

A NOTE ON MUMFORD-TATE GROUPS AND DOMAINS II

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ABSTRACT. We explain a higher structure on Kato-Usui compactification of Mumford-Tate domains as toric stacks. As a motivation the universal characteristic cohomology of Hodge domains can be described as cohomology of stacks which have better behaviour in general.

1. INTRODUCTION

Assume D is a period domain of pure Hodge structures defined by Griffiths, with the period map $(\Delta^*)^n \rightarrow \Gamma \backslash D$, where Γ is the monodromy group. We treat a partial compactification of $\Gamma \backslash D$ so that the period map is extended over Δ^n . Kato and Usui generalize the toroidal compactifications for any period domain D , which is not Hermitian symmetric in general and show these are moduli spaces of log Hodge structures. This partial compactification is given by using toroidal embedding associated to the cone generated by the data of the monodromies, [TH].

Mumford-Tate domain can be considered as a toric stack, by a definition using Stacky-fans. This structure is compatible with the Kato-Usui generalized toroidal compactification. In this way we may regard some additional data on MT-domains which preserves the stabilizers of the points under a Lie group action. There are several definitions of toric Stacks. A definition due to Lafforgue, defines a toric stack to be the stack quotient of a Toric variety by its torus. The second, Borisov-Chen-Smith, define smooth toric Deligne-Mumford stacks. These stacks have smooth toric varieties as their coarse moduli space. The third, due to Tyomkin, includes all toric varieties, which can be singular, [GS].

We define a toric stack to be the stack quotient of a normal toric variety X , by a subgroup G of its Torus T_0 , cf. [GS]. We begin with the ordinary definitions in the theory of toric varieties, and then try to upgrade the definition for higher structures. The main task of this text is to explain how to express the definition of a toric stack for period domains of Griffiths, Mumford-Tate domains, their boundary points

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via the Kato-Usui generalized toroidal compactification. As an application one can consider the universal characteristic cohomology of Mumford-Tate domains as a type of stack cohomology.

2. TORIC VARIETIES AND THE MOMENT MAP

In this section we briefly explain basic definitions in the theory of toric varieties mainly extracted from [WF]. Toric varieties primarily came up in connection with the study of compactification problems. This compactification simply says a toric variety X , is a normal variety that contains a torus T as a dense open subset, together with an action

$$(1) \quad T \times X \longrightarrow X$$

of T on X that extends the natural action of T on itself. The torus T is $\mathbb{C}^* \times \dots \times \mathbb{C}^*$. The simplest example is the projective space \mathbb{P}^n as the compactification of \mathbb{P}^n . There are several equivalent ways to construct such varieties, that we explain some of them below.

A toric variety may be constructed from a lattice N which is a collection of strongly convex rational polyhedral cones, σ in the real vector space $N_{\mathbb{R}} = N \otimes \mathbb{R}$. Let $M = \text{Hom}(N, \mathbb{Z})$ denote the dual lattice with dual pairing $\langle \cdot, \cdot \rangle$. If σ is a cone in N , let σ^\vee be its dual in $M_{\mathbb{R}}$. Then setting,

$$(2) \quad S_\sigma = \sigma^\vee \cap M = \{u \in M \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}$$

defines the toric variety as

$$(3) \quad U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma])$$

where by $\mathbb{C}[S_\sigma]$ we mean $\mathbb{C}[x^{[S_\sigma]}]$. If τ is a face of σ then the above procedure identifies $U_\tau \hookrightarrow U_\sigma$ as an open subset. Thus the faces of σ provide an open cover of U_σ . A basic example is to take the cone to be $0 \in \mathbb{R}^n$. It is a face of every other cone. The dual lattice M is generated by the standard generators $\mp e_1, \dots, \mp e_n$. Take X_1, \dots, X_n the elements corresponded to the dual basis in $\mathbb{C}[M]$. The $\mathbb{C}[M] = \mathbb{C}[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ which is the affine ring of the torus $U_0 = T = (\mathbb{C}^*)^n$. Thus every toric variety contains the torus as an open subset. if σ is the cone generated by e_1, \dots, e_n , then $\mathbb{C}[S_\sigma] = \mathbb{C}[X_1, \dots, X_n]$.

For any cone σ in a lattice N , the corresponding affine variety U_σ has a distinguished point which we denote by x_σ . This point in U_σ is given by a map of semi-groups

$$(4) \quad u \rightarrow \begin{cases} 1 & \text{if } u \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases}$$

This is well-defined since σ^\perp is a face of σ^\vee , which implies that the sum of two elements in σ^\vee can not be in σ^\perp unless both are in σ^\perp . The point x_σ is independent of $\gamma \supset \sigma$. A cone σ is called non-singular if it is generated by part of a basis for the lattice N . This implies the affine toric variety is non-singular.

Given a lattice N with dual M , we have the corresponding torus $T_N = \text{Hom}(M, \mathbb{G}_m)$. Then one knows that $\text{Hom}(\mathbb{G}_m, T_N) = \text{Hom}(\mathbb{Z}, N) = N$. The torus $T_N = \text{Spec}(\mathbb{C}[M]) = \text{Hom}(M, \mathbb{C}^*) = N \otimes \mathbb{C}^*$ acts on U_σ as follows. A point $t \in T_N$ is identified with a map $M \rightarrow \mathbb{C}^*$, and a point $x \in U_\sigma$ with a map $S_\sigma \rightarrow \mathbb{C}$ of semigroups. Then $t.x$ is the map of semigroups $u \rightarrow t(u)x(u)$. It follows that, every 1-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow T_N$ is given by a unique $v \in N$. We denote by λ_v the 1-parameter subgroup corresponding to v . Dually $\text{Hom}(T_N, \mathbb{G}_m) = \text{Hom}(N, \mathbb{Z}) = M$. Every character $\chi : T_N \rightarrow \mathbb{G}_m$ is given by a unique $u \in M$. The character corresponding to u can be identified with the function $z^u = \chi^u$ in the coordinate ring $\mathbb{C}[M] = \Gamma(T_N, \mathcal{O}^*)$. Then $\lambda_v(z)(u) = \chi^u(\lambda_v(z)) = z^{\langle u, v \rangle}$. Given a cone σ , for $v \in |\Delta|$ that contains v in its interior, $\lim_{z \rightarrow 0} \lambda_v(z) = x_\sigma$. If v is not in any cone of Δ this limit does not exist. In other words $\sigma \cap N$ is characterized as the set of v for which $\lambda_v(z)$ has a limit as $z \rightarrow 0$.

As with any set on which a group acts, a toric variety X is the disjoint union of its orbits by the action of the torus T_N . There is one such orbit O_τ for each cone $\tau \in \Delta$. If τ has maximal rank, then O_τ is a point x_τ . if $\tau = 0$ then $O_\tau = T_N$. O_τ is an open subvariety of its closure $V(\tau)$, which is a closed toric subvariety of X . $V(\tau)$ is the disjoint union of those orbits O_γ for which $\gamma \supset \tau$. Thus $O_\tau = \text{Hom}(\tau^\perp \cap M, \mathbb{C}^*)$. define N_τ to be the sublattice of N generated by $\tau \cap N$, and $N(\tau) = N/N_\tau$, $M(\tau) = \tau^\perp \cap M$. Then define

$$(5) \quad O_\tau = T_{N(\tau)} = \text{Hom}(M(\tau), \mathbb{C}^*) = \text{Spec}(\mathbb{C}[M(\tau)]) = N(\tau) \otimes \mathbb{C}^*$$

This is a torus of dimension $n - \dim(\tau)$, on which T_N acts by $T_N \rightarrow T_{N(\tau)}$. The star of τ is defined as the set of cones σ in Δ that contain τ as a face. Such cones σ are determined by their images in $N(\tau)$,

$$(6) \quad \bar{\sigma} = \sigma + (N_\tau)_\mathbb{R} / (N_\tau)_\mathbb{R} \subset N_\mathbb{R} / (N_\tau)_\mathbb{R} = (N_\tau)_\mathbb{R}$$

These cones $\{\bar{\sigma} : \tau \subset \sigma\}$ form a fan in $N(\tau)$, and we denote this fan by $Star(\tau)$. One knows that $V(\tau) = X(Star(\tau))$.

3. STACKS VS ARTIN AND DELIGNE-MUMFORD STACKS

In this section we discuss what is the point of view in comparing the ordinary notion of schemes with the corresponding stack object. We avoid of having a strict mathematical language, and try to give the intuitive idea behind the concept of a stack. This section is a brief from the article [BN] which I suggest to every one to look at.

Stacks were introduced by Grothendieck to provide a general framework for studying local-global phenomena in mathematics. The early stages of the development of the theory can be traced out in the Ph.D thesis of the students of Grothendieck's students. Later it was used by Deligne and Mumford, that what is now called Deligne-Mumford stack. Later E. Artin generalized Deligne-Mumford work in which it became a vital tool in algebraic geometry, specially in the study of quotient spaces. Every scheme is a Deligne-Mumford (DM-)stack and every DM-stack is an Artin stack, in which all are also called algebraic stacks. Algebraic stacks are a new breed of spaces for algebraic geometer, providing greater flexibility for performing constructions which are impossible in the category of schemes. The notions of analytic, differentiable, and topological stacks was introduced analogously in the corresponding categories.

A topological space is naturally a topological stack. Main classes of examples can be obtained from a topological group acting continuously on a topological space X , in which we associate what is called the quotient stack of the action, and is denoted by $[X/G]$. The quotient stack $[X/G]$ is better behaved than X/G , and retains much more information, specially when the action as fixed points or misbehaved orbits. For instance $[X/G]$, in some sense remembers all the stabilizer groups of the action, while X/G is blind to them. There is a natural morphism $\pi_{mod} : [X/G] \rightarrow X/G$ which allows us to compare the stack $[X/G]$ with the coarse moduli X/G . One can produce lots of examples by gluing quotient stacks. We call such stack locally quotient stack. Such topological stacks are called Deligne-Mumford. A topological stack is uniformizable if it is of the form $[X/G]$, where G is a discrete group acting properly discontinuously on a topological space X . Thus every DM-stack is locally uniformizable. There are examples of DM-stacks that are not globally uniformizable.

Every topological stack \mathcal{X} has an underlying topological space called coarse moduli space denoted X_{mod} . There is a natural functorial map $\pi_{mod} : \mathcal{X} \rightarrow X_{mod}$ called the

moduli map. Roughly X_{mod} is the best approximation of \mathcal{X} by a topological space. Assume that $\mathcal{X} = [X/G]$ is a quotient stack. Then there is a natural quotient map $q : X \rightarrow [X/G]$, and this maps makes X a principal G -bundle over $[X/G]$. So in particular $q : X \rightarrow [X/G]$ is a Serre fibration. the usual quotient map we know from topology is the composition $\pi_{mod} \circ q$. A basic example is when a topological group G acts on a point $*$, trivially. The quotient stack $[*/G]$ of this action is called classifying stack of G , and is denoted BG . Note that $(BG)_{mod}$ is a point. However BG is far from being a trivial object. More precisely, the map $* \rightarrow BG$ makes $*$ into a principal G -bundle over BG , and this universal G -bundle is universal. That is for every topological space T , the equivalence classes of morphisms $T \rightarrow BG$ are in bijection with the isomorphism classes of principal G -bundles over T . In this situation if G is discrete, the quotient map $* \rightarrow BG$ becomes the universal cover of BG .

We can think of \mathcal{X} as a topological space X_{mod} which at every point x is decorated with a topological group I_x . The group I_x is called the stabilizer or inertia group at x . These inertia groups are intertwined in an intricate way along X_{mod} . When \mathcal{X} is Deligne-Mumford, all I_x are discrete. At every point $x \in \mathcal{X}$ we have a pointed map $(BI_x, x) \rightarrow (\mathcal{X}, x)$. A basic example can be the sphere S^2 with an action of \mathbb{Z}_n as rotations fixing the north and south poles. the stack $[S^2/\mathbb{Z}_n]$ has an underlying space which is homeomorphic to a sphere, however $[S^2/\mathbb{Z}_n]$ remembers the stabilizers at the two fixed points, namely the north and the south poles. Thus $[S^2/\mathbb{Z}_n]$ is like $B\mathbb{Z}_n$ at the fixed points and in the remaining points is like the sphere.

A groupoid is a category such that any morphism between two objects is invertible. If a group acts on a set X one may form the action groupoid representing this groupoid action, by taking the objects to be the elements of X and the morphisms to be elements $g \in G$ such that $g.x = y$ and compositions to come from the binary operation in G . More explicitly the action groupoid is the set $G \times X$ (often denoted $G \ltimes X$) such that the source and target maps are $s(g, x) = x$, $t(g, x) = gx$. The action ρ is equivalently thought of as a functor $\rho : BG \rightarrow Sets$, from the group G regarded as a one-object groupoid, denoted by BG . This functors sends the single object to the set X . let Set_* be the category of pointed sets and $Set_* \rightarrow Sets$ be the forgetful functor. We can think of this as a universal set-bundle. Then the action groupoid is the pullback

$$(7) \quad \begin{array}{ccc} [X/G] & \longrightarrow & Set_* \\ \downarrow & & \downarrow \\ BG & \longrightarrow & Sets \end{array}$$

The notion of groupoids and their action can be generalized over arbitrary schemes, as Picard stacks with similar definitions, [FMN].

4. TORIC STACKS BY STACKY-FANS

Toric stacks are examples of locally quotient stacks having toric structures. There are several ways to express the definition, however we follow the reference [GS]. A toric stack is the stack quotient of a normal toric variety X , by a subgroup G of its torus T_0 . the stack $[X/G]$ has a dense open torus $T = T_0/G$ which acts on $[X/G]$. Such toric stacks have trivial generic stabilizers.

Definition 4.1. *A toric stack is an Artin Stack of the form $[X/G]$, together with the action of the torus $T = T_0/G$. A non-strict toric stack is an Artin stack which is isomorphic to an integral closed torus-invariant substack of a toric stack, i.e. is of the form $[Z/G]$, together with the action of the stacky torus $[T'/G]$.*

Taking G to be trivial gives rises to the definition of a Toric variety. Also taking $G = T_0$ gives the Lafforgue's definition stated in the introduction. Toric stacks in the sense of Tyomkin would also be toric stacks in our definition.

Definition 4.2. *A stacky fan is a pair (Σ, β) , where Σ is a fan on a lattice L and $\beta : L \rightarrow N$ is a homomorphism to a lattice N , so that $\text{coker}(\beta)$ is finite.*

A stacky fan gives rise to a toric stack as follows. Let X_Σ be the toric variety associated to Σ . The map $\beta^* : N^* \rightarrow L^*$ induces a homomorphism of Tori, $T_\beta : T_L \rightarrow T_N$, naturally identifying β with the induced map on lattices of 1-parameter subgroups. Since $\text{coker}(\beta)$ is finite, β^* is injective, so T_β is surjective. Let $G_\beta = \ker T_\beta$. Note that T_L is the torus on X_Σ , and $G_\beta \subset T_L$ is a subgroup.

Definition 4.3. *If (Σ, β) is a stacky fan, we define the toric stack $\mathcal{X}_{\Sigma, \beta}$ to be $[\mathcal{X}/G_\beta]$, with the torus $T_N = T_L/G_\beta$.*

Every toric stack arises from a stacky fan, since every toric stack is of the form $[X/G]$, where X is a toric variety and $G \subset T_0$ is a subgroup of its torus. Associated to X is a fan Σ on the lattice $L : \text{Hom}(\mathbb{G}_m, T_0)$. The surjection of tori $T_0 \rightarrow T_0/G$ induces a homomorphism of lattices of 1-parameter subgroups, $\beta : L \rightarrow N := \text{Hom}(\mathbb{G}_m, T_0/G)$. The dual homomorphism $\beta^* : N^* \rightarrow L^*$ is the induced homomorphism of characters. Since $T_0 \rightarrow T_0/G$ is surjective, β^* is injective, and the cokernel of β is finite. Thus (Σ, β) is a stacky fan and $[X/G] = \mathcal{X}_{\Sigma, \beta}$.

Example 4.4. A basic example is, take Σ to be an arbitrary fan on a lattice N , $L = N$, and $\beta = id$. The induced map $T_N \rightarrow T_N$ is also identity and $G_\beta = 0$. Then $\mathcal{X}_{\Sigma, \beta}$ is the toric variety X_Σ .

Example 4.5. Lets begin with $X_\Sigma = \mathbb{C}^2 \setminus (0, 0)$ and $\beta : \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} \mathbb{Z}$. The induced map by $\beta^* : \mathbb{Z} \rightarrow \mathbb{Z}^2$ with $\mathbb{G}_m^2 \rightarrow \mathbb{G}_m$ is given by $(s, t) \mapsto st^{-1}$. Thus $G_\beta = \mathbb{G}_m = \{(t, t)\} \subset \mathbb{G}_m^2$. Then $\mathcal{X}_{\Sigma, \beta} = \mathbb{P}^1$.

An integral T -invariant substack of $[X/G]$ is necessarily of the form $[Z/G]$, where $Z \subset X$ is an integral T_0 -invariant subvariety of X . The subvariety Z is naturally a toric variety whose torus T' is a quotient of T_0 . The quotient stack $[Z/G]$ contains a dense open stacky torus $[T'/G]$ which acts on $[Z/G]$.

In the definition of a toric stack one may consider N only be a finitely generated abelian group. In this case the toric stack is called non-strict. In this case we define $\mathcal{X}_{\Sigma, \beta}$ to be $[X_\Sigma/G_\beta]$ where the action of G_β on X_Σ is induced by the homomorphism $G_\beta \rightarrow T_L$.

A morphism of non-strict toric stacks is a morphism which restricts to a homomorphism of stacky tori and is equivariant with respect to that homomorphism. A morphism of non-strict stacky fans $(\Sigma, \beta : L \rightarrow N) \rightarrow (\Sigma', \beta' : L' \rightarrow N')$ is a pair of group homomorphisms $\Phi : L \rightarrow L'$ and $\phi : N \rightarrow N'$ so that $\beta' \circ \Phi = \phi \circ \beta$ and so that for every $\sigma \in \Sigma$, $\Phi(\sigma)$ is contained in a cone of Σ' . We draw this morphism as

$$\begin{array}{ccc} \Sigma & \rightarrow & \Sigma' \\ L & \xrightarrow{\Phi} & L' \\ \beta \downarrow & & \downarrow \beta' \\ N & \xrightarrow{\phi} & N' \end{array}$$

A morphism of non-strict stacky fans $(\Phi, \phi) : (\Sigma, \beta) \rightarrow (\Sigma', \beta')$ induces a morphism of toric varieties $X_\Sigma \rightarrow X_{\Sigma'}$ and a compatible morphism of groups $G_\beta \rightarrow G_{\beta'}$, so it induces a toric morphism of non-strict toric stacks $\mathcal{X}_{(\Phi, \phi)} : \mathcal{X}_{\Sigma, \beta} \rightarrow \mathcal{X}_{\Sigma', \beta'}$.

Definition 4.6. (*Fantastacks*) Let Σ be a fan on a lattice N , and let $\beta : \mathbb{Z}^n \rightarrow N$ a homomorphism with finite cokernel so that every ray of Σ contains some $\beta(e_i)$, and every $\beta(e_i)$ lies in the support of Σ . For a cone $\sigma \in \Sigma$, let $\hat{\sigma} = \text{cone}(\{e_i \mid \beta(e_i) \in \sigma\})$. We define the fan $\hat{\Sigma}$ on \mathbb{Z}^n to the fan generated by all $\hat{\sigma}$. Define $\mathcal{F}_{\Sigma, \beta} = \mathcal{X}_{\Sigma, \beta}$. Any stack isomorphic to $\mathcal{F}_{\Sigma, \beta}$ is called a *Fantastack*.

A simple example is to take Σ the trivial fan on $N = 0$, and $\beta : \mathbb{Z}^n \rightarrow N$ to be the zero map. Then $\hat{\Sigma}$ is the fan of \mathbb{C}^n , and $G_\beta = \mathbb{G}_m^n$. So $\mathcal{F}_{\Sigma, \beta} = [\mathbb{C}^n / \mathbb{G}_m^n]$. A sort of examples are any smooth toric variety X_Σ where $\beta : \mathbb{Z}^n \rightarrow N$ is constructed by sending the generators of \mathbb{Z}^n to the first lattice points along the rays of Σ . Then $X_\Sigma = \mathcal{F}_{\Sigma, \beta}$.

As a non-trivial example take $X_\Sigma = \mathbb{C}^4$ and assume $\beta^* : \mathbb{Z}^3 \rightarrow \mathbb{Z}^4$ is given by the matrix

$$\beta^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

then $\mathcal{F}_{\Sigma, \beta} = [\mathbb{C}^4 / \mathbb{G}_m]$ can be regarded as an stacky resolution of singularity, cf. [GS].

The cones of $\hat{\Sigma}$ are indexed by sets $\{e_{i_1}, \dots, e_{i_k}\}$ such that $\{\beta(e_{i_1}), \dots, \beta(e_{i_k})\}$ is contained a simple cone of Σ . It is then easy to identify which open subvariety of \mathbb{C}^n is represented by $\hat{\Sigma}$. Explicitly define the ideal

$$I_\Sigma = \left(\prod_{\beta(e_i) \notin \sigma} x_i \mid \sigma \in \Sigma \right)$$

Then $X_{\hat{\Sigma}} = \mathbb{C}^n \setminus V(I_\Sigma)$.

There exists an alternative definition for toric stacks generalizing the torus action to higher structures. In the following definition set $G_A := \text{Hom}(A, \mathbb{C}^*)$ for an abelian group (the group of characters).

Definition 4.7. *A Deligne-Mumford torus is a Picard stack over $\text{Spec}(\mathbb{C})$ which is obtained as a quotient $[G_{A^0}/G_{A^1}]$, with $\phi : A^0 \rightarrow A^1$ is a morphism of finitely generated abelian groups such that $\ker(\phi)$ is free and $\text{coker}(\phi)$ is finite.*

Proposition 4.8. [FMN] *Any Deligne-Mumford (DM) torus is isomorphic as Picard stack to $T \times BG$, where T is a torus and G is a finite abelian group.*

Definition 4.9. *A smooth toric Deligne-Mumford stack is a smooth separated DM-stack X together with an open immersion of a Deligne-Mumford torus $\iota : \mathcal{T} \hookrightarrow X$ with dense image such that the action of \mathcal{T} on itself extends to an action $\mathcal{T} \times X \rightarrow X$. In this case a morphism is a morphism of stacks which extends a morphism of Deligne-Mumford tori.*

A toric orbifold is a toric DM-stack with generically trivial stabilizer. A toric DM-stack is a toric orbifold iff its DM-torus is an ordinary torus.

Proposition 4.10. [FMN] *Let X be a toric DM-stack with DM-torus $T \times BG$. Then the morphism $G \times X \rightarrow I^{gen}(X)$ is an isomorphism.*

$I^{gen}(X) \subset I(X)$ is the subset of all top dimensional components in

$$I(X) = X \times_{X \times X} X = \{(x, g) | x \in X, g \in Aut(x)\}.$$

5. COHOMOLOGY OF DELIGNE-MUMFORD (DM)-STACKS

In this section we provide the definition of the de Rham cohomology of stacks for an application to characteristic cohomology of Mumford-Tate domains. This brief is taken from the lectures of K. Behrend at UBC [KB]. Differentiable stacks are stacks over the category of differentiable manifolds. They are the stacks associated to Lie groupoids. A groupoid $X_1 \rightrightarrows X_0$ is a Lie groupoid if X_0 and X_1 are differentiable manifolds, structure maps are differentiable, source and target maps are submersions. There is associated a simplicial nerve to a Lie groupoid namely

$$X_p := \overbrace{X_1 \times_{X_0} X_1 \times_{X_0} \dots \times_{X_0} X_1}^{p \text{ times}}$$

Then we get an associated co-simplicial object

$$\Omega^q(X_0) \rightarrow \Omega^q(X_1) \rightarrow \Omega^q(X_2) \rightarrow \dots \quad , \quad \partial = \sum_{i=0}^p (-1)^i \partial_i^*$$

the cohomology groups $H^k(X, \Omega^q)$ are called the Cech cohomology groups of the groupoid $X = [X_1 \rightrightarrows X_0]$ with values in the appropriate sheaf.

Definition 5.1. *Let \mathcal{X} be a differentiable stack. Then*

$$H^k(\mathcal{X}, \Omega^q) = H^k(X_1 \rightrightarrows X_0, \Omega^q)$$

for any Lie groupoid $X \rightrightarrows X_0$ giving an atlas for \mathcal{X} . In particular this defines

$$\Gamma(\mathcal{X}, \Omega^q) = H^0(\mathcal{X}, \Omega^q)$$

Example 5.2. *If G is a Lie group then $H^k(BG, \Omega^0)$, is the group cohomology of G calculated with differentiable cochains.*

Definition 5.3. *The double complex $A^{pq} := \Omega^q(X_p)$ is called the de Rham complex of the stack \mathcal{X} , and its cohomologies are called de Rham cohomologies of \mathcal{X} .*

One can show using a double fibration argument to prove that the de Rham cohomology is invariant under Morita equivalence and hence well defined for differentiable stacks. Thus $H_{DR}^n(X) = H_{DR}^n(X_1 \rightrightarrows X_0)$, for any groupoid atlas $X_1 \rightrightarrows X_0$ of the stack X .

When the stack X is a quotient stack there is a clear explanation of its cohomology as an equivariant cohomology. There is a well-known generalization of the de Rham complex to the equivariant case, namely the Cartan complex $\Omega_G^\bullet(X)$, defined by

$$(8) \quad \Omega_G^\bullet(X) := \bigoplus_{2k+i=n} (S^k \mathfrak{g}^\vee \otimes \Omega^i(X))^G$$

where $S^\bullet \mathfrak{g}^\vee$ is the symmetric algebra on the dual of the Lie algebra of G . The group G acts on \mathfrak{g} by adjoint representation on \mathfrak{g}^\vee and by pull back of differential forms on $\Omega^\bullet(X)$. The Cartan differential is $d_{DR} - \iota$ where ι is the tensor induced by the vector bundle homomorphism $\mathfrak{g}_X \rightarrow T_X$ coming from differentiating the action. If G is compact, the augmentation is a quasi-isomorphism, i.e

$$(9) \quad H_G^i(X) \xrightarrow{\cong} H^i(\text{Tot } \Omega_G^\bullet(X_\bullet))$$

for all i and the groupoid X_\bullet .

Proposition 5.4. [KB] *If the Lie group G is compact, there is a natural isomorphism*

$$H_G^i(X) \xrightarrow{\cong} H_{DR}^i(G \times X \rightrightarrows X) = H_{DR}^i([X/G])$$

As a corollary $H_{DR}^*(BG) = (S^{2*} \mathfrak{g}^\vee)^G$ for a compact lie group G .

Remark 5.5. [KB] *If G is not compact, then $H_{DR}([X/G])$ is still equal to equivariant cohomology. This fact holds for equivariant cohomology in general.*

As in the de Rham cohomology, every topological groupoid defines a simplicial nerve X_\bullet which gives rise to the double complex $C_q(X_p)$ of simplices. The total homology of this double complex are called singular homology of $X_1 \rightrightarrows X_0$. A simple example is to consider the transformation groupoid $G \times X \rightrightarrows X$ for a discrete group G . In this case we have the degenerate spectral sequence

$$(10) \quad E_{p,q}^2 = H_q(G, H_p(X)) \Rightarrow H_{p+q}(G \times X \rightarrow X)$$

When X is a point $H_p(G \times X \rightrightarrows X) = H_p(G, \mathbb{Z})$. In general there exists interpretation of the singular homology $H_*([X/G])$ in terms of the equivariant homology of X when the Lie group G acts continuously on X , exactly similar to de Rham cohomology case. There is also the dual notion of singular cohomologies of the stack X by replacing the double complex $C_q(X_p)$ with its dual $Hom(C_q(X_p), \mathbb{Z})$. Directly we obtain a pairing

$$(11) \quad H_k(X, \mathbb{Z}) \times H^k(X, \mathbb{Z}) \rightarrow \mathbb{Z}$$

Example 5.6. [KB] *The stack of triangles up to similarity may be represented by $S_3 \times \Delta_2 \rightrightarrows \Delta_2$. Thus the homology of the stack of triangles is equal to the homology of the symmetric group S_3 .*

Similarly the stack of Elliptic curves $M_{1,1}$ may be represented by the action of $Sl_2(\mathbb{Z})$, by the linear fractional transformations on the upper half plane in \mathbb{C} . Thus the homology of the stack of Elliptic curves is equal to the homology of $Sl_2(\mathbb{Z})$.

A 1-cycle is a sequence of oriented paths between the edges followed by invertible morphisms between successive ones. Explaining higher cycles is so difficult on stacks and we refer to [KB] for more discussions.

Example 5.7. [KB] *Lets consider a finite type DM-stack X of dimension 0. We can present X by a groupoid $X_1 \rightrightarrows X_0$, where both X_1 and X_0 are 0-dimensional, i.e just finite set of points. Then it is obvious that $X_1 \rightrightarrows X_0$ is equivalent to a disjoint union of groups: $X_0 = \{*_1, \dots, *_n\}$, $X_1 = \amalg_{i=1}^n G_i$, for finite groups G_i and one-points $*_i$. Then $H^0(X) = \mathbb{R}^n$ and all other cohomology groups vanishes. There is a canonical element $1 \in H^0(X)$, which together with the integral $\int_X : H^0(X) \rightarrow \mathbb{R}$ defines a canonical number $\int_X 1 \in \mathbb{R}$. To calculate $\int_X 1 \in \mathbb{R}$, note that $\rho(*_i) = \frac{1}{\#G_i}$ defines a partition of unity for $X_1 \rightrightarrows X_0$. Thus*

$$\int_X 1 = \sum_{i=1}^n \int_{*_i} \frac{1}{\#G_i} = \sum_{i=1}^n \frac{1}{\#G_i}$$

Theorem 5.8. [KB] *Let X be a topological Deligne-Mumford stack with coarse moduli space \bar{X} . Then the canonical morphism $X \rightarrow \bar{X}$ induces isomorphisms on \mathbb{Q} -valued cohomologies,*

$$H^k(\bar{X}, \mathbb{Q}) \xrightarrow{\cong} H^k(X, \mathbb{Q})$$

The theorem follows from 6.4, taking an open cover of X by quotient stacks $[U_i/G_i]$ and then applying Čech spectral sequence together with a similar formula to (7) on cohomologies.

We know $H^*(BGl_n, \mathbb{Z}) = \mathbb{Z}[t_1, \dots, t_n]$. $t_i \in \mathcal{H}^{2i}(BGl_n, \mathbb{Z})$ is called the universal chern class. Given a rank n -vector bundle E over a stack X , we get an associated morphism of stacks $f : X \rightarrow BGl_n$, such that the following diagram is commutative;

$$(12) \quad \begin{array}{ccc} B & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BGl_n \end{array} \quad \Leftrightarrow \quad \begin{array}{ccc} E & \longrightarrow & [\mathbb{C}^n/Gl_n] \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BGl_n \end{array}$$

B is the principal Gl_n -bundle of frames of E . The i -th chern class of the vector bundle E is defined by

$$(13) \quad c_i(E) := f^* t_i$$

We will use the following theorem on a discussion about characteristic cohomology of Mumford-Tate domains.

Theorem 5.9. *cf. [KB] If all the odd degree cohomologies of the stack X vanishes then $H^*(E \setminus X) = H^*(X)/c_n(E)$.*

6. PERIOD AND MUMFORD-TATE DOMAINS

Mumford-Tate groups are basic symmetry groups of Hodge structures. A Mumford-Tate domain D_M is by definition the orbit under the Mumford-Tate group M of a point in the period domain D classifying polarized Hodge structures with given Hodge numbers. In the classical weight 1-case, the quotient of a Mumford-Tate domain by an arithmetic group are the complex points of a Shimura variety.

To begin with let V be finite dimensional \mathbb{Q} -vector space, and Q a non-degenerate bilinear map $Q : V \otimes V \rightarrow \mathbb{Q}$ which is $(-1)^n$ -symmetric for some fixed n . A Hodge structure is given by a representation

$$\phi : \mathbb{U}(\mathbb{R}) \rightarrow \text{Aut}(V, Q)_{\mathbb{R}}, \quad \mathbb{U}(\mathbb{R}) = \left(\begin{array}{cc} a & -b \\ b & a \end{array} \right), \quad a^2 + b^2 = 1$$

It decomposes over \mathbb{C} into eigenspaces $V^{p,q}$ such that $\phi(t).u = t^p \bar{t}^q .u$ for $u \in V^{p,q}$, and $\overline{V^{p,q}} = V^{q,p}$. In general a not-necessarily polarized mixed Hodge structure is given by V as above together with a representation $\mathbb{S}(\mathbb{R}) = \mathbb{C}^* \rightarrow \text{Gl}(V_{\mathbb{R}})$. Then $V_{\mathbb{R}}$ decomposes into a direct sum of weight spaces $V_{\mathbb{R},n}$ on which $t \in \mathbb{R}^* \subset \mathbb{C}^*$ acts by t^n , and then under the action of S^1 , $V_{\mathbb{C},n}$ decomposes as above into a direct sum of $V_{\mathbb{C}}^{p,q}$, $p + q = n$.

It is well-known that $\text{Ad}(\phi) : \mathbb{U}(\mathbb{R}) \rightarrow \text{Aut}(\mathfrak{g}_{\mathbb{R}}, B)$ defines a weight 0 Hodge structure, where B is the Killing form. Then $\mathfrak{g}_{\mathbb{C}}$ has a decomposition

$$\mathfrak{g} = \mathfrak{g}_{\phi}^{-} \oplus \overline{\mathfrak{g}_{\phi}^{-}} \oplus \mathfrak{h}_{\phi}, \quad \mathfrak{g}^{-} := \bigoplus_{i>0} \mathfrak{g}_{\phi}^{-i,i}$$

and $T_{\phi, \mathbb{R}} D \otimes \mathbb{C} = \mathfrak{g}_{\phi}^{-} \oplus \overline{\mathfrak{g}_{\phi}^{-}}$, $T_{\phi} D = \mathfrak{g}_{\phi}^{-}$.

Definition 6.1. [GGK] *The Mumford-Tate group of the Hodge structure ϕ denoted $M_{\phi}(R)$ is the smallest \mathbb{Q} -algebraic subgroup of $G = \text{Aut}(V, Q)$ with the property*

$$\phi(\mathbb{U}(R)) \subset M_{\phi}(R)$$

M_{ϕ} is a simple, connected, reductive \mathbb{Q} -algebraic group. If $F^{\bullet} \in \check{D}$ the Mumford-Tate group $M_{F^{\bullet}}$ is the subgroup of $G_{\mathbb{R}}$ that fixes the Hodge tensors.

Assume $\Phi : S \rightarrow \Gamma \backslash D$ is a variation of Hodge structure and let T_1, \dots, T_n be generators of the monodromy group Γ . Then the partial compactification $\Gamma \backslash D_{\sigma}$ is given by the cone

$$\sigma = \sum_{j=1}^n \mathbb{R}_{\geq 0} N_j, \quad N_j = \log T_j$$

in the lie algebra $\mathfrak{g} := \text{Lie}(G_{\mathbb{C}}) \subset \text{Hom}(V, V)$. Here a boundary point is a nilpotent orbit associated to a face σ . Let σ be a nilpotent cone, and $F \in \check{D}$. Then $\exp(\sigma_{\mathbb{C}})F \subset \check{D}$ is called a nilpotent orbit if it satisfies

- $\exp(\sum_j iy_j N_j)F \in D$ for all $y_j \gg 0$;
- $NF^p \subset F^{p-1}$ for all $p \in \mathbb{Z}$.

Define the set of nilpotent orbits

$$D_{\sigma} := \{(\tau, Z) \mid \tau \text{ face of } \sigma, Z \text{ is a } \tau - \text{nilpotent orbit}\}$$

Set

$$\Gamma(\sigma)^{\text{gp}} = \exp(\sigma_{\mathbb{R}}) \cap G_{\mathbb{Z}}, \quad \Gamma(\sigma) = \exp(\sigma) \cap G_{\mathbb{Z}}$$

The monoid $\Gamma(\sigma)$ defines the toric variety

$$(14) \quad D_{\sigma} := \text{Spec}([\mathbb{C}[\Gamma(\sigma)^{\vee}]]_{\text{an}}) \cong \text{Hom}(\Gamma(\sigma)^{\vee}, \mathbb{C})$$

with the torus

$$(15) \quad T_{\sigma} := \text{Spec}(\mathbb{C}[\Gamma(\sigma)^{\vee \text{gp}}])_{\text{an}} \cong \text{Hom}(\Gamma(\sigma)^{\vee \text{gp}}, \mathbb{G}_m) \cong \mathbb{G}_m \otimes \Gamma(\sigma)^{\text{gp}}$$

We assume that Γ is strongly compatible with Σ , that is

- $Ad(\gamma).\sigma \in \Sigma, (\forall \gamma \in \Gamma, \sigma \in \Sigma)$
- $\sigma_{\mathbb{R}} = \mathbb{R}_{\geq 0} \langle \log \Gamma(\sigma) \rangle$.

Theorem 6.2. [KP] $\Gamma \setminus D_{M, \Sigma}$ is a logarithmic manifold, which is Hausdorff in the strong topology. For each $\sigma \in \Sigma$, the map

$$(16) \quad \Gamma(\sigma)^{\text{gp}} \setminus D_{M, \sigma} \rightarrow \Gamma \setminus D_{M, \Sigma}$$

is open and locally an isomorphism.

Theorem 6.3. [KP] *Let $D = G(\mathbb{R})/H$ be a Mumford-Tate domain, and $\Gamma \leq G(\mathbb{Q})$ a torsion-free non-co-compact congruence subgroup. Then the $\mathcal{B}(\sigma)$ parametrize all nilpotent orbits subordinate to a given nilpotent cone σ . Assume $B(N) = \{e^{\tau N} F^\bullet\}$ in \check{D} is non-empty. Then the Mumford-Tate group of the limit MHS $(F^\bullet, W(N)_\bullet)$ is a subgroup of the centralizer of G . It is equipped with a filtration*

$$M_{B(N)} = W_0 M_{B(N)} \supseteq W_{-1} M_{B(N)} \supseteq \dots$$

- $Gr_0^W M_{B(N)} = G_{B(N)}$ is reductive and is the Mumford-Tate group of a general $\phi_{split} := \oplus_i Gr_i^W(N) F^\bullet$.
- $Gr_k^W M_{B(N)}$ is abelian for $k < 0$.
- $W_{-1} M_{B(N)} = M_N := \exp\{im(ad(N) \cap \ker(ad(N)))\}$.
- $M_N(\mathbb{C}) \rtimes G_{B(N)}(\mathbb{R})$ acts transitively on $\check{B}(N)$ of the limit MHS.

When Γ is neat, there exist successive covers

$$\overline{B(N)} \rightarrow \dots \overline{B(N)}_{(k)} \rightarrow \dots \rightarrow \overline{B(N)}_{(1)} \rightarrow \overline{D(N)}$$

with $k > 1$, and intermediate Jacobian fibers at each stage, and also $\overline{D(N)}$ being discrete.

Theorem 6.4. [KP] *For a non-empty Kato-Usui boundary component $B(\sigma)$ of D_M associated to a nilpotent cone $\sigma \subset \mathfrak{m}_{\mathbb{Q}}$, the Mumford-Tate group $M_{B(\sigma)}$ satisfies the following:*

- $M_{B(\sigma)}$ is contained in the centralizer $Z(\sigma)$ of the cone.
- $W_{-1} M_{B(\sigma)} = M_\sigma$ is its unipotent radical.
- $Gr_0^W M_{B(\sigma)} = M_{B(\sigma)}/M_\sigma (\subset Z(\sigma)/M_\sigma)$ is the \mathbb{Q} -algebraic closure of the orbit $(Z_\sigma/M_\sigma)(\mathbb{R}) \cdot \phi_{split}$ which may be regarded as a set of polarized Hodge structure on $\oplus_k P_k$.

The first item in theorems 5.3 and 5.4 characterizes $M_{B(\sigma)}$ as a quotient of the centralizer $Z(\sigma)$.

7. HIGHER STRUCTURES ON MUMFORD-TATE DOMAINS

Assume $\phi : \mathbb{U} \rightarrow M$ be a Hodge structure, with the associated Mumford-Tate domain $D_M = M(\mathbb{R}).\phi$. Set $\mathfrak{m} = \text{Lie}(M)$. The boundary component associated to $\mathbb{Q}_{\geq 0}\langle N_1, \dots, N_r \rangle \subset \mathfrak{m}$ is

$$(17) \quad B_\sigma := \tilde{B}_\sigma / e^{(\sigma)\mathfrak{c}}$$

where

$$(18) \quad \tilde{B}_\sigma := \{F^\bullet \in \check{D} \mid \text{Ad}(e^\sigma).F^\bullet \text{ is a nilpotent orbit}\}$$

Kato-Usui define a generalization of the Hodge domains as follow,

$$D_{M,\sigma} := \coprod_{\sigma \in \Sigma} \{Z \subset \check{D}_M \mid Z \text{ is a } \sigma - \text{nilpotent orbit}\} = \coprod_{\sigma \in \Sigma} B(\sigma)$$

This always contain $B(\{0\}) = D_M$. In particular $D_{M,\sigma} = D_{M, \text{faces of } \sigma}$.

Let Σ be a fan in $\mathfrak{m} := \text{Lie}(M)$ where M is a Mumford-Tate group of Hodge structure. Assume $\Gamma \subset M(\mathbb{Z})$ is a neat subgroup of finite index and consider the monoid

$$\Gamma(\sigma) := \Gamma \cap \exp(\sigma_{\mathbb{R}})$$

whose group theoretic closure is $\Gamma(\sigma)^{\text{gp}}$. The same definitions as in (8) and (9) give a toric variety structure for D_M , denoted $D_{M,\sigma}$. Now consider the injective map $\beta = e : \log \gamma \mapsto \gamma$ restricted to $\Sigma = \mathbb{Z}\langle N_1, \dots, N_r \rangle$. We will consider e as an isomorphism onto its image $\mathfrak{m}_{\mathbb{Z}}$ a maximal lattice in \mathfrak{m} . According to what we explained in section 4, Def. 4.2 and 4.3 this establishes a toric stack structure on D_M .

Theorem 7.1. *The triple $(D_M, e : \Sigma_{\mathbb{Z}} \rightarrow \mathfrak{m}_{\mathbb{Z}}, \Sigma)$ is a toric fantastack.*

Proof. Most of the proof carries from the material previously encountered. First D_M is equipped with its Kato-Usui toric compactification defined by (14) and (15) compatible with the structure in Theorems 6.2 and 6.4 item 1. It follows that D_M has a toric variety structure both on the interior and the boundary points B_σ for various nilpotent cones in \mathfrak{m} . Second the map on the lattices e is an isomorphism, mentioning that the generic stabilizer is trivial in this case. \square

Let X be a complex manifold and $W \subset X$ a holomorphic sub-bundle. Then $I = W^\perp \subset T^*(X)$ is also holomorphic and let \bar{I} be its conjugate. Let $\mathcal{I}^{\bullet\bullet} \subset \mathcal{A}^{\bullet\bullet}$ be the differential ideal of sections of $I \oplus \bar{I}$ in the algebra of smooth differential forms on X . The characteristic cohomology of X denoted $H_{\mathcal{I}}^*(X)$ is the cohomology of the double complex $\subset \mathcal{A}^{\bullet\bullet}/\mathcal{I}^{\bullet\bullet}$.

There is a natural inclusion $T_{F^\bullet}\check{D} \subset \oplus_p \text{Hom}(F^p, V_{\mathbb{C}}/F^p)$. The canonical sub-bundle $W \subset T\check{D}$ given by the infinitesimal period relation is defined by

$$W_{F^\bullet} = T_{F^\bullet}\check{D} \cap (\oplus_p \text{Hom}(F^p, F^{p-1}/F^p))$$

The bundle $W \rightarrow \check{D}$ is acted by $G_{\mathbb{C}}$, and the action of $G_{\mathbb{R}}$ on $W \rightarrow D$ leaves invariant the metric given by Cartan killing form at each point. With the identification $T_\phi D = \oplus_{i>0} \mathfrak{g}^{-i,i}$ we have $W_\phi = \mathfrak{g}_\phi^{-1,1}$.

In the situation of defining the characteristic cohomology of varieties, take $X = D_M \subset D$ to be a Mumford-Tate domain, and $W_M \subset TD_M$ is the infinitesimal period relation. Denote by $\Lambda_M^{\bullet\bullet}$ the complex of $G(\mathbb{R})$ -invariant forms in $A^{\bullet\bullet}(D_M)/\mathcal{I}^{\bullet\bullet}$ with the operator $\delta : \Lambda_M^\bullet \rightarrow \Lambda^{\bullet+1}$. $H^*(\Lambda_M^\bullet, \delta_M)$ is called the universal characteristic cohomology.

Proposition 7.2. [GGK] *The universal characteristic cohomology i.e the cohomology of $A^{\bullet\bullet}(D_M)/\mathcal{I}^{\bullet\bullet}, \delta : \Lambda_M^\bullet \rightarrow \Lambda^{\bullet+1}) := H^*(\Lambda_M^\bullet, \delta_M)$ is equal to the equivariant cohomology $H^*(I^{\bullet\bullet})$.*

Proposition 7.3. *The universal characteristic cohomology is equal to the de Rham cohomology of the stack D_M , viz 5.3.*

Proposition 7.4. [GGK] $H^{2p-1}(\Lambda_M^\bullet, \delta_M) = 0$ and $H^{2p-1}(\Lambda_M^\bullet, \delta_M) = (\Lambda^{p,p})^{\text{m}^{0,0}}$.

In case $D_M = D$ and $M = G$, the universal characteristic cohomology is generated by the chern forms of Hodge bundles. According to the proposition 5.4 and the remark 5.5, the universal characteristic cohomology can be understood as the stack cohomology of the Mumford-Tate domains. The following question has been asked in [GGK];

Question: What are the conditions that the chern forms of Hodge bundles on D_M generate the characteristic cohomology? In case this holds what are the relations?.

The following theorem is generally true for the cohomology of vector bundle on stacks.

Theorem 5.9 says that one of the relations is given by the top chern class of Hodge bundle. Our hope is that the above question can be a little better worked out using cohomology of stacks, and more clearly understood.

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