 1$.

If $n_c(D^{(i)}) \geqq n_c(D_{i,j}) \geqq 2$, then the equality in (7.2) does not hold. If $n_c(D_{II}) \geqq 1$, then the equality in (7.4) does not hold. In any case, the equality in (2.2) does not hold. Thus the theorem is proved.

9. Construction of Herman rings

In this section, we discuss on the counter procedure of the surgery in paragraph 6. However, we do not attempt to give a general method, and state only two examples, according to (A), (B) of Theorem 5. The case (A) suffices to prove Theorem 6.

(A) *See* Figure 4. Suppose that rational functions $f_0, f_1, \ldots, f_p$ $(p \geqq 1)$ satisfy the following conditions:

(α) f_0 has Siegel disks $S_1, \ldots, S_p$ *of order* p with rotation number θ, where $f_0(S_i) = S_{i+1}$ $(i = 1, \ldots, p-1)$ and $f_0(S_p) = S_1$;

(β) the composite $f_p \circ \ldots \circ f_1$ has a Siegel disk S'_1 of order 1 with rotation number $-\theta$.

Choose a (real analytic) Jordan curve γ_1 in S_1 invariant for f_0^p, and γ'_1 in S'_1 invariant for $f_p \circ \ldots \circ f_1$. Define $S'_{i+1} = f_i(S'_i)$, $\gamma_{i+1} = f_0(\gamma_i)$ and $\gamma'_{i+1} = f_i(\gamma'_i)$, inductively.

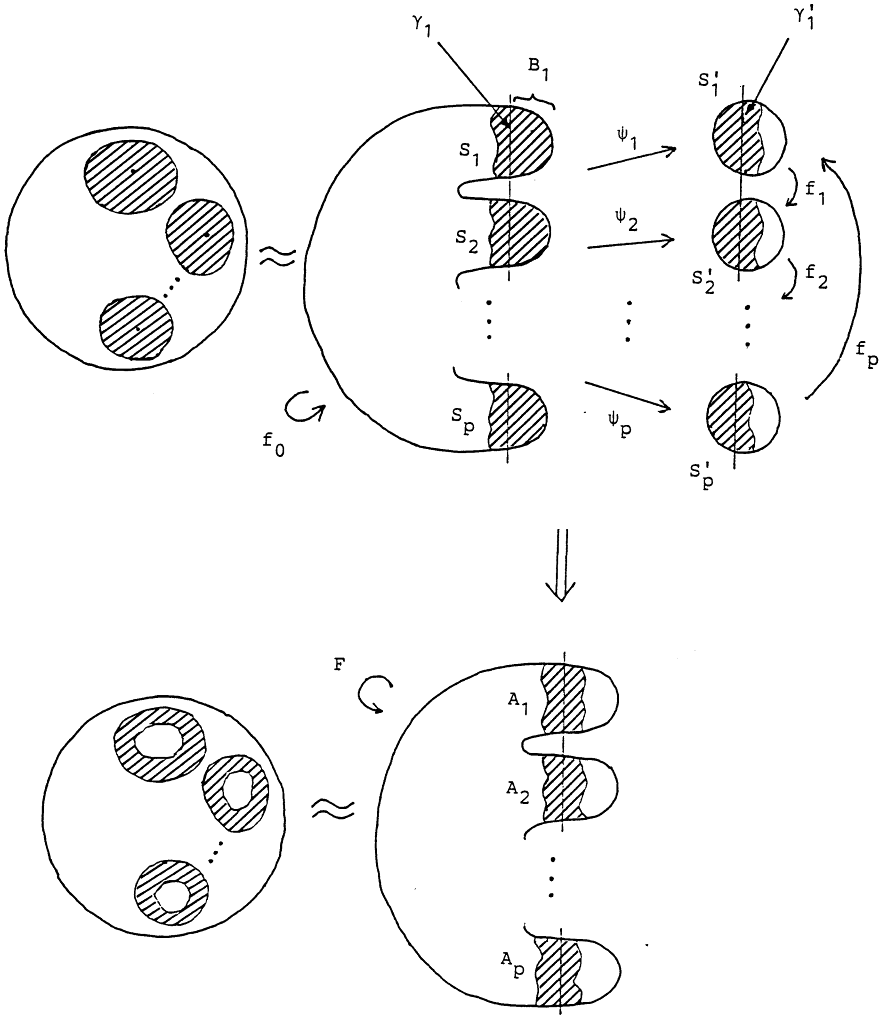

Fig. 4. — Surgery for case (A).

LEMMA 12. — *There exist quasi-conformal mappings* $\psi_1, \ldots, \psi_p : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ *satisfying* (*for each* $i=1, \ldots, p$)

(i) $\psi_i(\gamma_i)=\gamma'_i$;

(ii) $\psi_{i+1} \circ f_0 = f_i \circ \psi_i$ *on* γ_i, *where* $\psi_{p+1}=\psi_1$;

(iii) ψ_i *is conformal in a neighborhood of* $\bar{\mathbb{C}}-(S_i \cap \psi_i^{-1}(S'_i))$.

Proof. — By the definition of Siegel disks, there exists a real analytic diffeomorphism $\psi'_1 : \gamma_1 \to \gamma'_1$ satisfying

$$\psi'_1 \circ f_0^p = (f_p \circ \ldots \circ f_1) \circ \psi'_1.$$

Define $\psi'_i : \gamma_i \to \gamma'_i$, $i=2, \ldots, p$ so that (ii) holds with ψ_i replaced by ψ'_i. Let B_i (resp. B'_i) be the component of $\bar{\mathbb{C}}-\gamma_i$ (resp. $\bar{\mathbb{C}}-\gamma'_i$), entirely contained in S_i (resp. S'_i). Take conformal mappings $\psi_i'': \bar{\mathbb{C}}-\gamma_i \to \bar{\mathbb{C}}-\gamma_i'$ (which may be discontinuous on γ_i) such that $\psi_i''(B_i)=\bar{\mathbb{C}}-\bar{B}_i'$ and $\psi_i''(\bar{\mathbb{C}}-\bar{B}_i)=B_i'$. Note that the extension of $\psi_i''|_{B_i}$ (or $\psi_i''|_{\bar{\mathbb{C}}-\bar{B}_i}$) to γ_i and ψ_i' have the same orientations. Finally, modify each ψ_i'' to obtain the desired ψ_i satisfying: $\psi_i|_{\gamma_i}=\psi_i'$ and $\psi_i=\psi_i''$ in some neighborhood of $\bar{\mathbb{C}}-(S_i \cap \psi_i''^{-1}(S_i'))$. □

Now, define a mapping $g: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ by

$$g=\begin{cases} f_0 & \text{on } \bar{\mathbb{C}}-\bigcup_i B_i \\ \psi_{i+1}^{-1} \circ f_i \circ \psi_i & \text{on } B_i. \end{cases}$$

It is easily seen that g is continuous, and moreover, quasi-regular.

Let

$$E_0=\bigcup_{i=1}^{p}(S_i-\bar{B}_i), \qquad \Phi_0=\mathrm{id}_{E_0},$$

$$E_i=B_i \cap \psi_i^{-1}(S_i'), \qquad \Phi_i=\psi_i|_{E_i} \qquad (i=1, \ldots, p),$$

and $N=1$. Obviously, $g(E)=E$, where $E=\bigcup_{i=0}^{p} E_i$. As each ψ_i is conformal in a neighborhood of $\bar{\mathbb{C}}-(E \cup \gamma_i)$, $g_{\bar{z}}=0$ a. e. on $\bar{\mathbb{C}}-g^{-1}(E)$. All the conditions in Lemma 1 are satisfied. So there exists a quasi-conformal mapping φ such that $F=\varphi \circ g \circ \varphi^{-1}$ is a rational function. Write $A_i=\varphi(S_i \cap \psi_i^{-1}(S_i'))$. It is easily seen that $A_1, \ldots, A_p$ form an HR-cycle of F of order p with rotation number θ.

To finish the proof of (A), we take rational functions of the forms

$$f_0(z)=z^2+c_0, \qquad f_1(z)=z^2+c_1 \qquad \text{and} \qquad f_2=\ldots=f_p=\mathrm{id}_{\bar{\mathbb{C}}},$$

satisfying (α) and (β) above, for suitable θ. (Such c_i exist. *See* § 1.5.) Counting its critical points, we conclude that the obtained rational function F is of degree 3.

(B) We state only a sketch of the proof, since its details are quite similar to the case (A).

First, choose a rational function

$$f_1(z)=\lambda z(z-1)^2,$$

where $\lambda=e^{2\pi i\theta}$, $\theta \in \mathbb{R}$, such that f_1 has a Siegel disk S_1 with center 0. Let γ_1 be an invariant curve in S_1, and $\gamma_2=f_1^{-1}(\gamma_1)-\gamma_1$. Then γ_2 is a Jordan curve, and $f_1|_{\gamma_2}: \gamma_2 \to \gamma_1$ is a covering of degree 2. Let D_1 be the region bounded by γ_1 and γ_2.

For $0<r<1$, define the following subsets of $\bar{\mathbb{C}}$:

$$D_0'=\{r<|z|<1/r\}, \qquad D_1'=\{r^2<|z|<r\}, \qquad D_2'=\{|z|<r^2\},$$

$$\gamma_1'=\{|z|=r^2\}, \qquad \gamma_2'=\{|z|=r^1\}.$$

Let $f_0(z)=1/z^2$, $A(z)=e^{-i\pi\theta}z/r^4$ and $B(z)=e^{2\pi i\theta}.z$.

Construct a qc-mapping $\psi : D_1 \to D'_1$ satisfying:

(i) $\psi(\gamma_1)=\gamma'_1$, and $\psi \circ f_1 = B \circ \psi$ on γ_1;

(ii) $\psi(\gamma_2)=\gamma'_2$, and $\psi \circ f_1 = A^{-1} \circ f_0 \circ \psi$ on γ_2;

(iii) ψ is conformal in a neighborhood of $D_1 - f_1^{-1}(S_1)$.

If r is sufficiently small, such ψ exist. (Consider the modulus of D_1 and D'_1.) Next, extend ψ to the component of $\bar{\mathbb{C}} - D_1$ bounded by γ_2, as a qc-mapping onto $\{r \leqq |z|\}$, so that

(iv) ψ^{-1} is conformal in $\{r_1 < |z|\}$, where r_1 satisfies $r_1 > r$ and $\{r^2 < |z| \leqq r_1^2\} \subset \psi(D_1 \cap S_1)$.

Define g on $\{|z| < r^{-1}\}$ by

$$g = \begin{cases} f_0 & \text{on } D'_0, \\ A \circ \psi \circ f_1 \circ \psi^{-1} & \text{on } \bar{D}'_1, \\ A \circ B & \text{on } D'_2, \end{cases}$$

and on $\{|z| \geqq r^{-1}\}$ by $g = C \circ g \circ C$, where $C(z) = 1/\bar{z}$.

Then, as in (A), there exist a qc-mapping φ such that $f = \varphi \circ g \circ \varphi^{-1}$ is a rational finctions of degree 4, which has an HR-cycle of order 2 as in Figure 5.

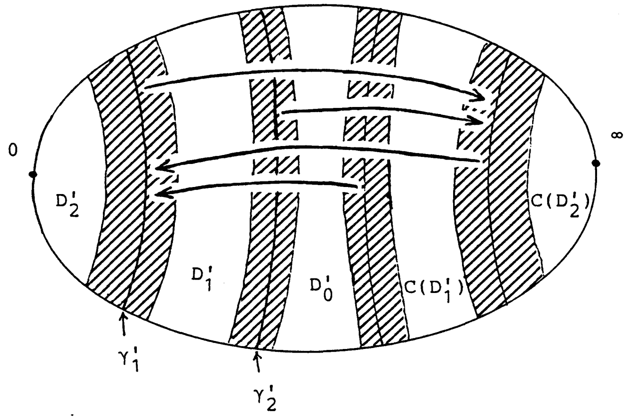

Fig. 5. — Surgery for Case (B).

Remark. — The case (B) gives an example for which the N in Lemma 6 cannot be 0. So it was necessary in paragraph 6 to cut the Riemann sphere by the inverse images of γ_A, not only by γ_A themselves.

Note that we had to pay attention on the modulus of D_1, in Case (B). The moduli will raise a difficulty when one glues up some multiply connected pieces.

Proof of Theorem 6. — Note that if f has a Siegel disk of order p with rotation number θ, then $\overline{f^p(\bar{z})}$ has a Siegel disk of order 1 with rotation number $-\theta$. Hence the theorem follows from Proposition 2 in paragraph 6 and the above (A).

10. Proof of theorem 4

For c a critical point of f, consider the following property:

(pp) c is strictly preperiodic, i. e. $f^{n+p}(c)=f^{n}(c)$ for some n, $p\geqq 1$, but c itself is not periodic, and moreover $f^{n}(c)$ is a repulsive periodic point.

Then c is contained in the Julia set. Let $n_{pp}(f)$ be the number of critical points of f satisfying (pp).

The proof of Theorem 2 and Corollary 2 implies

$$n_{\mathrm{AB}}+n_{\mathrm{PB}}+n_{\mathrm{SD}}+n_{\mathrm{Cremer}}+2\,n_{\mathrm{HR}}+n_{pp}\leqq 2\,(d-1), \tag{10.1}$$

for a rational functions of degree d. In fact, owing to Lemma 4, the perturbations in paragraph 4 can be constructed so that if c is a critical point of f satisfying (pp), then c^{ε} satisfies (pp) for f_{ε}.

Hence, for the proof of Theorem 4, it suffices to prove that for given set of nonnegative integers m_{AB}, m_{rat}, m_{SD}, m_{Cremer}, m_{HR} and m_{pp}, satisfying

$$m_{\mathrm{AB}}+m_{\mathrm{rat}}+m_{\mathrm{SD}}+m_{\mathrm{Cremer}}+2\,m_{\mathrm{HR}}+m_{pp}=2\,(d-1) \tag{10.2}$$

and

$$m_{\mathrm{HR}}\leqq d-2, \tag{10.3}$$

there exists a rational function f of degree d such that $n_{\mathrm{AB}}(f)=m_{\mathrm{AB}}$, etc. [We need only to show $n_{\mathrm{AB}}(f)\geqq m_{\mathrm{AB}}$ etc., since (10.1) and $n_{\mathrm{rat}}\leqq n_{\mathrm{PB}}$ imply the equalities.]

First, consider the case $m_{\mathrm{HR}}=0$.

STEP 1. — Let p, $q\geqq 1$ be integers relatively prime with $d-1$, and λ_1 (resp. λ_2) be a p-th (resp. q-th) prime root of unity, such that $\lambda_1\neq\bar{\lambda}_2$. Define

$$f_{\varepsilon}(z)=z\,.\,\frac{\lambda_1(1+\varepsilon_1)+z^{d-1}}{1+\lambda_2(1+\varepsilon_2)\,z^{d-1}},$$

where $\varepsilon=(\varepsilon_1,\varepsilon_2)\in\mathbb{C}^2$ is small. Comparing the expansions of $f_0^p\circ f_0$ and $f_0\circ f_0^p$, one easily gets for $\varepsilon=(0,0)$

$$f_0^p(z)=z\,.\,[1+c_0\,z^{np}+O(z^{np+1})]\quad\text{as}\quad z\to 0, \tag{10.4}$$

where $c_0\neq 0$, $n\geqq 1$. Let $\mathrm{R}(z)=\exp(2\pi i/m)\,.\,z$, where we write $m=d-1$ for simplicity. Since m and p are relatively prime and $f_0\circ\mathrm{R}=\mathrm{R}\circ f_0$, n is a multiple of m. By the flower theorem (*see* [3]), f_0 has np parabolic basins with the limit point 0, which form n cycles. Combining with a similar expansion at ∞, we conclude that $n=m$, since f_0 has at most $2m$ PB-cycles.

It is easily seen that

$$f^p_{(\varepsilon_1,\varepsilon_2)}(z)=z\,.\,[(1+\varepsilon_1)^p+c(\varepsilon_2)\,z^{mp}+O(\varepsilon_1 z)+O(z^{mp+1})]\quad\text{as}\quad z,\ \varepsilon_1,\ \varepsilon_2\to 0, \tag{10.5}$$

where $c(\varepsilon_2)$ is a holomorphic function of ε_2, with $c(0)=c_0$. Let $f_\varepsilon^p(z)=z.\mathrm{F}(z,\varepsilon)$ and consider the equation

$$(10.6) \qquad \mathrm{F}(z,\varepsilon)=1$$

near $z=0$, for small ε. Clearly, (10.6) has mp roots

$$\zeta_{j,k}^\varepsilon=\exp\left(\frac{2\pi k}{m}i\right).f_\varepsilon^j(\zeta^\varepsilon) \qquad (j=0,\ldots,p-1,\ k=0,\ldots,m-1).$$

Define $\alpha_1(\varepsilon)=(f_\varepsilon^p)'(\zeta^\varepsilon)$. Then α_1 is independent of the choice of ζ^ε and holomorphic, since

$$[\alpha_1(\varepsilon)]^m=\prod_{j,k}(f_\varepsilon)'(\zeta_{j,k}^\varepsilon)$$

is a rational function of the coefficients in (10.6). Define similarly $\alpha_2(\varepsilon)$ for q-periodic points near $z=\infty$.

A simple computation shows that

$$\alpha_1(\varepsilon_1,\varepsilon_2)=1-mp^2\varepsilon_1+o(\varepsilon_1) \quad \text{as } \varepsilon_1,\ \varepsilon_2\to 0.$$

Thus we obtain

$$\frac{\partial(\alpha_1,\alpha_2)}{\partial(\varepsilon_1,\varepsilon_2)}(0)=\begin{pmatrix}-mp^2 & 0\\ 0 & -mq^2\end{pmatrix},$$

hence $(\varepsilon_1,\varepsilon_2)\to(\alpha_1,\alpha_2)$ is a local diffeomorphism with $\alpha_i(0)=1$. Let θ_1, θ_2 be irrational numbers satisfying the Diophantine condition (1.1) and sufficiently close to 0. Then there exists $\varepsilon=(\varepsilon_1,\varepsilon_2)$ such that $\alpha_j(\varepsilon)=\exp(2\pi i\theta_j)$ $(j=1,2)$. Obviously, $f=f_\varepsilon$ has $2(d-1)$ Siegel cycles of order p or q. We may take $p=1$ or $q=1$.

STEP 2. — Consider a rational function f with $n_{\mathrm{SD}}(f)\geqq 2$. Let $\{z_i: i=0,\ldots,p-1\}$ and $\{z'_j: j=0,\ldots,q-1\}$ be Siegel cycles of f. Suppose that the rotation number of the SD-cycle with centers $\{z_i\}$ satisfies (1.1).

We may assume that $f(\infty)=z_0$ and $z_{p-1}\neq\infty$. Take a polynomial $h(z)$ such that

$$h(z'_j)=0,\ h'(z'_j)=1;$$

$h(z)=h'(z)=0$, if z is a non-repulsive periodic point of f other than $\{z'_j\}$;

$h(z)=0$, if z is a forward orbit of a critical point satisfying (pp).

Define H_ε, g_ε and V_ε as in paragraph 4, but now $\varepsilon\in\mathbb{C}$. By Siegel [15], there exist δ, $\varepsilon_0>0$ such that if $|\varepsilon|<\varepsilon_0$, g_ε has a Siegel disk S_ε containing $\{z:|z-z_0|<\delta\}$. Let $\mathrm{E}_\varepsilon=\mathrm{S}_\varepsilon\cup\ldots\cup g_\varepsilon^{p-1}(\mathrm{S}_\varepsilon)$. Since $g_\varepsilon(\bar{\mathrm{V}}_\varepsilon)\subset\{|z-z_0|<\delta\}\subset\mathrm{E}_\varepsilon$ for small ε, the same argument as paragraph 4 yields f_ε and φ_ε. The multiplicator of $(z'_j)^\varepsilon=\varphi_\varepsilon(z'_j)$ is $(1+\varepsilon)^p.(f^p)'(z'_j)$. Hence the cycle $\{z'_j\}$ can be perturbed as one likes, with other non-repulsive cycles and critical points satisfying (pp) unchanged. Thus we can reduce n_{SD} by one, and increase n_{AB} (or n_{rat}, n_{Cremer}) by one. (Concerning the Cremer cycle, *see* paragraph 1.5.)

STEP 3. — Consider again g_ε and f_ε above. We show that for suitable ε, a critical point c^ε newly comes to satisfy (*pp*) for f_ε, hence $n_{pp}(f_\varepsilon) \geqq n_{pp}(f)+1$.

Assume that such ε does not exist. Let us consider a family of analytic functions

$$\{ g_\varepsilon|_{U} : |\varepsilon| < \varepsilon_1 \},$$

where we take $U = \{ |z| < R \}$ and ε_1 satisfying $0 < \varepsilon_1 < \varepsilon_0$, $V_\varepsilon \subset \bar{\mathbb{C}} - U$ and $g_\varepsilon(\bar{\mathbb{C}} - U) \subset S_\varepsilon$ for $|\varepsilon| < \varepsilon_1$. Let ζ be a repulsive periodic point of f, not contained in any orbit of critical points. There exist $\varepsilon_2 < \varepsilon_1$ and an analytic function $\zeta(\varepsilon)$ of ε in $\{ |\varepsilon| < \varepsilon_2 \}$ such that $\zeta(0) = \zeta$ and $\zeta(\varepsilon)$ is a repulsive periodic point of $g_\varepsilon|_U$. By the assumption, we obtain branches of $\{ g_\varepsilon^{-n}(\zeta(\varepsilon)) \}$, analytic in $\{ |\varepsilon| < \varepsilon_2 \}$, since $g_\varepsilon^n(\bar{\mathbb{C}} - U)$ are contained in E_ε and do not intersect with $\zeta(\varepsilon)$. As Lemma III.2 in Mañé-Sad-Sullivan [14], one can prove that $\{ g_\varepsilon \}$, hence $\{ f_\varepsilon \}$ are J-stable. This contradicts with the fact that the multiplicator of $(z'_j)^\varepsilon$ actually varies with ε.

Thus we can increase n_{pp} by one.

STEP 4. — Let f_0 be a rational function of degree d with $n_{SD}(f_0) \geqq 1$. Write

$\mathscr{R}_d$ = the space of all rational functions of degree d, which is embedded in $\mathbb{CP}^{2d+1}$ as an open set (by considering coefficients). Fix a Siegel cycle $z_0, \ldots, z_{p-1}$ of f_0.

Define hypersurfaces of relations H_ζ and H'_c as follows. Let ζ be an indifferent periodic point of period q with multiplicator μ. There exist small neighborhoods U of ζ in $\bar{\mathbb{C}}$ and W of f_0 in $\mathscr{R}_d$, such that

$$H_\zeta = \{ f \in W \mid f \text{ has a unique } q\text{-periodic point } z \text{ in } U, \text{ and its multiplicator is } \mu \}$$

is an analytic variety in W. Let c be a critical point of f_0 satisfying (*pp*) with $f_0^{n+q}(c) = f_0^n(c)$. There exist small neighborhoods U′ of c in $\bar{\mathbb{C}}$ and W of f_0 in $\mathscr{R}_d$, such that

$$H'_c = \{ f \in W \mid f \text{ has a unique critical point } c' \text{ in } U', \text{ and } c' \text{ satisfies } f^{n+q}(c') = f^n(c') \}$$

is an anlytic variety in W. Define the intersection

$$X = (\cap H_\zeta) \cap (\cap H'_c),$$

for all indifferent periodic points ζ of f_0 other than z_i and for all critical points c of f_0 satisfying (*pp*), with common W. Then X is an analytic variety in W.

On the other hand, if W is small enough, there exists an analytic function $z(f)$ of f in W, satisfying

$$f^p(z(f)) = z(f) \qquad \text{and} \qquad z(f_0) = z_0.$$

Let $\alpha(f)$ be the multiplicator of $z(f)$ for f, which is a holomorphic function of $f \in W$.

This α is not constant, since we can perturb f_0 as in paragraph 4 to make $z(f)$ attractive, with the relations corresponding to H_ζ and H'_c unchanged. So $\alpha|_X$ is an open map. Hence, considering the multiplicator, one can pertub z_0 as one likes to reduce n_{SD}

by one and increase n_{AB} (or n_{rat}, n_{Cremer}) by one. We may use this method instead of Step 2.

Combining Step 1-4, we can prove the assertion in the case $m_{AB}+m_{rat}+m_{SD}+m_{Cremer}>0$ and $m_{HR}=0$. If $m_{pp}=2(d-1)$, it suffices to see the function

$$z \to \lambda . \left(\frac{z-2}{z}\right)^d, \qquad \text{where} \quad \lambda=\frac{2}{1-\zeta}, \quad \zeta^d=1, \quad \zeta \neq 1.$$

Thus, we get the conclusion in all the cases where $m_{HR}=0$.

Now, consider the case $m_{HR}>0$. Let $M=m_{HR}$.

Suppose that a rational function f_0 (or f_M) has a Siegel disk of order 1 with rotation number $-\theta_1$ (resp. θ_M) which satisfies (1.1). Let $\theta_2, \ldots, \theta_{M-1}$ be irrational numbers satisfying (1.1) and $\lambda_i \neq \lambda_{i+1}$ $(i=1, \ldots, M-1)$, where $\lambda_j=\exp(2\pi i \theta_j)$. Set

$$f_j(z)=z . \frac{\lambda_j+z}{1+\bar{\lambda}_{j+1} z} \qquad (j=1, \ldots, M-1).$$

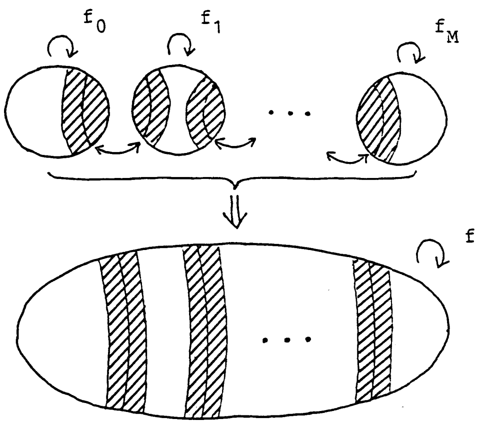

Fig. 6. — Surgery yielding M Herman rings.

Each f_j has two Siegel disks of order 1 with rotation numbers θ_j and $-\theta_{j+1}$. Glue up $f_0, \ldots, f_M$ by a surgery as in (A) of paragraph 9 (see *Fig.* 6). Obviously, the obtained rational function f has M Herman rings of order 1 with rotation number $\theta_1, \ldots, \theta_M$; hence

(10.7) $$n_{HR}(f) \geqq M.$$

Moreover,

(10.8) $$n_{SD}(f) \geqq n_{SD}(f_0)+n_{SD}(f_M)-2$$

and

(10.9) $$n_{AB}(f) \geqq n_{AB}(f_0)+n_{AB}(f_M), \quad \text{etc.}$$

On the other hand, counting critical points, one obtains

$$2(\deg f-1)=2(\deg f_0-1)+2(\deg f_M-1)+2(M-1).$$

Assume that the equality in (10.1) holds for f_0 and for f_M. Combining the above results, we easily deduce that the equalities in (10.7)-(10.9) hold. Using the result for the case $m_{HR}=0$, we can choose suitable f_0 and f_M so that f satisfies

$$\deg f=d \quad \text{and} \quad n_{AB}(f)=m_{AB}, \quad \text{etc.}$$

Thus the proof of Theorem 4 is completed.

Remark. — If one needs a superattractive basin, it can be made from attractive basins by a surgery as Example in paragraph 3.

Notice that we can construct f so that all its Herman rings and at least $d-M-1$ cycles of non-repulsive periodic points are of order (or period) 1.

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