A NOTE ON MUMFORD-TATE GROUPS AND DOMAINS

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ABSTRACT. We use some classical results on nonabelian Galois cohomology to explain Mumford-Tate groups and Domains in Hodge theory, as automorphism groups and moduli of polarized Hodge structures.

1. INTRODUCTION

In this short note we remark some connections between non-abelian Galois cohomology H^1 and basic symmetry groups of Hodge tensors. The observation is comparison of classical results on Galois cohomology in 'Local fields' of J. Serre and Mumford-Tate groups in 'Mumford-Tate Domains' by M. Green-P Griffiths-M. Kerr. We first briefly introduce non-abelian cohomology of groups as the cohomology of groups with coefficients in non-commutative G-module, as a set with a distinguished element, namely trivial element. Then we explain a Galois descent procedure on Hodge tensors which connects the above theory to Hodge theory. Mumford-Tate groups can be understood as basic symmetry groups of Hodge structures. Mumford-Tate domains parametrize the set of Hodge structures whose generic points have a fixed Mumford-Tate group.

2. NON-ABELIAN COHOMOLOGY

Let G be a group and A (not necessarily abelian) another group on which G acts on the left. Write A multiplicatively. $H^0(G, A)$ is by definition the group A^G of elements of A fixed by G. A 1-cocycle would be a map $s \mapsto a_s$ from $G \to A$ such that $a_{st} = a_s.s(a_t)$. Two cocycles a_s and b_s are equivalent if there exists $a \in A$ such that $b_s = a^{-1}.a_s.s(a)$ for all $s \in G$. This defines an equivalence relation and the quotient set is denoted by $H^1(G, A)$. It is a pointed set with a distinguished element of the unit cocycle $a_s = 1$. Here $s(-)$ means the action of $s \in G$, and a_s is the value of the cocycle a at $s \in G$. These two definitions agree with the usual definitions of cohomology of G when A is abelian. These constructions are also functorial in A and G. If $0 \to A \to B \to C \to 0$ is an exact sequence of non-abelian G-modules, then we have the following exact sequence of pointed sets

$$H^0(G, A) \to H^0(G, B) \to H^0(G, C) \to H^1(G, A) \to H^1(G, B) \to H^1(G, C)$$

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when the group G is Galois group of a (not necessarily finite) field extension and A a topological G-module such that

$$A = \bigcup A^H$$

when H runs through the open normal subgroups of G, we define

$$H^1(G, A) := \lim_{\rightarrow} H^1(G/H, A^H)$$

There is a simple geometric interpretation for $H^1(G, A_K)$: It is the set of classes of principal homogeneous spaces for A, defined over k which have a rational point over K.

As an example, take $G = \mathbb{Z}/2 = Gal(\mathbb{C}/\mathbb{R})$ acting naturally on a set of tensors $A_{\mathbb{R}}$. Then any element in $H^1(\mathbb{Z}/2, A_{\mathbb{C}})$ is determined by an involution map $-1 : a_{-1} \mapsto * \in A_{\mathbb{R}}$. This shows

$$H^1(\mathbb{Z}/2, A_{\mathbb{C}}) = A_{\mathbb{R}}$$

3. Galois descent for Hodge tensors

If V be a vector space over k, provided with a fixed tensor of x of type (p, q), i.e. $x \in \bigotimes^p V \otimes \bigotimes^q V^*$ where V^* is the dual of V. two pairs $(V, x), (V, x')$ are called k-isomorphic if there is a k-linear isomorphism $f : V \to V'$ such that $f(x) = x'$. Denote by A_k the group of these automorphisms. Let K/k be a Galois extension with Galois group G. Write $E_{V,x}(K, k)$ for the set of k-isomorphism classes that are K-isomorphic to (V, x). The group G acts on V_K by $s.(x \otimes \lambda) = x \otimes s.\lambda$. It also acts on $f : V \to V'$ by $s.f = s \circ f \circ s^{-1}$. If we put

$$p_s = f^{-1} \circ s \circ f \circ s^{-1}, \qquad s \in G$$

the map $s \mapsto p_s(f)$ is a 1-cocycle in $H^1(G, A_K)$.

Theorem 3.1. *The map $\theta : E_{V,x}(K/k) \to H^1(G, A_K)$ defined by*

$$f \mapsto p_s(f)$$

is a bijection.

Theorem 3.2. *The set $H^1(G, Aut(Q, K))$ is in bijective correspondence with the classes of quadratic k-forms that are K-isomorphic to Q.*

The above definition can be generalized in this way that instead of considering a single tensor $T^{p,q} = \bigotimes^p V \otimes \bigotimes^q V^*$ one may consider a a sum of such tensors that is a subset T as

$$T \subset T^{\bullet,\bullet} = \oplus_{p,q} T^{p,q}$$

The proofs will proceed exactly the same and we obtain

Theorem 3.3. *Assume T is a set of tensors for a vector space V. The map $\theta : E_{V,T}(K/k) \to H^1(G, A_{T,K})$ defined by*

$$f_T \mapsto p_s(f_T)$$

is a bijection, where $A_{T,K}$ is the group of K-automorphisms of all the tensors in T.

4. Mumford-Tate groups and Domains

Mumford-Tate groups are basic symmetry groups of Hodge structures. To begin with let V be finite dimensional $\mathbb{Q}$-vector space, and Q a non-degenerate bilinear map $Q : V \otimes V \to \mathbb{Q}$ which is $(-1)^n$-symmetric for some fixed n. A Hodge structure is given by a representation

$$\phi : \mathbb{U}(\mathbb{R}) \to Aut(V, Q)_{\mathbb{R}}, \qquad \mathbb{U}(\mathbb{R}) = \left(\begin{array}{cc} a & -b \\ b & a \end{array} \right), \ a^2 + b^2 = 1$$

It decomposes over $\mathbb{C}$ into eigenspaces $V^{p,q}$ such that $\phi(t).u = t^p \bar{t}^q.u$ for $u \in V^{p,q}$, and $\overline{V^{p,q}} = V^{q,p}$. The Hodge tensors $Hg_\phi^{a,b}$ are given by the subspace of $T^{a,b}$ such that $\mathbb{U}(R)$ acts trivially. Set

$$Hg_\phi^{\bullet,\bullet} = \oplus_{a,b} Hg_\phi^{a,b}$$

The period domain D, associated to the above data is the set of all polarized Hodge structures ϕ with a given Hodge numbers. The real Lie group $G(\mathbb{R})$ acts transitively on D. The compact dual $\check{D}$ of D is the set of flags $F^\bullet = \{F^n \subset ... \subset F^0 = V_{\mathbb{C}}\}$ with $\dim F^p = \sum_{r \geq p} h^{r,s}$ and where the first Hodge-Riemann bilinear relation, $Q(F^p, F^{n-p+1}) = 0$ holds . The Mumford-Tate group of the Hodge structure ϕ denoted $M_\phi(R)$ is the smallest $\mathbb{Q}$-algebraic subgroup of $G = Aut(V, Q)$ with the property

$$\phi(\mathbb{U}(R)) \subset M_\phi(R)$$

M_ϕ is a simple, connected, reductive $\mathbb{Q}$-algebraic group. If $F^\bullet \in \check{D}$ the Mumford-Tate group $M_{F^\bullet}$ is the subgroup of $G_{\mathbb{R}}$ that fixes the Hodge tensors in $Hg_{F^\bullet}^{\bullet,\bullet}$.

5. Relation with non-abelian cohomology

We are going to investigate the relation between nonabelian cohomology discussed in sec. 2, and classifying spaces for Hodge structures. By definition $H^0(G_{\mathbb{R}}, Hg_{F^\bullet}^{\bullet,\bullet}) = M_{F^\bullet}$. The relation $H^1(\mathbb{Z}/2, G_{\mathbb{C}}) = G_{\mathbb{R}}$ is trivial. By applying Theorem 3.3 to the above construction we get the following results.

Theorem 5.1. $H^1(\mathbb{Z}/2, M_{F^\bullet}) = M_\phi$.

This follows from the example at the end of sec 2, and the fact that M_ϕ is the subgroup of G that fixes pointwise the algebra of Hodge tensors, cf. [MGK].

Theorem 5.2. $H^1(Gal(\mathbb{C}/\mathbb{Q}), Aut(Q,\mathbb{C})) = \check{D}$

Follows from Theorem 3.3, noting that $\check{D} = G_{\mathbb{C}}/Stab_{G_{\mathbb{C}}}(F_0)$.

Theorem 5.3. $H^1(Gal(\mathbb{R}/\mathbb{Q}), Aut(Q,\mathbb{R})) = D$

This also follows from Theorem 3.3, and similar identity $D = G_{\mathbb{R}}/Stab_{G_{\mathbb{R}}}(F_0)$. . We have the short exact sequence

$$0 \to \mathbb{Z}/2 \to Gal(\mathbb{C}/\mathbb{Q}) \to Gal(\mathbb{R}/\mathbb{Q}) \to 0$$

where the second non-zero map is the restriction

$$\begin{aligned} 0 \to H^1(\mathbb{Z}/2, Aut(Q,\mathbb{C})^{Gal(\mathbb{R}/\mathbb{Q})}) &\to H^1(Gal(\mathbb{C}/\mathbb{Q}), Aut(Q,\mathbb{C})) \\ &\to H^1(Gal(\mathbb{R}/\mathbb{Q}), Aut(Q,\mathbb{C}))^{\mathbb{Z}/2} \end{aligned}$$

where the second map is the restriction. The first map is called inflation map. The first item is $Aut(Q,\mathbb{R})$ by the discussion in section 1, and the second item is $\check{D}$ by Theorem 5. Thus we have

$$0 \to Aut(Q,\mathbb{R}) \to \check{D} \to H^1(Gal(\mathbb{R}/\mathbb{Q}), Aut(Q,\mathbb{C}))^{\mathbb{Z}/2}$$

as exact sequence of sets with distinguished unit elements.

Theorem 5.4. $H^1(\mathbb{U}_{\mathbb{R}}, G_{\mathbb{R}}) = D$, *where* $G_{\mathbb{R}}$ *acts on* $G_{\mathbb{R}} = Aut(Q,\mathbb{R})$ *by* $g : T \mapsto g^{-1}Tg$.

Proof. The cocycle condition is equivalent to $\mathbb{U}_{\mathbb{R}} \to G_{\mathbb{R}}$ being a homomorphism and the boundary condition is when two such homomorphism are conjugate by an element of $G_{\mathbb{R}}$. Then, the theorem is consequence of that, D is isomorphic to the set of conjugacy classes of the isotropy group of a fixed Hodge structure. □

For each point $\phi \in D$, the adjoint representation

$$Ad\phi : \mathbb{U}(\mathbb{R}) \to Aut(\mathfrak{g}_{\mathbb{R}}, B)$$

induces a Hodge structure of weight 0 on $\mathfrak{g}$. This Hodge structure is polarized by the killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. and it is a sub-Hodge structure of $\check{V} \times V$.

Theorem 5.5. $H^1(\mathbb{U}(\mathbb{R}), Aut(\mathfrak{g}_{\mathbb{R}}, B)) = D$, *where* $\mathbb{U}_{\mathbb{R}}$ *acts by adjoint representation, and* B *is the killing form.*

Proof. Checking the cocycle condition shows that cocycles are $Hom(\mathbb{U}_{\mathbb{R}}, Aut(B))$, and the coboundray condition becomes when two such homomorphisms are conjugate by an automorphism of B. Regarding B as a tensor then the theorem follows from the known fact that M_ϕ is the subgroup of G with the property that M_ϕ-stable subspaces $W \subset T_\phi^{a,b}$ are exactly the sub-Hodge structures of these tensor space. □

References

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