ON THE MIXED HODGE STRUCTURE ASSOCIATED TO HYPERSURFACE SINGULARITIES

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Abstract. Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a germ of hyper-surface with isolated singularity. By the work of Saito M. and P. Deligne the extension of the VMHS associated to cohomologies of the fibers of an isolated singularity in the affine space (Also called vanishing cohomology in this case) after the extension, on the puncture defines another MHS whose Hodge filtration is automatically opposite to the former Hodge filtration (Limit Hodge filtration of Schmid in pure case and Steenbrink in mixed case). Now the polarization deforms to Grothendieck residue pairing modified by a varying sign on the Hodge graded pieces. This also proves the existence of a Riemann-Hodge bilinear relation for Grothendieck pairing and allow to calculate the signature of Grothendieck pairing. The above form polarizes the complex VHS of G. Pearlstein et al.

Introduction

One of the important subject of study in Hodge theory and D-modules is the behaviour of the underlying variation of (mixed) Hodge structures in the extensions. We will consider the VMHS associated to isolated hypersurface singularities in the affine space $\mathbb{C}^{n+1}$. The mixed Hodge structure would be the Steenbrink limit mixed Hodge structure. Classically there are two equivalent ways to define this MHS. One method which is actually due to J. Steenbrink himself is by applying a spectral sequence argument to the resolution of the singularity in projective fibration followed with Invariant cycle theorem. Another method which is equivalent to the first is to define it by the structure of lattices in the Gauss-Manin system associated to VMHS on the punctured disc. We are mainly interested to the extension of the Gauss-Manin system over the puncture both in the algebraic and analytic case. The extension of the MHS, is then a consequence of the Riemann-Hilbert correspondence. We explain the extension by gluing vector bundles which is called minimal extension in the literature. It mainly involves the fact that, different V-lattices can be glued with the Brieskorn lattices in the other chart. Classically this is equivalent to the Fourier-Laplace transform of perverse sheaves. the observation made by P. Deligne and M. Saito is the two Hodge filtration on the vanishing cohomology, one defined by the Brieskorn lattices and the second defined by the V-lattices are opposite. This

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corresponds to a choice of good basis of the Gauss-Manin system which defines the desired isomorphism. We try to study the polarization in the new chart after the extension. The method we use is to formulate a special presentation of K. Saito higher residue pairing due to the works of F. Pham.

Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic germ with isolated singularity at $0 \in \mathbb{C}^{n+1}$ and $f : X \to T$ a representative for its Milnor fibration. The stack on 0 of the extended Gauss-Manin module can be naturally identified with the module of relative differentials Ω_f. This is a direct consequence isomorphisms between V-lattices and the Brieskorn lattices, which also pairs the Hodge filtration and the (actually a re-indexed) V-filtration in an opposite way. By the works of P. Deligne and M. Saito any such opposite pair correspond s to choice of a good section $s : H''/\partial_t^{-1}H'' \to H''$ which is also corresponds to choose a good basis of the vanishing cohomology $H^n(X_\infty, \mathbb{C})$. A classical example of this situation is when the variation of MHS is mixed Tate or Hodge-Tate. Then the opposite filtration would be the weight filtration indexed by rationals. We show a modification of Grothendieck residue defines a polarization for Ω_f. The correspondence between polarization form and Grothendieck residue pairing is proved by using asymptotic integrals defined by f which define sections of the relative cohomology $H^n(X, X_t)$. This allows us to obtain a simpler presentation of K. Saito higher residue pairing. Following C. Sabbah one can explain the conjugation operator on the sections of the Gauss-Manin system. An application to the Grothendieck pairing to Jacobians of curves is also given.

1. Steenbrink limit Hodge filtration

In this section we review the Steenbrink MHS defined in 6.3.3 to fix the notation. Suppose we have an isolated singularity holomorphic germ $f : \mathbb{C}^{n+1} \to \mathbb{C}$. By the Milnor fibration Theorem we can always associate to f a C^∞-fiber bundle over a small punctured disc T'. The corresponding cohomology bundle $\mathcal{H}$, constructed from the middle cohomologies of the fibers defines a variation of mixed Hodge structure on T'. The Brieskorn lattice is defined by,

$$\mathcal{H}^{(0)} = H'' = f_* \frac{\Omega^{n+1}_{X,0}}{df \wedge d\Omega^{n-1}_{X,0}}$$

The Brieskorn lattice is the stack at 0 of a locally free $\mathcal{O}_T$-module $\mathcal{H}''$ of rank μ with $\mathcal{H}''_{T'} \cong \mathcal{H}$, and hence $H'' \subset (i_*\mathcal{H})_0$, with $i : T' \hookrightarrow T$. The regularity of the Gauss-Manin connection proved by Brieskorn and Malgrange implies that $H'' \subset \mathcal{G}$.

Theorem 1.1. *(Malgrange)(see* [SCHU] *1.4.10)*
$H'' \subset V^{-1}$

The Leray residue formula can be used to express the action of ∂_t in terms of differential forms by

$$\partial_t^{-1} s[d\omega] = s[df \wedge \omega]$$

.
where, $\omega \in \Omega_X^n$. In particular , $\partial_t^{-1}.H'' \subset H''$, and

$$\frac{H''}{s.H''} \cong \frac{\Omega_{X,0}^{n+1}}{df \wedge \Omega_{X,0}^n} \cong \frac{\mathbb{C}\{\underline{z}\}}{(\partial(f))}. \tag{1}$$

In this chapter we use the notation

$$\Omega_f \cong \frac{\Omega_{X,0}^{n+1}}{df \wedge \Omega_{X,0}^n} \tag{2}$$

for the module of relative differentials of the map f. The Hodge filtration on $H^n(X_\infty, \mathbb{C})$ is defined by

$$F^p H^n(X_\infty)_\lambda = \psi_\alpha^{-1} \partial_t^{n-p} Gr_V^{\alpha+n-p} \mathcal{H}^{(0)}. \tag{3}$$

where ψ_α is the (Deligne) nearby map. Therefore,

$$Gr_F^p H^n(X_\infty, \mathbb{C})_\lambda = Gr_V^{\alpha+n-p} \Omega_f \tag{4}$$

where Gr_V^β is defined as follows,

Definition 1.2. *(cf.* [KUL] *page 110) The V-filtration on Ω_f is defined by*

$$V^\alpha \Omega_f = pr(V^\alpha \cap H'') \tag{5}$$

Clearly $V^\alpha \Omega_f = \oplus_{\beta \geq \alpha} \Omega_f^\beta$ and $\Omega_f \cong \oplus Gr_V^\alpha \Omega_f$ hold.

2. Theorem of Varchenko on multiplication by f

A theorem of A. Varchenko, shows the relation between the operator N, on vanishing cohomology and multiplication by f on Ω_f.

Theorem 2.1. [SC2]
The maps $Gr(f)$ and $N = \log M_u \in End(H^n(X_\infty, \mathbb{C}))$ have the same Jordan normal forms.

Proof. The map N is a morphism of mixed Hodge structures of type $(-1, -1)$. Hence, all the powers of N are strictly compatible with the filtration F (with the appropriate shift). This implies the existence of a splitting of the Hodge filtration, i.e a grading of $H^n(X_\infty, \mathbb{C})$ which has F as its associated filtration, such that N becomes a graded morphism of degree -1. In particular, one concludes that N and its induced endomorphism $Gr_F N$ of degree -1 of $Gr_F H^n(X_\infty, \mathbb{C})$, have the same Jordan normal forms.

We have a canonical isomorphism

$$Gr_F H^n(X_\infty, \mathbb{C}) = \bigoplus_{-1<\alpha\leq 0} Gr_F C^\alpha$$

and the corresponding endomorphism

$$N_{F,\alpha} : Gr_F^p C^\alpha \to Gr_F^{p-1} C^\alpha$$

are given by

$$N_{p,\alpha}(x) = -2\pi i(t\partial_t - \alpha)x \cong -2\pi i.t\partial_t x \qquad (\mod F^p)$$

On the other hand, it is immediately seen that for $\beta \in \mathbb{Q},\ \beta = n - p + \alpha$ with $p \in \mathbb{Z}$ and $-1 < \alpha \leq 0$, the map

$$\partial_t^{n-p} : V^\beta \cap F^n \mathcal{H}_{X,0} \to V^\alpha / V^{>\alpha} = C^\alpha$$

induces an isomorphism from $Gr_\beta^V \Omega_f \to Gr_F^p C^\alpha$, and the diagram

$$\begin{array}{ccc} Gr_\beta^V \Omega_f & \xrightarrow{Gr(f)} & Gr_{\beta+1}^V \Omega_f \\ \partial_t^{n-p} \Big\downarrow & & \partial_t^{n-p+1} \Big\downarrow \\ Gr_F^p C^\alpha & \xrightarrow{N_{p,a}} & Gr_F^{p-1} C^\alpha \end{array}$$

commutes up to a factor of $-2\pi i$. Hence $Gr(f)$ and $Gr_F N$ have the same Jordan normal form.

□

3. Integrals along Lefschetz thimbles

Consider the function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with isolated singularity at 0, and a holomorphic differential $(n+1)$-form ω given in a neighborhood of the critical point. We shall study the asymptotic behaviour of the integral,

$$\int_\Gamma e^{\tau f} \omega \tag{6}$$

for large values of the parameter τ, namely a complex oscilatory integral. In the long exact homology sequence of the pair (X, X_t) where X is a tubular neibourhood of the singular fiber X_0 in the Milnor ball,

$$... \to H_{n+1}(X) \to H_{n+1}(X, X_t) \overset{\partial_t}{\to} H_n(X_t) \to H_n(X) \to ... \tag{7}$$

X is contractible. Therefore, we get an isomorphism $\partial_t : H_{n+1}(X, X_t) \cong H_n(X_t)$, and similar in cohomologies, i.e. $H^{n+1}(X, X_t) \cong H^n(X_t)$. Now if ω is a holomorphic differential $(n+1)$-form on X, and let $\Gamma \in H_{n+1}(X, X_t)$, we have the following;

Proposition 3.1. *(cf.* [AGV] *Theorems 8.6, 8.7, 8.8, 11.2)*
Assume $\omega \in \Omega^{n+1}$, and let $\Gamma \in H_n(X, X_t)$. Then

$$\int_\Gamma e^{-\tau f}\omega = \int_0^\infty e^{-t\tau} \int_{\Gamma\cap\{f=t\}} \frac{\omega}{df}_{|X_t} dt = e^{\tau . f(0)} \int_{\Gamma\cap\{f=t\}} \frac{\omega}{df}_{|X_t} \tag{8}$$

for $Re(\tau)$ large, and in this way can also be expressed as $\sum \tau^\alpha \log \tau^k A_{\alpha,k}$ in that range.

By theorem 3.1, we identify the cohomology classes $\int_\Gamma e^{-\tau f}\omega$ and $\int_{\Gamma\cap\{f=t\}} \frac{\omega}{df}_{|X_t}$ via integration on the corresponding homology cycles. We can also choose Γ such that its intersection with each Milnor fiber has compact support, and its image under f is the positive real line, [PH].
The asymptotic integral

$$I(\tau) = \int e^{\tau f}\phi dx_0 ... dx_n, \qquad \tau \to \pm\infty$$

satisfies

$$\frac{d^p}{d\tau^p} I = \int e^{\tau f} f^p \phi dx_0 ... dx_n$$

In case f is analytic then it has an asymptotic expansion

$$I(\tau) = \sum_{\alpha,p,q} c_{\alpha,p,q}(f)\tau^{\alpha-p}(\log\tau)^q, \qquad \tau \to +\infty$$

for finite number of rational numbers $\alpha < 0$, $p \in \mathbb{N}$, $0 \leq q \leq n-1$. Then $\phi \to c_{\alpha,p,q}(\phi)$ is a distribution with support contained in the support of f, [MA].

Remark 3.2. *(see* [PH] *page 27) We have the formula;*

$$I(\tau) = (2\pi)^{n/2}(Hess f)^{-1/2} f(0)\tau^{-n/2}[1 + O(1/\tau)]$$

Summarizing, the form $e^{-\tau f}\omega$ (for τ large enough) and the form $\frac{\omega}{df}\ |_{X_t}$, define the same cohomology classes via integration on cycles.

Proposition 3.3. *(*[AGV] *lemma 11.4, 12.2, and its corollary) There exists a basis $\omega_1, ..., \omega_\mu$ of Ω_f such that the corresponding Leray residues $\omega_1/df, ..., \omega_\mu/df$ define a basis for the sections of vanishing cohomology.*

4. Extension of the Gauss-Manin system

In this section we explain the extension of the Gauss-Manin system of an isolated hypersurface singularity. We conclude that the module Ω_f (module of relative $(n+1)$-differential forms of f) can be canonically regarded as the fiber over the puncture. This is an example of minimal extension of polarized variation of mixed Hodge structure, and should not be confused with Deligne extension. For simplicity, we explain it for the algebraic Gauss-Manin system associated to a polynomial f. The conclusion also holds in the analytic setting i.e. with f holomorphic, see for instance [MA] or [S1]. The difference is that in analytic set up one needs to consider the completions of the modules we are considering with respect to appropriate filtrations. Thus we follow ([SA6] page 6, [SA8] sec 3, see also [SA5]) in order to explain the gluing procedure which defines an extension of the Gauss-Manin system $\mathcal{G}$. By this we mean to glue the Gauss-Manin system $\mathcal{G}$ defined before with another one defined in a chart around 0. We are interested to understand the fiber $\mathcal{G}$ on 0 after the extension.

The Gauss-Manin system $\mathcal{G} := R^n f_* \mathbb{C}_{X'}$ (see Theorem 6.3.7) of a polynomial or holomorphic map $f : X' \to T'$ is a module over the ring $\mathbb{C}[\tau, \tau^{-1}]$, where τ is a new variable, and comes equipped with a connection, that we view as a $\mathbb{C}$-linear morphism $\partial_\tau : \mathcal{G} \to \mathcal{G}$ satisfying Leibnitz rule

$$\partial_\tau(\phi.g) = \frac{\partial \phi}{\partial \tau}.g + \phi \partial_\tau(g)$$

We put $\tau = t^{-1}$, and consider (τ, t) as coordinates on $\mathbb{P}^1$. Then $\mathcal{G}$ is a $\mathbb{C}[t, t^{-1}]$-module with connection and $\partial_\tau = -t^2 \partial_t$, [SA6], [SA5], [SA8].

Let $\Omega^{n+1}[\tau, \tau^{-1}]$ be the space of Laurent polynomials with coefficients in Ω^{n+1}. According to its very definition (cf. Theorem 6.3.7, [?] sec. 10.4, [SAI6] lemma 2.4), the Gauss-Manin System is given by;

$$\mathcal{G} = \frac{\Omega^{n+1}[\tau, \tau^{-1}]}{(d - \tau df \wedge)\Omega^{n+1}[\tau, \tau^{-1}]},$$

$$(d - \tau df \wedge) \sum_k \eta_k \tau^k = \sum_k (d\eta_k - df \wedge \eta_{k-1}) \tau^k$$

The action of the connection ∇_τ on $\mathcal{G}$ i.e. the $\mathbb{C}[\tau]\langle \partial_\tau \rangle$-module structure on G, is first defined on the image of Ω^{n+1} by

$$\partial_\tau[\omega] = [f\omega]$$

and then extended to $\mathcal{G}$ using the Leibnitz rule

$$\partial_\tau(\tau^p[\omega]) = p\tau^{p-1}[\omega] + \tau^p[f\omega]$$

In order to extend it as a rank μ-vector bundle on $\mathbb{P}^1$, one is led to study lattices i.e. $\mathbb{C}[\tau]$, and $\mathbb{C}[t]$-submodules which are free of rank μ.
In the chart t, the Brieskorn lattice

$$\mathcal{H}^{(0)} := image(\Omega^{n+1}[\tau^{-1}] \to \mathcal{G}) = \frac{\Omega^{n+1}[t]}{(td - df\wedge)\Omega^{n+1}[t]}$$

is a free $\mathbb{C}[t]$ module of rank μ. It is stable by the action of $\partial_\tau = -t^2\partial_t$. Therefore ∂_t is a connection on $\mathcal{G}$ with a pole of order 2. We consider the increasing exhaustive filtration $\mathcal{H}^{(p)} := \tau^p \mathcal{H}^{(0)}$ of $\mathcal{G}$.

In the chart τ, there are various natural lattices indexed by $\mathbb{Q}$, we denote them by V^α, with $V^{\alpha-1} = \tau V^\alpha$. On the quotient space $C_\alpha = V^\alpha / V^{>\alpha}$ there exists a nilpotent endomorphism $(\tau\partial_\tau - \alpha)$.

The space $\oplus_{\alpha\in[0,1[} C_\alpha$ is isomorphic to $H^n(X_\infty, \mathbb{C})$, and $\oplus_{\alpha\in[0,1[} F^p C_\alpha$ is the limit MHS on $H^n(X_\infty, \mathbb{C})$. A basic isomorphism can be constructed, as

$$\begin{array}{c} \dfrac{\mathcal{G}_p \cap V^\alpha}{\mathcal{G}_{p-1} \cap V^\alpha + \mathcal{G}_p \cap V^{>\alpha}} = Gr_F^{n-p}(C_\alpha) \\ \tau^p \Big\downarrow \cong \\ \dfrac{V^{\alpha+p} \cap \mathcal{G}_0}{V^\alpha \cap \mathcal{G}_{-1} + V^{>\alpha} \cap \mathcal{G}_0} = Gr^V_{\alpha+p}(\mathcal{G}_0/\mathcal{G}_{-1}) \end{array}$$

Thus, the gluing is done via the isomorphisms,

$$Gr_F^{n-p}(H_\lambda) \cong Gr^V_{\alpha+p}(\mathcal{H}^{(0)}/\tau^{-1}.\mathcal{H}^{(0)})$$

where $\lambda = exp(2\pi i\alpha)$ and we have chosen $-1 \leq \alpha < 0$ (cf. [SA3], [SA5], [SA6]). We have

$$\frac{\mathcal{H}^{(0)}}{\tau^{-1}.\mathcal{H}^{(0)}} = \frac{\Omega^{n+1}}{df \wedge \Omega^n} = \Omega_f \tag{9}$$

canonically. We conclude that;

Definition 4.1. *The identity*

$$\frac{\mathcal{H}^{(0)}}{\tau^{-1}.\mathcal{H}^{(0)}} = \frac{\Omega^{n+1}}{df \wedge \Omega^n} = \Omega_f$$

defines the extension fiber of the Gauss-Manin system of the isolated singularity $f : \mathbb{C}^{n+1} \to \mathbb{C}$.

The same conclusion can be obtained when f is a holomorphic germ, However one needs to consider the completions of the modules involved, (see [MA] page 422 or [S1]). In this way for $f : \mathbb{C}^{n+1} \to \mathbb{C}$ we have;

$$\frac{\hat{H}''}{\tau^{-1}\hat{H}''} \cong \Omega_f$$

.

By identifying the sections with those of relative cohomology, this formula is a direct consequence of the formula

$$\int_{\Gamma} e^{-\tau f} d\omega = \tau \int_{\Gamma} e^{-\tau f} df \wedge \omega, \qquad \omega \in \Omega_X^n$$

We refer to [MA] page 422 for details.

5. MHS on the extended fiber

Asuume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a germ of isolated singularity. In this section we build an isomorphism $\Phi : H^n(X_\infty, \mathbb{C}) \to \Omega_f$ which depends on an identification $Gr_V \Omega_f \cong \Omega_f$, and allows us to equip a mixed Hodge structure on Ω_f, [H1], [SAI6]. This also motivates the definition of opposite filtrations on a vector space. It is based on the following theorem.

Proposition 5.1. *(*[H1] *prop. 5.1) Assume $\{(\alpha_i, d_i)\}$ is the spectrum of a germ of isolated singularity $f : \mathbb{C}^{n+1} \to \mathbb{C}$. There exists elements $s_i \in C^{\alpha_i}$ with the properties*

(1) *$s_1, ..., s_\mu$ project onto a $\mathbb{C}$-basis of $\bigoplus_{-1<\alpha<n} Gr_V^\alpha H''/Gr_V^\alpha \partial_t^{-1} H''$.*
(2) *$s_{\mu+1} := 0$; there exists a map $\nu : \{1, ..., \mu\} \to \{1, ..., \mu, \mu+1\}$ with $(t - (\alpha_i + 1)\partial_t^{-1})s_i = s_{\nu(i)}$*
(3) *There exists an involution $\kappa : \{1, ..., \mu\} \to \{1, ..., \mu\}$ with $\kappa = \mu + 1 - i$ if $\alpha_i \neq \frac{1}{2}(n-1)$ and $\kappa(i) = \mu + 1 - i$ or $\kappa(i) = i$ if $\alpha_i = \frac{1}{2}(n-1)$, and*

$$P_S(s_i, s_j) = \pm \delta_{(\mu+1-i)j} . \partial_t^{-1-n}$$

.

Condition (1) implies

$$Gr_V^\alpha \partial_t^q H'' = \bigoplus_{\alpha_i - p = \alpha,\ p \leq q} \mathbb{C}.\partial_t^p.s_i$$

Condition (2) can be replaced by

[(2')] $(t - (\alpha_i + 1)\partial_t^{-1})s_i \in \bigoplus_{\alpha_j = \alpha_i + 1} \mathbb{C}.s_j$

in which case the involution $\kappa(i) = \mu - i + 1$ for any i.

The proof of the proposition 5.1 concerns with the construction of a $\mathbb{C}$-linear isomorphism as follows. Suppose,

$$H^n(X_\infty, \mathbb{C}) = \bigoplus_{p,q,\lambda} I_\lambda^{p,q}$$

is the Deligne-Hodge bigrading, and generalized eigen-spaces of vanishing cohomology, and also $\lambda = \exp(-2\pi i \alpha)$ with $\alpha \in (-1, 0]$. Consider the isomorphism obtained by composing the three maps,

$$\Phi_\lambda^{p,q} : I_\lambda^{p,q} \xrightarrow{\hat{\Phi}_\lambda} Gr_V^{\alpha+n-p} H'' \xrightarrow{pr} Gr_V^\bullet H''/\partial_t^{-1} H'' \xrightarrow{\cong} \Omega_f \tag{10}$$

where

$$\hat{\Phi}_\lambda^{p,q} := \partial_t^{p-n} \circ \psi_\alpha | I_\lambda^{p,q}$$

$$\Phi = \bigoplus\nolimits_{p,q,\lambda} \Phi_\lambda^{p,q}, \qquad \Phi_\lambda^{p,q} = pr \circ \hat{\Phi}_\lambda^{p,q}$$

ψ_α is the nearby functor.

Lemma: The map Φ is a well-defined $\mathbb{C}$-linear isomorphism.

We list some of the properties of the map ϕ as follows;

- $\hat{\Phi}_\lambda^{p,q}$ takes values in $C^{\alpha+n-p}$. By the formula $F^p = \oplus_{r\geq p} I^{r,s}$, any cohomology class in $I_\lambda^{p,q}$, is of the form $\psi_\alpha^{-1}[\partial_t^{n-p}h'' + V^{>\alpha}] = \psi_\alpha^{-1}\partial_t^{n-p}[h'' + V^{>\alpha+n-p}]$, for $h'' \in H''$, cf. def. 6.3.3. By substituting in the formula it explains the image of $\hat{\Phi}_\lambda^{p,q}$.
- Taking two different representatives $\omega_1, \omega_2 \in \Omega_X^{n+1}$ for h'' does not effect on the class $h'' + V^{>\alpha+n-p}$. Because by identifying H'' with its image in V^{-1}, the difference $\omega_1 - \omega_2$ belongs to $V^{>\alpha+n-p}$.
- The map Φ is obviously a $\mathbb{C}$-linear isomorphism because both of the ψ_α and ∂_t^{-1} are $\mathbb{C}$-linear isomorphisms on the appropriate domains, and
$$\Phi(I_\lambda^{p,q}) \subset \Phi(F^pH^n(X_\infty)_\lambda) \subset V^\alpha \cap H''/V^{\alpha+1} \hookrightarrow gr_V^\alpha H''/\partial_t^{-1}H''$$
- The definition of Φ fixes an isomorphism $Gr_V^\bullet \Omega_f \cong \Omega_f$. On the eigen space H_λ this corresponds to a choice of sections of $Gr_V^\alpha(V^\alpha \cap H'') \to Gr_V^\alpha H''/\partial_t^{-1}H''$ for $-1 \leq \alpha < 0$.

Definition 5.2. *(MHS on Ω_f) The mixed Hodge structure on Ω_f is defined by using the isomorphism Φ. This means that*

$$W_k(\Omega_f) = \Phi W_k H^n(X_\infty, \mathbb{Q}), \qquad F^p(\Omega_f) = \Phi F^p H^n(X_\infty, \mathbb{C})$$

and all the data of the Steenbrink MHS on $H^n(X_\infty, \mathbb{C})$ such as the $\mathbb{Q}$ or $\mathbb{R}$-structure is transformed via the isomorphism Φ to that of Ω_f. Specifically; in this way we also obtain a conjugation map

$$\bar{.} : \Omega_{f,\mathbb{Q}} \otimes \mathbb{C} \to \Omega_{f,\mathbb{Q}} \otimes \mathbb{C}, \qquad \Omega_{f,\mathbb{Q}} := \Phi H^n(X_\infty, \mathbb{Q}) \tag{11}$$

defined from the conjugation on $H^n(X_\infty, \mathbb{C})$ via this isomorphism.

The basis discussed in 5.1 is usually called a good basis. The condition (1) correspond to the notion of opposite filtrations. Two filtrations F and U on H are called opposite (cf. [SAI6] sec. 3) if

$$Gr_p^F Gr_U^q H = 0, \qquad \text{for } p \neq q$$

When one of the filtrations is decreasing say $\{F^p\}$ and the other increasing say $\{U_q\}$ then this is equivalent to

$$H = F^p \oplus U_{p-1}, \qquad \forall p, \tag{12}$$

Similarly, two decreasing filtrations F and U are said to be opposite if F is opposite to the increasing filtration $U'_q := U^{k-q}$, [P2].
In our situation this amounts to a choice of a section $s : H''/\partial_t^{-1}H'' \to H''$ of the projection $pr : H'' \to H''/\partial_t^{-1}H''$ such that the submodule generated by $Image(s)$ is $\oplus_\alpha(H'' \cap C^\alpha)$. Note also that $V^\alpha H''$ is the submodule generated by $s(V^\alpha\Omega_f)$.

Proposition 5.3. *(*[SAI6] *prop. 3.5) The filtration*

$$U^pC^\alpha := C^\alpha \cap V^{\alpha+p}$$

is opposite to the filtration Hodge filtration F on $\mathcal{G}$.

By this theorem the two filtrations F^p and $U'_q := U^{n-q} = \psi^{-1}(\oplus_\alpha C^\alpha \cap V^{\alpha+n-q})$ are two opposite filtrations on $H^n(X_\infty, \mathbb{C})$. This says

$$F^pH^n(X_\infty, \mathbb{C})_\lambda \cong U'_pH^n(X_\infty,)$$

A standard example of such a situation is when the variation of MHS namely $\mathcal{H}$ is mixed Tate (also called Hodge-Tate). By definition a mixed Tate Hodge structure H is when $Gr^W_{2l-1}H = 0$ and $Gr^W_{2l}H = \oplus_i\mathbb{Q}(-n_i)$. In that case one easily shows the Deligne-Hodge decomposition becomes

$$\bigoplus_p (W_{2p} \cap F^p)H = H_{\mathbb{C}}$$

and the two filtrations F and W are opposite.

Proposition 5.4. *(*[SAI6] *Theorem 3.6) There is a 1-1 correspondence between the opposite filtrations and the sections $s : H''/\partial_t^{-1}H'' \to H''$ compatible with the conditions of theorem 5.1.*

The data of an opposite filtration in a VMHS is equivalent to give a linear subspace $\mathcal{L} \subset \mathcal{G}$ such that:

- $\mathcal{G} = \mathcal{H}^{(0)} \oplus \mathcal{L}$ and
- $t^{-1} : \mathcal{L} \to \mathcal{L}$.
- $t\partial_t : \mathcal{L} \to \mathcal{L}$.

It is the same as choosing a section to the projection (cf. [SAI6], [H1], [LLS]);

$$\mathcal{H}^{(0)} \to \mathcal{H}^{(0)}/t.\mathcal{H}^{(0)}. \tag{13}$$

Example 5.5. *This example is taken from* [SAI6]*. If n is even, the duality S on $H^n(X_\infty, \mathbb{C})_1$ is anti-symmetric. Assume $H^n(X_\infty, \mathbb{C})_1 = H' \oplus H''$ as a direct sum of MHS, compatible with S and N, where*

$$H' = \oplus_{0\le i\le 3}H'_i, \qquad H'_i = \mathbb{Q}(-i-(n-2)/2),$$
$$NH'_i = H'_{i-1}(i > 0)$$
$$H'' = \oplus_{0\le i\le 2}H''_i, \qquad H''_i = \mathbb{Q}(-i-(n-2)/2),$$
$$NH''_2 = H''_1,\ NH''_1 = NH''_0 = 0$$

and

$$S(H',H'')=0$$
$$S(H'_i,H'_j)\neq 0, \qquad \textit{only when } i+j=3$$
$$S(H''_i,H''_j)\neq 0, \qquad \textit{only when } i+j=2$$

Then we have $H'_i = F^pW_{2p}H'$ for $p=i+(n-2)/2$ and we obtain a filtration U opposite to F on $H'\oplus H''$ compatible with S. If we choose generators as $H'_i =< e_i >$, $H''_i =< f_i >$ such that $S(e_0,e_3)=S(f_1,f_2)$. For the corresponding section we have

$$P_S(Im(v),Im(v))\subset \mathbb{C}\partial_t^{-1-n}$$

The situation explained in this example appears for the singularity $f=x^{10}+y^{10}+z^{10}+w^{10}+(xyzw)^2+v^2$, cf. [SAI6] *page 60.*

Remark 5.6. *(*[SAI6] *page 42) By definition we have the isomorphism*

$$H^n(X_\infty,\mathbb{C})_\lambda \cong gr_V^\alpha H'', \qquad -1\leq \alpha<0$$

It is compatible with the Hodge filtrations;

$$F^pH^n(X_\infty,\mathbb{C})_\lambda \cong \partial_t^{n-p}gr_V^\alpha H''$$

In general

$$H''\cap V^\alpha\mathcal{G}\neq (H''\cap V^\alpha)+(H''\cap V^{>\alpha})$$

This is why we have to take $Gr_V^\alpha \cong C^\alpha$.

Remark 5.7. *The complex structure defined on Ω_f via $\Phi: H^n(X_\infty)\cong\Omega_f$ is not unique, and it depends to the good basis chosen, or the section of $H''\to H''/\partial_t^{-1}H''$. However it does not affect the polarization, discussed in the next section.*

6. Polarization form on extension

Assume $f:\mathbb{C}^{n+1}\to\mathbb{C}$ is a germ of isolated singularity. We use the isomorphism $\Phi: H^n(X_\infty,\mathbb{C})\to\Omega_f$ introduced in the previous section to express a correspondence between polarization form on vanishing cohomology and the Grothendieck pairing on Ω_f.

Theorem 6.1. *Assume $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$, is a holomorphic germ with isolated singularity at 0, with $f:X\to T$ the associated Milnor fibration. Embed the Milnor fibration in a projective fibration $f_Y:Y\to T$ of degree d (with d large enough), by inserting possibly a singular fiber over 0. Then, the isomorphism Φ makes the following diagram commutative up to a complex constant;*

$$\begin{array}{ccc} \widehat{Res}_{f,0} : \Omega_f \times \Omega_f & \longrightarrow & \mathbb{C} \\ \Big\downarrow{\scriptstyle (\Phi^{-1},\Phi^{-1})} & & \Big\downarrow{\scriptstyle \times *} \\ S : H^n(X_\infty) \times H^n(X_\infty) & \longrightarrow & \mathbb{C} \end{array} \qquad * \neq 0 \tag{14}$$

where,

$$\widehat{Res}_{f,0} = res_{f,0}\ (\bullet, \tilde{C}\ \bullet)$$

and $\tilde{C}$ is defined relative to the Deligne decomposition of Ω_f, via the isomorphism Φ. If $J^{p,q} = \Phi^{-1} I^{p,q}$ is the corresponding subspace of Ω_f, then

$$\Omega_f = \bigoplus_{p,q} J^{p,q} \qquad \tilde{C}|_{J^{p,q}} = (-1)^{p(d-1)/d} \tag{15}$$

In other words;

$$S(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \times\ \ res_{f,0}(\omega, \tilde{C}.\eta), \qquad 0 \neq * \in \mathbb{C} \tag{16}$$

Part of this proof is given in [CIR] for homogeneous fibrations in the context of mirror symmetry, see also [PH].

Proof. Before starting the proof lets mention that the map Φ is classically used to correspond the mixed Hodge structure on $H^n(X_\infty, \mathbb{C})$ and Ω_f. We only prove the correspondence on polarizations.
Step 1: Choose a $\mathbb{C}$-basis of the module $\Omega(f)$, namely $\{\phi_1, ..., \phi_\mu\}$, where $\phi_i = f_i.d\underline{x}$. We identify the class $[e^{-f/t}\phi_i]$ with a cohomology class in $H(X_t)$. We may also choose the basis $\{\phi_i\}$ so that the forms $\{\eta_i = e^{(-f/t)}\phi_i\}$ correspond to a basis of vanishing cohomology, by the formula

$$\int_\Gamma e^{-\tau f} \omega = \int_0^\infty e^{-t\tau} \int_{\Gamma \cap X_t} \frac{\omega}{df} \mid_{X_t} \tag{17}$$

Step 2: In this step, we assume the Poincarè product is non-degenerate. By this assumption we may also assume f is homogeneous of degree d and ϕ_i's are chosen by homogeneous basis of Ω_f. Consider the deformation

$$f_s = f + \sum_{i=0}^{n} s_i x_i$$

and set

$$S_{ij}(s, z) := \langle [e^{-f_s/z}\phi_i], [e^{+f_s/z}\phi_j] \rangle.$$

The cup product is the one on the relative cohomology, and we may consider it in the projective space $\mathbb{P}^{n+1}$. The perturbation f_s and also the Saito form S_{ij} are weighted homogeneous. This can be seen by choosing new weights, $\deg(x_i) = 1/d$, $\deg(s_i) = 1-1/d$, and $\deg(z) = 1$ then the invariance of the product with respect to the change of variable $x \to \lambda^{1/d}x,\ z \to \lambda z$, shows that $S_{ij}(s,z)$ is weighted homogeneous. We show that S_{ij} is some multiple of $\widehat{Res}_{f,0}$.

$$S_{ij}(s,z) := (-1)^{n(n+1)/2}(2\pi iz)^{n+1}(Res_f(\phi_i,\phi_j) + O(z)).$$

Suppose that s is generic so that $x \to Re(f_s/z)$ is a Morse function. Let $\Gamma_1^+, ..., \Gamma_\mu^+$, (resp. $\Gamma_1^-, ..., \Gamma_\mu^-$) denote the Lefschetz thimbles emanating from the critical points $\sigma_1, ..., \sigma_\mu$ of $Re(f_s/t)$ given by the upward gradient flow (resp. downward). Choose an orientation so that $\Gamma_r^+.\Gamma_s^- = \delta_{rs}$. We have

$$S_{ij}(s,z) = \sum_{r=1}^{\mu}(\int_{\Gamma_r^+} e^{-f_s/z}\phi_i)(\int_{\Gamma_r^-} e^{f_s/z}\phi_j)\text{-}$$

For a fixed argument of z we have the stationary phase expansion as $z \to 0$.

$$(\int_{\Gamma_r^+} e^{-f_s/z}\phi_i) \cong \pm\frac{(2\pi z)^{(n+1)/2}}{\sqrt{Hess f_s(\sigma_r)}}(f_i(\sigma_r) + O(z))$$

where $\phi_i = f_i(x)d\underline{x}$. Therefore,

$$S_{ij}(s,z) = (-1)^{n(n+1)/2}(2\pi iz)^{n+1}\sum_{r=1}^{\mu}(\frac{f_i(\sigma_r)f_j(\sigma_r)}{Hess(f_s)(\sigma_r)} + O(z))$$

where the lowest order term in the right hand side equals the Grothendieck residue. As this holds for an arbitrary argument of z, and S_{ij} is holomorphic for $z \in \mathbb{C}^*$; the conclusion follows for generic s. By analytic continuation the same holds for all s. By homogenity we get,

$$S_{ij}(0,z) = (-1)^{n(n+1)/2}(2\pi iz)^{n+1}Res_f(\phi_i,\phi_j). \tag{18}$$

Note that there appears a sign according to the orientations chosen for the integrals; however this only modifies the constant in the theorem. Thus, we have;

$$S_{ij}(0,1) = (-1)^{n(n+1)/2}(2\pi i)^{n+1}Res_f(\phi_i,\phi_j). \tag{19}$$

Step 3: The sign appearing in residue pairing is caused by compairing the two products

$$(e^{-f}\phi_i, e^{-f}\phi_j), \qquad (e^{-f}\phi_i, e^{+f}\phi_j). \tag{20}$$

Assume we embed the fibration in a projective one as before, replacing f with a homogeneous polynomial germ of degree d. We can consider a change of variable as $I : z \to e^{\pi i/d}z$ which changes f by $-f$. Thus this map is an involution on the value of f. The cohomology bases in $Gr_F^p Gr_{n+1}^W H^n \subset I^{p,q}$ can be charcterized by the degree of forms. If $\phi_i = f_i dz$, with f_i homogeneous, with $p \leq deg(\phi_i) = l(\phi_i) < p+1$. This shows the cohomology class $e^{-f}\phi_j$ after the this change of variable is replaced by $c_p.e^{+f}\phi_j$ where $c_p \in \mathbb{C}$ only depends to the Hodge filtration (defined by degree of forms). By the above change of variable we obtain;

$$(e^{-f}\phi_i, I^* e^{-f}\phi_j) = (e^{-f}\phi_i, (-1)^{\deg \phi_j/d} e^{+f}\phi_j) \tag{21}$$

because $I^d = id$, if we iterate I^*, d times we obtain;

$$(e^{-f}\phi_i, e^{-f}\phi_j) = res_{f,0}(a\ , (-1)^{(d-1)\deg \phi_j/d}.b)$$

The Riemann-Hodge bilinear relations in $H_{\neq 1}$ implies that, the products of the forms under consideration is non-zero except when the degrees of ϕ_i and ϕ_j sum to n. This explains the formula in $H_{\neq 1}$. The above argument will still hold when the form is replaced by $(\bullet, N_Y \bullet)$, by the linearity of N_Y. Thus, we still have the same result on $H_{\neq 1}$.
Step 4: In case the Poincarè product is degenerate, we still assume f is homogeneous but we change the cup product by applying N_Y on one component. The same relation can be proved between the level form $(\bullet, N_Y \bullet)$, and the corresponded local residue, i.e.

$$S_Y(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = *.\ \widehat{Res}(a\ , \tilde{C}.b), \qquad * \neq 0,\ a, b \in \Omega_f$$

implies

$$S_Y(\Phi^{-1}(\omega), N_Y.\Phi^{-1}(\eta)) = *.\ \widehat{Res}(a\ , \mathfrak{f}.\tilde{C}.b), \qquad * \neq 0, \quad a, b \in \Omega_f$$

where $\mathfrak{f}$ is the nilpotent transformation corresponded to N_Y via Φ. The proof is complete.

$\square$

Remark 6.2. *([PH] page 37) Setting*

$$\psi_s^i(\omega, \tau) = \int_{\Gamma(i)} e^{-\tau f}\omega$$

$$\bar{\psi}_s^i(\omega', \tau) = \int_{\Gamma'(i)} e^{+\tau f}\omega'$$

with $\zeta = \frac{\omega}{df}$, $\zeta' = \frac{\omega'}{df}$, *the expression (which is the same as in the proof)*

$$P_s([\zeta], [\zeta'])(\tau) = \sum_{i=1}^{\mu} \psi_s^i(\tau, \omega)\bar{\psi}_s^i(\tau, \omega') = \sum_{r=0}^{\infty} P_s^r([\zeta], [\zeta'])(\tau).\tau^{-n-r} \tag{22}$$

is a presentation of K. Saito higher residue pairing.

Corollary 6.3. *Assume $f:\mathbb{C}^{n+1}\to\mathbb{C}$ is an isolated singularity germ. The polarization form of the MHS of vanishing cohomology and the modified residue pairing on the extended fiber Ω_f are given by the same matrix in corresponding bases.*

7. Riemann-Hodge bilinear relations for Ω_f

The isomorphism $\Phi: H^n(X_\infty,\mathbb{C})\to\Omega_f$ transforms the mixed Hodge structures already defined for $H^n(X_\infty)$ to Ω_f. It makes a correspondence between the Deligne-Hodge decompositions and also the Lefschetz decompositions. We use this to organize the polarization on the fiber Ω_f.

Theorem 7.1. *Assume $f:\mathbb{C}^{n+1}\to\mathbb{C}$ is a holomorphic isolated singularity germ. The modified Grothendieck residue provides a polarization for the extended fiber Ω_f, via the aforementioned isomorphism Φ. Moreover, there exists a unique set of forms $\{\widehat{Res}_k\}$ polarizing the primitive subspaces of $Gr_k^W\Omega_f$ providing a graded polarization for Ω_f.*

Proof. Because $H^n(X_\infty)$ is graded polarized, hence using theorem 6.1 Ω_f is also graded polarized via the isomorphism Φ. By the Mixed Hodge Metric theorem, the Deligne-Hodge decomposition;

$$\Omega_f=\bigoplus_{p,q}J^{p,q} \tag{23}$$

is graded polarized and there exists a unique hermitian form; $\mathcal{R}$ with,

$$i^{p-q}\mathcal{R}(v,\bar{v})>0,\qquad v\in J^{p,q} \tag{24}$$

and the decomposition is orthogonal with respect to $\mathcal{R}$. Here the conjugation is that in (11). This shows that the polarization forms $\{\widehat{Res}_l\}$ are unique if exist.
Let $N:=\log M_u$ be the logarithm of the unipotent part of the monodromy for the Milnor fibration defined by f. We have

$$H^n(X_\infty)=\bigoplus_r N^rP_{l-2r},\qquad P_l:=\ker N^{l+1}:Gr_l^WH^n\to Gr_{-l-2}^WH^n$$

and the level forms

$$S_l:P_l\otimes P_l\to\mathbb{C},\qquad S_l(u,v):=S(u,N^lv)$$

polarize the primitive subspaces P_l. By using the isomorphism Φ, similar type of decomposition exists for Ω_f. That is the isomorphic image $P_l':=\Phi^{-1}P_l$ satisfies

$$\Omega_f=\bigoplus_r N^rP'_{l-2r},\qquad P_l':=\ker \mathfrak{f}^{l+1}:Gr_l^W\Omega_f\to Gr_{-l-2}^W\Omega_f$$

and the level forms

$$\widehat{Res}_l:P_l'\otimes P_l'\to\mathbb{C},\qquad \widehat{Res}_l:=\widehat{Res}(u,\mathfrak{f}^lv)$$

polarize the primitive subspaces P'_l, where $\mathfrak{f}$ is the map induced from multiplication by f on $Gr_l^W \Omega_f$. Specifically, this shows

- $\widehat{Res}_l(x,y) = 0, \qquad x \in P'_r, y \in P'_s, r \neq s$
- $\widehat{Res}_l(C_l x, \mathfrak{f}^l \bar{x}) > 0, \qquad 0 \neq x \in P'_l$

  where C_l is the corresponding Weil operator.

□

Remark 7.2. *Let $\mathcal{G}$ be the Gauss-Manin system associated to a polarized variation of Hodge structure $(\mathcal{L}_{\mathbb{Q}}, \nabla, F, S)$ of weight n, with $S : \mathcal{L}_{\mathbb{Q}} \otimes \mathcal{L}_{\mathbb{Q}} \to \mathbb{Q}(-n)$ the polarization. Then we have the isomorphism*

$$\bigoplus_{k \in \mathbb{Z}} Gr_F^k \mathcal{G} \to \bigoplus_{k \in \mathbb{Z}} Hom_{\mathcal{O}_X}(Gr_F^{n-k}\mathcal{G}, \mathcal{O}_X) \tag{25}$$

given by (up to a sign factor) $\lambda \to S(\lambda, -)$, for $\lambda \in Gr_F^k \mathcal{G}$.

The following corollary is easily obtained in the course of the proof of Theorem 7.1.

Corollary 7.3. *The polarization S of $H^n(X_\infty)$ will always define a polarization of Ω_f, via the isomorphism Φ. In other words S is also a polarization in the extension.*

The Riemann-Hodge bilinear relations for the MHS on Ω_f and its polarization $\widehat{Res}$ would be that of an opposite MHS to $(H^n(X_\infty), S)$.

Corollary 7.4. *(Riemann-Hodge bilinear relations for Ω_f) Assume the holomorphic isolated singularity Milnor fibration $f : X \to T$ can be embedded in a projective fibration of degree d with $d >> 0$. Suppose $\mathfrak{f}$ is the corresponding map to N on $H^n(X_\infty)$, via the isomorphism Φ. Define*

$$P_l = PGr_l^W := \ker(\mathfrak{f}^{l+1} : Gr_l^W \Omega_f \to Gr_{-l-2}^W \Omega_f)$$

Going to W-graded pieces;

$$\widehat{Res}_l : Gr_l^W \Omega_f \otimes_{\mathbb{C}} Gr_l^W \Omega_f \to \mathbb{C} \tag{26}$$

is non-degenerate and according to Lefschetz decomposition

$$\Omega_f = \bigoplus_r \mathfrak{f}^r P_{l-2r}$$

we will obtain a set of non-degenerate bilinear forms,

$$\widehat{Res}_l \circ (id \otimes \mathfrak{f}^l) : PGr_l^W \Omega_f \otimes_{\mathbb{C}} PGr_l^W \Omega_f \to \mathbb{C}, \tag{27}$$

$$\widehat{Res}_l = res_{f,0}\ (id \otimes \tilde{C}.\ \mathfrak{f}^l) \tag{28}$$

where $\tilde{C}$ is as in 6.1, such that the corresponding hermitian form associated to these bilinear forms is positive definite. In other words,

- $\widehat{Res}_l(x,y) = 0, \qquad x \in P_r,\ y \in P_s,\ r \neq s$
- *If $x \neq 0$ in P_l,*

$$res_{f,0}\ (C_l x, \tilde{C}.\ \mathfrak{f}^l.\bar{x}) > 0$$

where C_l is the corresponding Weil operator.

Proof. This follows directly from 6.1 and 7.1. □

Note that the map

$$A_f = \frac{\mathcal{O}_X}{\partial f} \to \Omega_f, \qquad f \mapsto f dx_0 ... dx_n$$

is an isomorphism. Thus, the above corollary would state similarly for A_f.

Remark 7.5. *(*[DW] *page 53, 54, prop. 2.6 -* [SA7] *sec. 5) Consider the map*

$$F : \Omega_X^{n+1} \to i_* \bigcup_z Hom\big(H_n(X, f^{-1}(\eta.\frac{z}{|z|}), \mathbb{Z}) \cong \oplus_i \mathbb{Z}\Gamma_i, \mathbb{C}\big), \qquad \mathcal{H} := Im(F)$$

$$\omega \mapsto [z \to (\Gamma_i \to \int_{\Gamma_i} e^{-t/z}\omega)],$$

where Γ_i are the classes of Lefschetz thimbles. The vector bundle $\mathcal{H}$ is exactly the Fourier-Laplace transform of the cohomology bundle $R^n f_* \mathbb{C}_{S^*} = \cup_t H^n(X_t, \mathbb{C})$, equipped with a connection with poles of order at most two at ∞.

$$(\cup_t H^n(X_t, \mathbb{C}), \nabla) \leftrightarrows (\mathcal{H}, \nabla')$$

The modified Grothendieck residue

$$\widehat{Res}_{f,0} = res_{f,0}(\bullet, \tilde{C}\bullet)$$

with $\tilde{C}$ as in 6.1, is the Fourier-Laplace transform of the polarization S on $H^n(X_\infty, \mathbb{C})$.

8. Real Structure vs real splitting

In this section we show the possibility to modify the Hodge filtration in the commutative diagram of Theorem 6.1 in a way to obtain a real split MHS. In the following we work with a MHS (H, F, W) and $\mathfrak{g} = \mathfrak{gl}(H) = End_{\mathbb{C}}(H)$. We begin by the following theorem.

Theorem 8.1. *(*[CKS] *sec. 2) Given a mixed Hodge structure (W, F), there exists a unique $\delta \in \mathfrak{g}_{\mathbb{R}}^{-1,-1}(W, F)$ s.t. $(W, e^{-i\delta}.F)$ is a mixed Hodge structure which splits over $\mathbb{R}$.*

In the course to prove Theorem 8.1 one shows the existence of a unique $Z \in \mathfrak{g}^{-1,-1}$ such that

$$\overline{J^{p,q}} = e^{Z}.J^{p,q}, \qquad \bar{Z} = -Z$$

The operation Z obviously preserves the weight filtration. We write $Z = -2i\delta$. Define another Hodge filtration by setting

$$\tilde{F} := e^{i.\delta}.F$$

Since $\delta \in \mathfrak{g}_{\mathbb{R}}^{-1,-1} \subset W_{-2}^{\mathfrak{gl}}$, this element leaves W invariant and acts trivially on the quotient Gr_l^W. Therefore both $F, \tilde{F}$ induce the same filtrations on $Gr_l^W H$. Now it is clear that

$$e^{-i.\delta}.J^{p,q} = e^{i.\delta}.\overline{J^{p,q}}$$

gives a real splitting for H.

This non-trivial fact specifically applies to the mixed Hodge structure of $H^n(X_\infty),\ \Omega_f$. It means that a modification of Hodge filtration of both MHS provides a real splitting in the Theorem 6.1. Another

$$\tilde{C}_1 := Ad(e^{-i.\delta}).\tilde{C} = Ad(e^{i.\delta}).\overline{\tilde{C}}, \qquad Ad(g) : X \mapsto gXg^{-1},\ Ad : G \to Gl(\mathfrak{g})$$

is a real transformation (notation of theorem 6.1).

Proposition 8.2. *The bigrading $J_1^{p,q}$ defined by $J_1^{p,q} := e^{-i.\delta}.J^{p,q}$ is split over $\mathbb{R}$. The operator $\tilde{C}_1 = e^{-i.\delta}.\tilde{C} : \Omega_f \to \Omega_f$ defines a real structure on Ω_f.*

This says if $\Omega_{f,1} = \oplus_{p<q} J_1^{p,q}$ then

$$\Omega_f = \Omega_{f,1} \oplus \overline{\Omega_{f,1}} \oplus \bigoplus_p J_1^{p,p}, \qquad \overline{J_1^{p,p}} = J_1^{p,p}$$

The statement of theorem 6.1 is valid when the operator $\tilde{C}$ is replaced with $\tilde{C}_1$;

$$S(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \times \ \mathrm{res}_{f,0}(\omega, \tilde{C}_1.\eta), \qquad 0 \neq * \in \mathbb{C} \tag{29}$$

and this equality is defined over $\mathbb{R}$.

There is one more word to be mentioned here. That is in the construction of theorem 6.1, although the map $\tilde{C}$ seems to be a linear map between vector spaces, however it can also be thought of as a bundle map of the corresponding VMHS. In this way it is a C^∞-map, because the decomposition into Deligne-Hodge sub-bundles is in general C^∞. In proposition 8.2 it is this bundle map that considered.

By a real structure on the polarized extended Gauss-Manin system (M, K) where K is a sesqui-linear skew hermitian non-degenerate form, we mean an anti-linear isomorphism $\kappa : M \cong M$ such that

$$\kappa^2 = id, \qquad K(\kappa \bullet, \kappa \bullet) = \overline{K(\bullet, \bullet)}$$

and $\nabla\kappa = 0$. Here K is the polarization. The form defined by

$$g(\bullet, \bullet) = K(\bullet, \kappa \bullet)$$

is symmetric and also holomorphic. Moreover, g is real on $\ker(\kappa - id)$, (cf. [DW], expose 1, C. Sabbah, page 4). In this sense

$$S(\omega, \eta) = *. \widehat{\mathrm{Res}}(\Phi(\omega), \Phi(\eta)) = *. \mathrm{res}_{f,0}(\Phi(\omega), \kappa.\Phi(\eta)), \qquad * \neq 0$$

Another interesting fact is that, a polarization of the form

$$K : \mathcal{H}' \otimes_{\mathcal{O}} \overline{\mathcal{H}} \to \mathbb{C}[t, t^{-1}]$$

induces an isomorphism

$$\mathcal{H}'^{\vee} \cong_{\mathcal{O}} \overline{\mathcal{H}}$$

We can glue the above bundles by this isomorphism obtained from the polarization. Thus, the process of gluing is equivalent to polarization. Therefore, in former situation we have

$$\mathcal{H}^{(0)\vee} \cong \overline{\mathcal{G}_\infty}, \qquad \Rightarrow \qquad \Omega_f^{\vee} \cong \overline{H^n(X_\infty, \mathbb{C})}$$

as PVMHS, and PMHS respectively. The corresponding connections are given by

$$\nabla' : \mathcal{H}' \to \frac{1}{z}\Omega^1 \otimes \mathcal{H}', \qquad \nabla : \overline{\mathcal{H}} \to z\Omega^1 \otimes \overline{\mathcal{H}}$$

respectively, [DW] exp. 1, pages 12, 13.

9. Opposite Filtrations arising in VMHS's

The concept of opposite filtrations plays an important role in the study of asymptotic behaviour of a VMHS and Mirror symmetry. In this section we compare the theorems 6.1 and 7.1 with some results in asymptotic Hodge theory due to G. Pearlstein and J. Fernandez, [P2] developing some works of P. Deligne, [D2]. We begin by the following definition;

Definition 9.1. *Let S be a complex manifold, and A a sub-field of $\mathbb{R}$. A pure, polarized A-Hodge structure of weight k over S consists of, a local system of finite dimensional A-vector spaces $\mathcal{V}_A$ over S equipped with a decreasing Hodge filtration $\mathcal{F}$ of $\mathcal{V} = \mathcal{V}_A \otimes \mathcal{O}_S$ by holomorphic sub-bundles, and a flat $(-1)^k$-symmetric bilinear form $Q : \mathcal{V} \times \mathcal{V} \to A$ such that*

- *$\mathcal{F}$ and $\bar{\mathcal{F}}$ are k-opposed.*
- *$\mathcal{F}$ is horizontal, i.e. $\nabla(\mathcal{F}) \subset \mathcal{F} \otimes \Omega^1_S$*
- *Q polarizes each fiber of $\mathcal{V}$.*

A variation of graded polarized A-mixed Hodge structure is defined analogously having horizontality for F, and a collection of $(Gr_A^{\mathcal{W}}, \mathcal{F}Gr_k^{\mathcal{W}}, Q_k)$ of pure polarized A-Hodge structures.

Theorem 9.2. *(P. Deligne) Let $\mathcal{V} \to \triangle^{*n}$ be a variation of pure polarized Hodge structure of weight k, for which the associated limiting mixed Hodge structure is Hodge-Tate. Then the Hodge filtration $\mathcal{F}$ pairs with the shifted monodromy weight filtration $\mathcal{W}[-k]$, of $\mathcal{V}$, to define a Hodge-Tate variation over a neighborhood of 0 in $\triangle^{*n}$.*

Theorem 9.3. *(P. Deligne) Let $\mathcal{V}$ be a variation of mixed Hodge structure, and*

$$\mathcal{V} = \bigoplus_{p,q} I^{p,q}$$

denotes the C^∞-decomposition of $\mathcal{V}$ to the sum of C^∞-subbundles, defined by point-wise application of Deligne theorem. Then the Hodge filtration $\mathcal{F}$ of $\mathcal{V}$ pairs with the increasing filtration

$$\bar{\Phi}_q = \sum_k \bar{\mathcal{F}}^{k-q} \cap \mathcal{W}_k \tag{30}$$

to define an un-polarized $\mathbb{C}VHS$.

Remark 9.4. *Given a pair of increasing filtrations A and B of a vector space V one can define the convolution $A * B$ to be the increasing filtration*

$$A * B = \sum_{r+s=q} A_r \cap B_s. \tag{31}$$

In particular for any F setting $F_r^\vee = F^{-r}$, then the increasing filtration $\bar{\Phi}$ is given by the formula

$$\bar{\Phi} = \overline{\mathcal{F}^\vee} * \mathcal{W} \tag{32}$$

Theorem 9.5. *(G. Pearlstein-J. Fernandez)*[P2] *Let $\mathcal{H}$ be an admissible variation of graded polarized mixed Hodge structures with quasi-unipotent monodromy, and $\mathcal{V} = \oplus I^{p,q}$ the decomposition relative to the limiting mixed Hodge structure. Define*

$$\Psi_p = \bigoplus_{a \leq p} I^{p,q} \tag{33}$$

and $\mathfrak{g}_-^\Psi = \{\alpha \in \mathfrak{g}_{\mathbb{C}} | \alpha(\Psi_p) \subset \Psi_{p-1}\}$, then;

(a) Ψ *is opposite to F_∞. Moreover, relative to the decomposition*

$$\mathfrak{g} = \bigoplus_{r,s} \mathfrak{g}^{r,s} \tag{34}$$

(b) *If $\psi(s) : \triangle^{*n} \to \check{D}$ is the associated untwisted period map, then in a neighborhood of the origin it admits a unique representation of the form*

$$\psi(s) = e^{\Gamma(s)}.F_\infty \tag{35}$$

where $\Gamma(s)$ is a $\mathfrak{g}_-^{\psi}$-valued function.

(c) *Ψ is independent of the coordinate chosen for F_∞. Moreover,*

$$\Psi = \overline{F_{nilp}^{\vee}} * W = \overline{F_\infty^{\vee}} * W. \tag{36}$$

Here above F_{nilp} is an arbitrary element in the nilpotent orbit of the limit Hodge filtration corresponded to the nilpotent cone (i.e. positive linear combination) of the logarithms of the generators of the monodromy group, i.e $F_{nilp} = exp(z_1N_1 + ... + z_rN_r)F_\infty$ where N_k are logarithms of different local monodromies, cf. [P2].

Theorem 9.6. *Let $\mathcal{V}$ be an admissible variation of polarized mixed Hodge structure associated to a holomorphic germ of an isolated hyper-surface singularity. Set*

$$\Psi = \overline{F_\infty^{\vee}} * W. \tag{37}$$

Then Ψ extends to a filtration $\underline{\Psi}$ of $\mathcal{V}$ by flat sub-bundles, which pairs with the limit Hodge filtration $\mathcal{F}$ of $\mathcal{V}$, to define a polarized $\mathbb{C}$-variation of Hodge structure, on a neighborhood of the origin.

Proof. The first part of the theorem that the two filtrations F_∞ and Ψ pair opposite together in a neighborhood of 0 was shown in sec. 5, see also [P2]. The way that it is polarized is the content of Theorems 6.1, 7.1 and 7.4. ☐

Remark 9.7. [P2] *Associated to a MHS (F, W) the inclusion*

$$\bigoplus_{s \le q} I^{r,s} \subset \bigoplus_k W_k \cap F^{k-q} = \bigoplus_{s \le q} \bar{I}^{r,s} \tag{38}$$

is easily verified. For VMHS, $\mathcal{V}$, the Griffiths transversality for $\mathcal{F}$ induces a similar one for the increasing filtration $\Phi_\bullet$;

$$\nabla \Phi_q \subset \Omega^1 \otimes \Phi_{q+1}. \tag{39}$$

To the C^∞-vector bundle

$$E = \bigoplus_p \mathcal{U}^p, \qquad \mathcal{U}^p = \bigoplus_q I^{p,q} \tag{40}$$

$\mathcal{F}, \bar{\Phi}$ are the two filtrations associated. Then Griffiths transversality is equivalent to saying that the decomposition defines a complex variation of Hodge structure.

The polarization of a complex variation of Hodge structure will probably be interpreted to mean a parallel hermitian form which makes the system of Hodge bundles $\mathcal{U}^p$ orthogonal, and becomes positive definite on multiplying the form by $(-1)^p$ on $\mathcal{U}^p$. Suppose that in the situation of Theorem 8.6 there is any such hermitian form $\mathcal{R}$. Then, on the one hand since $\mathcal{R}$ and Ψ are flat, so is the orthogonal complement of Ψ_{p-1} in Ψ_p. On the other hand, the way things have been setup, the orthogonal complement of Ψ_{p-1} in Ψ_p is exactly

$$\mathcal{U}^p = \Psi_p \cap F^p \tag{41}$$

But this is the system of Hodge bundles, and so the Hodge filtration is also flat.

10. Primitive elements

The extension of an PVMHS's may be explained by the solvability of the Poincarè-Birkhoff problem associated to the Gauss-manin systems. One step in solving the Poincarè-Birkhoff problem for the Gauss-Manin system of f is to glue different lattices in the Gauss-Manin vector bundle to obtain a vector bundles over $\mathbb{C}P(1)$. In this section we briefly follow C. Sabbah in [SA2] to explain the primitive elements in the variation of MHS of vanishing cohomology.

Assume $\mathcal{G}$ the associated Gauss-Manin system of the isolated singularity $f : \mathbb{C}^{n+1} \to \mathbb{C}$ and $\phi_1, ..., \phi_\mu$, a frame basis for $\mathcal{G}$ and (α, s_α) is the spectral pairs of f. According to [SA2] it is possible to choose the basis in a way that we have the following recursive relations;

$$\phi_{s(i)+k} = \partial_t^{-k} \prod_\alpha (t\partial_t - \alpha)^j \phi_{s(i)}, \qquad 1 \leq i \leq r, \ 1 \leq k \leq k_i. \tag{42}$$

for specific numbers $0 \leq s(1), ..., s(r) \leq \mu$. In this way we reach a set of forms $\phi_{s(i)}$ indexed by spectral numbers which produce other basis elements by applying the operators $t\partial_t - \alpha$ successively. They also describe $Gr_l^W Gr_\alpha^V \mathcal{G}$ concretely. These forms are called primitive elements relative to the nilpotent operator induced by $t\partial_t - \alpha$ on C_α. They provide information about the Jordan blocks structure in $\mathcal{G}$. If we denote the Jordan block as

$$B_k := \langle N^j[\omega_{k_0}] \mid j = 0, ..., \nu_k \rangle,$$

then it holds that;

$$\overline{B_{\alpha,l}} = \begin{cases} B_{1-\alpha,\nu_k-l}, & \alpha \in]0,1[\\ B_{0,\nu_k-l}, & \alpha = 0 \end{cases}$$

cf. [SA2].

Proposition 10.1. [SA5], [SA2] *There is a 1-1 correspondence between opposite filtrations on $H^n(X_\infty)_\mathbb{C}$ and free, rank μ, $\mathbb{C}[t]$-submodules $\mathcal{G}_\infty$ on which the connection is logarithmic where $\mathcal{G}_0, \mathcal{G}_\infty$ define a trivial vector bundle on $\mathbb{P}^1$.*

The submodule $\mathcal{G}_\infty$ in Proposition 9.5.2 is given by;

$$\mathcal{G}_\infty = \mathbb{C}[t]\langle [\omega_0], ..., \partial_t^{-s_0}[\omega_0], ..., \partial_t^{-s_r}[\omega_r] \rangle.$$

The primitive elements provide the good bases of the Brieskorn module. They also prove the existence of a solution to The Poincarè-Birkhoff problem. In such a basis the matrix of the operator t has the form;

$$t = A_0 + A_1 \partial_t^{-1} \tag{43}$$

where A_0, A_1 are square matrices of size μ and A_1 is a diagonal matrix. It holds (cf. [SA5], [H1]) that, in such a basis the K. Saito higher residue form [S1] takes the form

$$K_f(\eta_i, \eta_j) = \delta_{\kappa(i)j}.\partial_t^{-n-1}, \tag{44}$$

where δ is the Kronecker delta and κ is an involution of the set $\{1, ..., \mu\}$ as index set of a specific basis of $\mathcal{G}$ namely $\{\eta_i\}_{i=1}^{\mu}$.

11. Family of curve Jacobians

Let V be a complex vector space and Λ a discrete lattice of maximal rank. Let $\Pi = (\pi_{ij})$ be the $2n \times n$ matrix such that

$$dx_i = \sum_{\alpha} \pi_{i\alpha} dz_\alpha + \bar{\pi}_{i\alpha} d\bar{z}$$

A necessary and sufficient condition for the complex torus $M = V/\Lambda$ to be an abelian variety is given by the well-known Riemann conditions. M is an abelian variety iff there exists an integral skew symmetric matrix Q such that

$$^t\Pi.Q\Pi = 0$$

and

$$-\sqrt{-1}^t\Pi.Q\bar{\Pi} > 0$$

In terms of the matrix $\tilde{\Pi} = (\Pi, \bar{\Pi})$

$$-\sqrt{-1}^t\Pi.Q\bar{\Pi} = \begin{pmatrix} H & 0 \\ 0 & -^tH \end{pmatrix}$$

where $H > 0$. These conditions can also be written in terms of the inverse matrix $\tilde{\Omega} = \begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix}$ similarly. There exists a basis for Λ such that the matrix of Q in this basis is of the form

$$Q = \begin{pmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{pmatrix}, \qquad \Delta_\delta = \begin{pmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{pmatrix}, \quad \delta_i \in \mathbb{Z}$$

One can modify this process more to choose a complex basis $e_1, ..., e_n$ such that $\Omega = (\delta_\delta, Z)$ with Z symmetric and $\text{Im}Z > 0$, [G3].

Theorem 11.1. [G3] *$M = V/\Lambda$ is an abelian variety iff there exists an integral basis for Λ and a complex basis for V such that,*

$$\Omega = (\Delta_\delta, Z)$$

with Z symmetric and $ImZ > 0$.

Then the form

$$\omega = \sum_{i=0}^{n} \delta_i \; dx_i \wedge dx_{n+i}$$

namely the (invariant harmonic) Hodge form is non-degenerate and provides an embedding of M in projective space. The form ω is also called a polarization of M, and δ_i's are called elementary divisors of ω. When $\delta_\alpha = 1$ the abelian variety is called principally polarized.

The basic example of a principally polarized abelian variety is the Jacobian variety of a complex Riemann surface S of genus g. It is given by the choice of a basis $\delta_1, ..., \delta_{2g}$ for $H_1(S, \mathbb{Z})$ and a basis $\omega_1, ..., \omega_g$ for $H^0(S, \Omega^1)$, we have

$$\mathcal{I}(S) = \frac{\mathcal{C}^g}{\mathbb{Z}\{\lambda_1, ..., \lambda_{2g}\}}$$

where λ_i are the columns of the matrix

$$\lambda_i =^t \Big(\int_{\delta_i} \omega_1, ..., \int_{\delta_g} \omega_g\Big)$$

We may choose the bases such that

$$\int_{\lambda_i} \omega_\alpha = \delta_{i\alpha}, \qquad 1 \leq i, \alpha \leq g$$

Then the period matrix would be of the form

$$\Omega = (I, Z)$$

Thus $\mathcal{I}(S)$ is an abelian variety principally polarized given in terms of the basis $\{dx_i\}$ for $H^1(\mathcal{I}(S), \mathbb{Z})$ dual to the basis $\{\lambda_i\} \in H_1(\mathcal{I}(S), \mathbb{Z})$, by

$$\omega = \sum dx_\alpha \wedge dx_{n+\alpha}$$

Geometrically $\mathcal{I}(S) = H^0(S, \Omega^1)^\vee / H_1(S, \mathbb{Z})$, where $H_1(S, \mathbb{Z})$ is embedded in $H^0(S, \Omega^1)^\vee$ by integration. Then the polarization form $\omega \in H^2(\mathcal{I}(S), \mathbb{Z}) = Hom_{\mathbb{Z}}(\bigwedge^2 H_1(S, \mathbb{Z}), \mathbb{Z})$ is the skew symmetric bilinear form

$$H_1(S, \mathbb{Z}) \otimes H_1(S, \mathbb{Z}) \to \mathbb{Z}$$

given by intersection of cycles, [G3]. Thus we have shown the following important fact;

Theorem 11.2. [G3] *Let C be a smooth projective curve over the field $\mathbb{C}$, and $J(C)$ its Jacobian. Then we have a canonical isomorphism $H^1(C,\mathbb{C}) = J(C)$, such that the Poincare duality of $H^1(C,\mathbb{C})$ is identified with the polarization of $J(C)$, given by the Θ-divisor.*

Suppose now that

$$\mathcal{A}_s = J^1(H_s^1) = H_{s,\mathbb{Z}}^1 \setminus H_{s,\mathbb{C}}^1 / F^0 H_{s,\mathbb{C}}^1$$

$$J(\mathcal{H}) = \bigcup_{s \in S^*} J^1(H_s)$$

is the family of Jacobians associated to the variation of Hodge structure in a projective degenerate family of algebraic curves (here we have assumed the Hodge structures have weight -1). Then the fibers of this model would be principally polarized abelian varieties. The polarization of each fiber is given by the Poincare product of the middle cohomology of the curves, via a holomorphic family of Θ-divisors. Then as a matter similar to the previous sections the extended fiber is a Jacobian of the opposite Hodge filtration. In this way the extended fiber would be an abelian variety and principally polarized, with some Θ-divisor.

Theorem 11.3. *The extension of a degenerate holomorphic family of Θ-divisors polarizing the Jacobian of curves in a projective fibration, is a Θ-divisor polarizing the extended Jacobian.*

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