

POSITIVITY OF HOCHSTER THETA

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ABSTRACT. M. Hochster defines an invariant namely $\Theta(M, N)$ associated to two finitely generated module over a hyper-surface ring $R = P/f$, where $P = k\{x_0, \dots, x_n\}$ or $k[X_0, \dots, X_n]$, for k a field and f is a germ of holomorphic function or a polynomial, having isolated singularity at 0. This invariant can be lifted to the Grothendieck group $G_0(R)_{\mathbb{Q}}$ and is compatible with the chern character and cycle class map, according to the works of W. Moore, G. Piepmeyer, S. Spiroff, M. Walker. They prove that it is semi-definite when f is a homogeneous polynomial, using Hodge theory on Projective varieties. It is a conjecture that the same holds for general isolated singularity f . We give a proof of this conjecture using Hodge theory of isolated hyper-surface singularities when $k = \mathbb{C}$. We apply this result to give a positivity criteria for intersection multiplicity of proper intersections in the variety of f .

INTRODUCTION

One of the important ways to generalize known facts about smooth algebraic varieties in algebraic geometry, is to try to extend them over singular varieties. Most of the valid methods in the smooth category are hard to be worked out in the singular category. An example of this is the definition of intersection multiplicity as the *Tor*-formula in algebraic geometry as an Euler characteristic. The Riemann-Roch formula provides a definition of this characteristic class by the chern character map on the K -group. The useful observation here is the intersection theory is in fact a theory on K -groups. In the singular set up both of the definitions of K -groups and the chern character become very complicated in the first glance. although trying to generalize the homology theory over singular varieties provides some type of solution for this case, trying to extend the natural concepts from the smooth category to the singular one remains still complicated and difficult.

In this short note we provide a very basic example of this situation and try to approach to the solution using asymptotic Hodge theory. In this way we employ some standard methods in the extensions of polarized mixed Hodge structure to answer questions about singularities in algebraic geometry. Thus the reader may divide the text into three part, the first is a singular set up in algebraic geometry,

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the second is on variation of mixed Hodge structures, and the third is an application of the results mentioned in part 2.

1. MODULES OVER HYPERSURFACE RINGS

A hyper-surface ring is a ring of the form $R := P/(f)$, where P is an arbitrary ring and f a non-zero divisor. Localizing we may assume P is a local ring of dimension $n+1$. As according to the title we assume $P = \mathbb{C}\{x_0, \dots, x_n\}$ and f a holomorphic germ, or $P = \mathbb{C}[x_0, \dots, x_n]$ and then f would be a polynomial. Then we are mainly interested to study finitely generated modules over these rings. A matrix factorization of f in P is a pair of matrices A and B such that $AB = BA = f \cdot id$. It is equivalent to the data of a pair of finitely generated free P -modules

$$d_0 : X^0 \rightleftarrows X^1 : d_1, \quad d_0 d_1 = d_1 d_0 = f \cdot id$$

It is a basic fact, discovered by D. Eisenbud, that the R -modules have a minimal resolution that is eventually 2-periodic. Specifically, In a free resolution of such a module M , we see that after n -steps we have an exact sequence of the following form.

$$(1) \quad 0 \rightarrow M' \rightarrow F_{n-1} \rightarrow F_{n-2} \rightarrow \dots \rightarrow F_0 \rightarrow M \rightarrow 0$$

where the F_i are free R -modules of finite rank and $depth_R(M') = n$. If $M' = 0$ then M has a free resolution of finite length., If $M' \neq 0$, then M' is a maximal Cohen-Macaulay module, that is $depth_R(M') = n$. So "up to free modules" any R -module can be replaced by a maximal Cohen-Macaulay module. If M is a maximal Cohen-Macaulay R -module that is minimally generated by p elements, its resolution as P -module has the form

$$\begin{array}{ccccccccc} 0 & \rightarrow & P^p & \xrightarrow{A} & P^p & \rightarrow & M & \rightarrow & 0 \\ & & & & \downarrow B & & \downarrow & & \\ & & & & \swarrow & & \downarrow & & 0 \\ 0 & \rightarrow & P^p & \xrightarrow{A} & P^p & \rightarrow & M & \rightarrow & 0 \end{array}$$

where A is some $p \times p$ matrix with $det(A) = f^q$. The fact that multiplication by f acts as 0 on M produces a matrix B such that $A \cdot B = B \cdot A = f \cdot I$, where I is the identity matrix. In other words we find a matrix factorization (A, B) of f determined uniquely up to base change in the free module P^p , by M . This matrix factorization not only determines M but also a resolution of M as R -module.

$$\dots \rightarrow R^p \rightarrow R^p \rightarrow R^p \rightarrow M \rightarrow 0$$

So a minimal resolution of M looks in general as follows

$$\dots \rightarrow G \rightarrow F \rightarrow G \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_0 \rightarrow M \rightarrow 0$$

As a consequence all the homological invariants like $Tor_k^R(M, N)$, $Ext_R^k(M, n)$ are 2-periodic, [BVS], [EP].

2. HOCHSTER THETA FUNCTION

Definition 2.1. (*Hochster Theta pairing*) The theta pairing of two R -modules M and N over a hyper-surface ring $R/(f)$ is

$$\Theta(M, N) := l(\text{Tor}_{2k}^R(M, N)) - l(\text{Tor}_{2k+1}^R(M, N)), \quad k \gg 0$$

This definition makes sense as soon as the length appearing are finite. This certainly happens if R has an isolated singular point.

Example 2.2. [BVS] Take $f = xy - z^2$, $M = \mathbb{C}[[xyz]]/(x, y)$. A matrix factorization (A, B) associated to M is given by

$$A = \begin{pmatrix} y & -z \\ -z & x \end{pmatrix}, \quad B = \begin{pmatrix} x & z \\ z & y \end{pmatrix}$$

And $\text{Tor}_k^R(M, M)$ is the homology of the complex

$$\dots \rightarrow \mathbb{C}[[y]]^2 \rightarrow \mathbb{C}[[y]]^2 \rightarrow \mathbb{C}[[y]] \rightarrow 0$$

where

$$\alpha = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$$

So we find that $\Theta(M, M) = 0$.

Hochster theta pairing is additive on short exact sequences in each argument, and thus determines a \mathbb{Z} -valued pairing on $G(R)$, the Grothendieck group of finitely generated R -modules. One loses no information by tensoring with \mathbb{Q} and so often theta is interpreted as a symmetric bilinear form on the rational vector space $G(R)_{\mathbb{Q}}$. It is basic that Theta would vanish if either M or N be Artinian or have finite projective dimension [MPSW], [BVS].

Theorem 2.3. [BVS] When $M = \mathcal{O}_Y = R/I$, $N = \mathcal{O}_Z = R/J$, where $Y, Z \subseteq X_0$ are the sub-varieties defined by the ideals I, J respectively, then

$$\Theta(\mathcal{O}_Y, \mathcal{O}_Z) = i(0; Y, Z)$$

in case that $Y \cap Z = 0$. Here $i(0; ,)$ is the ordinary intersection multiplicity in \mathbb{C}^{n+1} .

By additivity over short exact sequences and the fact that any module admits a finite filtration with sub-quotients of the form R/I , knowing $\Theta(\mathcal{O}_Y, \mathcal{O}_Z)$ determines $\Theta(M, N)$ for all modules M, N .

Theorem 2.4. [BVS] Assume $f \in \mathbb{C}[[x_1, \dots, x_{2m+2}]]$ is a homogeneous polynomial of degree d , and $X_0 = f^{-1}(0) \in \mathbb{C}^{2m+2}$ and $T = V(f) \in \mathbb{P}^{2m+1}$ the associated projective cone of degree d . Let Y and Z be also co-dimension m cycles in T . If Y, Z intersect transversely, then

$$\Theta(\mathcal{O}_Y, \mathcal{O}_Z) = -\frac{1}{d}[[Y]].[[Z]]$$

Where $[[Y]] := d[Y] - \text{deg}(Y).h^m$ is the primitive class of $[Y]$, with $h \in H^1(T)$ the hyperplane class.

The primitive class of a cycle Y is the projection of its fundamental class $[Y] \in H^m(T)$ into the orthogonal complement to h^m with respect to the intersection pairing into $H^{2m}(T) = \mathbb{C}$. As $h^m.h^m = d = \text{deg}(T)$ and $[Y].h^m = \text{deg}(Y)$ the description of the primitive class follows. Substituting the claim can be reformulated

$$\Theta(\mathcal{O}_Y, \mathcal{O}_Z) = -\frac{1}{d}[[Y]].[[Z]] = -d[Y].[Z] + \text{deg}(Y)\text{deg}(Z)$$

Where $[Y].[Z]$ denotes the intersection form on the cohomology of the projective space, [BVS].

When f in consideration is a homogeneous polynomial of degree d , such that $X := \text{Proj}(R)$ is a smooth k -variety, the Theta pairing is induced, via chern character map, from the pairing on the primitive part of de Rham cohomology

$$\frac{H^{(n-1)/2}(X, \mathbb{C})}{\mathbb{C}.\gamma^{(n-1)/2}} \times \frac{H^{(n-1)/2}(X, \mathbb{C})}{\mathbb{C}.\gamma^{(n-1)/2}} \rightarrow \mathbb{C}$$

given by

$$(a, b) \rightarrow (\int_X a \cup \gamma^{(n-1)/2})(\int_X a \cup \gamma^{(n-1)/2}) - d(\int_X a \cup b)$$

where γ is the class of a hyperplane section and theta would vanish for rings of this type having even dimensions. When $n = 1$ by γ^0 we mean $1 \in H^0(X, \mathbb{C})$, [MPSW].

Theorem 2.5. [MPSW] *For R and X as above, if n is odd there is a commutative diagram*

$$(2) \quad \begin{array}{ccc} G(R)_{\mathbb{Q}}^{\otimes 2} & \longleftarrow & \left(\frac{K(X)_{\mathbb{Q}}}{\alpha} \right)^{\otimes 2} \\ \Theta \downarrow & & \downarrow (ch^{n-1/2})^{\otimes 2} \\ \mathbb{C} & \longleftarrow_{\theta} & \left(\frac{H^{(n-1)/2}(X, \mathbb{C})}{\mathbb{C}.\gamma^{(n-1)/2}} \right)^{\otimes 2} \end{array}$$

If However f is not quasi-homogeneous, there no longer will be a projective variety to do intersection theory on.

Theorem 2.6. [BVS] *Let $f \in P = \mathbb{C}[[x_0, \dots, x_n]]$ define an isolated singularity and let M, N be $R = P/(f)$ -modules.*

- (1) *If n is even then $\Theta(M, N) = 0$*
(2) *If $n = 2m + 1$ is odd, then*

$$\Theta(\mathcal{O}_Y, \mathcal{O}_Z) = lk(ch(M), ch(N))$$

Here $ch : K^0(L) \rightarrow H^{ev}(L)$ is the chern character. Only the m -component $ch^m \in H^{2m}(L, \mathbb{Q})$ contributes to linking form so, $\theta(\mathcal{O}_Y, \mathcal{O}_Z) = lk(ch^m(M), ch^m(N))$.

Remark 2.7. *The linking form is a restriction of the Seifert form of the singularity,*

$$L : H_n(X_t) \times H_n(X_t) \rightarrow \mathbb{Z}, \quad (\alpha, \beta) \rightarrow l(\alpha, h_{1/2}(\beta))$$

where $h_{1/2}$ is half monodromy rotation.

Theorem 2.8. [MPSW] For R and X as above and n odd the restriction of the pairing $(-1)^{(n+1)/2}\Theta$ to

$$\text{im}(ch^{\frac{n-1}{2}}) : K(X)_{\mathbb{Q}}/\alpha \rightarrow \frac{H^{(n-1)/2}(X, \mathbb{C})}{\mathbb{C} \cdot \gamma^{\frac{n-1}{2}}}$$

is positive definite. i.e. $(-1)^{(n+1)/2}\Theta(v, v) \geq 0$ with equality holding if and only if $v = 0$. In this way θ is semi-definite on $G(R)$.

Proof. [MPSW] Define

$$W = H^{n-1}(X(\mathbb{C}), \mathbb{Q}) \cap H^{\frac{n-1}{2}, \frac{n-1}{2}}(X(\mathbb{C}))$$

It is classical that the image of $ch^{(n-1)/2}$ is contained in W . Define $e : W/\mathbb{Q} \cdot \gamma^{(n-1)/2} \hookrightarrow H^{n-1}(X, \mathbb{Q})$ by

$$e(a) = a - \frac{\int_X a \cup \gamma^{(n-1)/2}}{d} \cdot \gamma^{(n-1)/2} \in W$$

We know that $\theta(a, b) = -d \cdot I^{coh}(e(a), e(b))$ Now the theorem follows from the polarization properties of cup product on cohomology of projective varieties. \square

3. HODGE THEORY AND RESIDUE FORM

Assume $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a germ of isolated singularity. We choose a representative $f : X \rightarrow T$ over a small disc T according to the milnor fibration theorem. It is possible to embed the Milnor fibration $f : X \rightarrow T$ into a compactified (projective) fibration $f_Y : Y \rightarrow T$ such that the fiber Y_t sits in \mathbb{P}^{n+1} for $t \neq 0$. The projective fibration f_Y has a unique singularity at $0 \in Y_0$ over $t = 0$. Then, there exists a short exact sequence

$$(3) \quad 0 \rightarrow H^n(Y_0, \mathbb{Q}) \rightarrow H^n(Y_t, \mathbb{Q}) \xrightarrow{i^*} H^n(X_t, \mathbb{Q}) \rightarrow 0, \quad t \neq 0.$$

We have $H^n(Y_0, \mathbb{Q}) = \ker(M_Y - id)$, by the invariant cycle theorem, where M_Y is the monodromy of f_Y . The form $S_Y := (-1)^{n(n-1)/2} I_Y^{coh} : H^n(Y_t, \mathbb{Q}) \times H^n(Y_t, \mathbb{Q}) \rightarrow \mathbb{Q}$ is the polarization form of pure Hodge structure on $H^n(Y_t, \mathbb{C})$, $t \in T'$ (the punctured disc). W. Schmid (resp. J. Steenbrink) has defined a canonical MHS on $H^n(Y_t, \mathbb{Q})$ (resp. on $H^n(Y_t, \mathbb{Q})$) namely limit MHS, which make the above sequence an exact sequence of MHS's.

In the short exact sequence, the map i^* is an isomorphism on $H^n(Y_t, \mathbb{Q})_{\neq 1} \rightarrow H^n(X_t, \mathbb{Q})_{\neq 1}$ giving $S = (-1)^{n(n-1)/2} I^{coh} = (-1)^{n(n-1)/2} I_Y^{coh} = S_Y$ on $H^n(X_t, \mathbb{Q})_{\neq 1}$.

The above short exact sequence restricts to the following one,

$$(4) \quad 0 \rightarrow \ker\{N_Y : H^n(Y_t, \mathbb{Q})_1 \rightarrow H^n(Y_t, \mathbb{Q})_1\} \rightarrow H^n(Y_t, \mathbb{Q})_1 \rightarrow H^n(X_t, \mathbb{Q})_1 \rightarrow 0$$

So $a, b \in H^n(X_t, \mathbb{Q})_1$ have pre-images $a_Y, b_Y \in H^n(Y_t, \mathbb{Q})_1$ and

$$(5) \quad S(a, b) = S_Y(a_Y, (-N_Y)b_Y)$$

is independent of the lifts of a_Y, b_Y , by the fact that N_Y is an infinitesimal isometry for S_Y . The equation (10) defines the desired polarization on $H^n(X_t, \mathbb{Q})_1$. The polarization form S is M -invariant, non-degenerate, $(-1)^n$ -symmetric on $H^n(X_t, \mathbb{Q})_{\neq 1}$ and $(-1)^{n+1}$ -symmetric on $H^n(X_t, \mathbb{Q})_1$, [H1].

Suppose,

$$H^n(X_\infty, \mathbb{C}) = \bigoplus_{p,q,\lambda} I_\lambda^{p,q}$$

is the Deligne-Hodge bigrading, and generalized eigen-spaces of vanishing cohomology, and also $\lambda = \exp(-2\pi i\alpha)$ with $\alpha \in (-1, 0]$. Consider the isomorphism obtained by composing the three maps,

$$(6) \quad \Phi_\lambda^{p,q} : I_\lambda^{p,q} \xrightarrow{\hat{\Phi}_\lambda} Gr_V^{\alpha+n-p} H'' \xrightarrow{pr} Gr_V^\bullet H'' / \partial_t^{-1} H'' \xrightarrow{\cong} \Omega_f$$

where

$$\begin{aligned} \hat{\Phi}_\lambda^{p,q} &:= \partial_t^{p-n} \circ \psi_\alpha | I_\lambda^{p,q} \\ \Phi &= \bigoplus_{p,q,\lambda} \Phi_\lambda^{p,q}, \quad \Phi_\lambda^{p,q} = pr \circ \hat{\Phi}_\lambda^{p,q} \end{aligned}$$

where $I^{p,q}$ stands for the bigrading, ∂_t is the Gauss-Manin connection and ψ_α is the nearby map cf. [1]. The map Φ is obviously an isomorphism because both of the ψ_α and ∂_t^{-1} are isomorphisms, cf. [H1], [V].

Definition 3.1. (MHS on Ω_f) We define a mixed Hodge structure on Ω_f using the isomorphism Φ . This means that all the data of the Steenbrink MHS on $H^n(X_\infty, \mathbb{C})$ such as the \mathbb{Q} or \mathbb{R} -structure, the weight filtration $W_\bullet \Omega_{f,\mathbb{Q}}$ and the Hodge filtration $F^\bullet \Omega_{f,\mathbb{C}}$ is defined via the isomorphism Φ . Specifically; in this we obtain a conjugation map

$$(7) \quad \bar{\cdot} : \Omega_{f,\mathbb{Q}} \otimes \mathbb{C} \rightarrow \Omega_{f,\mathbb{Q}} \otimes \mathbb{C}, \quad \Omega_{f,\mathbb{Q}} := \Phi^{-1} H^n(X_\infty, \mathbb{Q})$$

defined from the conjugation on $H^n(X_\infty, \mathbb{C})$ via this isomorphism.

Recall that the limit (Steenbrink) mixed Hodge structure, is defined by

$$F^p H^n(X_\infty, \mathbb{C})_\lambda = \psi_\alpha^{-1} (Gr_V^\alpha \partial_t^{n-p} H'')$$

This justifies the power of ∂_t^{-1} applied in the definition of Φ .

Theorem 3.2. [1] Assume $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$, is a holomorphic germ with isolated singularity at 0, with $f : X \rightarrow T$ the associated Milnor fibration. Embed the Milnor fibration in a projective fibration $f_Y : Y \rightarrow T$ of degree d (with d large

enough), by inserting possibly a singular fiber over 0. Then, the isomorphism Φ makes the following diagram commutative up to a complex constant;

$$(8) \quad \begin{array}{ccc} \widehat{Res}_{f,0} : \Omega_f \times \Omega_f & \longrightarrow & \mathbb{C} \\ \downarrow (\Phi^{-1}, \Phi^{-1}) & & \downarrow \times * \\ S : H^n(X_\infty) \times H^n(X_\infty) & \longrightarrow & \mathbb{C} \end{array} \quad * \neq 0$$

where,

$$\widehat{Res}_{f,0} = res_{f,0} (\bullet, \tilde{C} \bullet)$$

and \tilde{C} is defined relative to the Deligne decomposition of Ω_f , via the isomorphism Φ . If $J^{p,q} = \Phi^{-1} I^{p,q}$ is the corresponding subspace of Ω_f , then

$$(9) \quad \Omega_f = \bigoplus_{p,q} J^{p,q} \quad \tilde{C}|_{J^{p,q}} = (-1)^{p(d-1)/d}$$

In other words;

$$(10) \quad S(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \times res_{f,0}(\omega, \tilde{C}.\eta), \quad 0 \neq * \in \mathbb{C}$$

Remark 3.3. Let \mathcal{G} be the Gauss-Manin system associated to a polarized variation of Hodge structure $(\mathcal{L}_{\mathbb{Q}}, \nabla, F, S)$ of weight n , with $S : \mathcal{L}_{\mathbb{Q}} \otimes \mathcal{L}_{\mathbb{Q}} \rightarrow \mathbb{Q}(-n)$ the polarization. Then we have the isomorphism

$$(11) \quad \bigoplus_{k \in \mathbb{Z}} Gr_F^k \mathcal{G} \rightarrow \bigoplus_{k \in \mathbb{Z}} Hom_{\mathcal{O}_X}(Gr_F^{n-k} \mathcal{G}, \mathcal{O}_X)$$

given by (up to a sign factor) $\lambda \rightarrow S(\lambda, -)$, for $\lambda \in Gr_F^k \mathcal{G}$.

Theorem 3.4. Assume $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is a holomorphic isolated singularity germ. The modified Grothendieck residue provides a polarization for the extended fiber Ω_f , via the aforementioned isomorphism Φ . Moreover, there exists a unique set of forms $\{\widehat{Res}_k\}$ polarizing the primitive subspaces of $Gr_k^W \Omega_f$ providing a graded polarization for Ω_f .

Corollary 3.5. (Riemann-Hodge bilinear relations for Ω_f) Assume the holomorphic isolated singularity Milnor fibration $f : X \rightarrow T$ can be embedded in a projective fibration of degree d with $d \gg 0$. Suppose \mathfrak{f} is the corresponding map to N on $H^n(X_\infty)$, via the isomorphism Φ . Define

$$P_l = PGr_l^W := \ker(\mathfrak{f}^{l+1} : Gr_l^W \Omega_f \rightarrow Gr_{-l-2}^W \Omega_f)$$

Going to W -graded pieces;

$$(12) \quad \widehat{Res}_l : Gr_l^W \Omega_f \otimes_{\mathbb{C}} Gr_l^W \Omega_f \rightarrow \mathbb{C}$$

is non-degenerate and according to Lefschetz decomposition

$$\Omega_f = \bigoplus_r \mathfrak{f}^r P_{l-2r}$$

we will obtain a set of non-degenerate bilinear forms,

$$(13) \quad \widehat{Res}_l \circ (id \otimes \mathfrak{f}^l) : PGr_l^W \Omega_f \otimes_{\mathbb{C}} PGr_l^W \Omega_f \rightarrow \mathbb{C},$$

$$(14) \quad \widehat{Res}_l = res_{f,0} (id \otimes \tilde{C} \cdot \mathfrak{f}^l)$$

where \tilde{C} is as in 3.2, such that the corresponding hermitian form associated to these bilinear forms is positive definite. In other words,

- $\widehat{Res}_l(x, y) = 0$, $x \in P_r$, $y \in P_s$, $r \neq s$
- If $x \neq 0$ in P_l ,

$$res_{f,0} (C_l x, \tilde{C} \cdot \mathfrak{f}^l \bar{x}) > 0$$

where C_l is the corresponding Weil operator, and the conjugation is as in (7).

Note that the map

$$A_f = \frac{\mathcal{O}_X}{\partial f} \rightarrow \Omega_f, \quad f \mapsto f dx_0 \dots dx_n$$

is an isomorphism. Thus, the above corollary would state similarly for A_f .

4. HOCHSCHILD HOMOLOGY AND MATRIX FACTORIZATION

The category of matrix factorizations of f over R , namely $MF(R, f)$; is defined to be the differential $\mathbb{Z}/2$ -graded category, whose objects are pairs (X, d) , where $X = X^0 \oplus X^1$ is a free $\mathbb{Z}/2$ -graded R -module of finite rank equipped with an R -linear map d of odd degree satisfying $d^2 = f \cdot id_X$. Here the degree is calculated in $\mathbb{Z}/2$. Regarding to the first definition

$$d = \begin{pmatrix} 0 & d_0 \\ d_1 & 0 \end{pmatrix}, \quad d^2 = f \cdot id$$

The morphisms $MF(X, X')$ are given by $\mathbb{Z}/2$ -graded R -module maps from X to X' (or equivalent between the components X^0 and X^1) provided that the differential is given by

$$(15) \quad d(f) = d_{X'} \circ f - (-1)^{|f|} f \circ d_X.$$

Here d_X or d'_X may be considered as the matrix given above or to be separately d_0 and d_1 , and also it is evident that $d(f)^2 = 0$. By choosing bases for X^0 and X^1 we reach to the former definition, [EP].

Remark 4.1. *There exist exotic versions of the definition of matrix factorization which allows the free modules to be of infinite ranks, denoted $MF^\infty(R, f)$. Another is the definition may be shifted to be stated for sheaves and schemes rather than rings. Then, we would begin with a scheme X , noetherian, separated of finite dimension, and f would be a function in the structure sheaf \mathcal{O}_X locally a non-zero divisor, and X^0, X^1 would be coherent sheaves (or analogously in other set up be quasi-coherent sheaves). Also the derived version of the definition, that is to consider the object up to homotopy equivalences, in order to study the categories of matrix factorization, [D], [PV], [EP].*

The Hochschild homology of $MF(R, f)$ is equivalent to the Hochschild homology of the commutative ring $R = P/f$. The Hochschild chain complex of $MF(R, f)$ is quasi-isomorphic to the Koszul complex of the regular sequence $\partial_0 f, \dots, \partial_n f$. In particular the Hochschild homology (and also the Hochschild cohomology) of 2-periodic dg-category $MF(R, f)$ is isomorphic to the module of relative differentials or the Jacobi ring of f , [D].

Theorem 4.2. *(T. Dyckerhoff) [D], [PV] The canonical bilinear form on the Hochschild homology of category of matrix factorizations $\mathcal{C} = MF(P, f)$ of f , after the identification*

$$(16) \quad HH_* MF(P, f) \cong A_f \otimes dx[n]$$

coincides with

$$(17) \quad \langle g \otimes dx, h \otimes dx \rangle = (-1)^{n(n-1)/2} res_{f,0}(g, h)$$

5. CHERN CHARACTER (DENIS TRACE MAP) FOR ISOLATED HYPERSURFACE SINGULARITIES

The chern character or Denis trace map is a ring homomorphism

$$(18) \quad ch : K_0(X) \rightarrow HH_0(X) \cong \Omega_f$$

where K' is free abelian group on the isomorphism classes of finitely generated modules modulo relations obtained from short exact sequences. The construction of chern character map or chern classes is functorial w.r.t flat pull back. In the special case of $i : X \hookrightarrow Y$ the compactification, the following diagram commutes,

$$(19) \quad \begin{array}{ccccc} K'_0(Y_0) & \xrightarrow{ch_Y} & HH_0(Y_0) \cong \Omega_f^Y & \xrightarrow{\Phi_Y^{-1}} & H^n(Y_\infty) \\ i^* \downarrow & & \downarrow i^* & & \downarrow i^* \\ K'_0(X_0) & \xrightarrow{ch_X} & HH_0(X_0) \cong \Omega_f^X & \xrightarrow{\Phi_X^{-1}} & H^n(X_\infty). \end{array}$$

Given a matrix factorization (A, B) for a maximal Cohen-Macaulay M , one can find de Rham representatives for the chern classes. Consider $\mathbb{C}[[x_0, \dots, x_n]]$ as a $\mathbb{C}[[t]]$ -module with t acting as multiplication by f . Denote by Ω^p the module of germs of p -forms on \mathbb{C}^{n+1} , and let $\Omega_f^p = \Omega^p / (df \wedge \Omega^{p-1})$. One puts $\omega(M) = dA \wedge dB$. The components of the chern character

$$(20) \quad ch_M := tr(\exp(\omega(M))) = \sum_i \frac{1}{i!} \omega^i(M)$$

are well-defined classes

$$(21) \quad \omega^i(M) = tr((dA \wedge dB)^i) \in \Omega_f^{2i} / (df \wedge \Omega^{2i-1})$$

There are however odd degree classes

$$\eta^i(M) := tr(AdB(dA \wedge dB)^i) \in \Omega_f^{2i+1} / \Omega_f^{2i}$$

The group $\Omega_f^{2i+1} / d\Omega_f^{2i}$ can be identified with the cyclic homology $HC_i(P/\mathbb{C}\{t\})$. They fit into the following short exact sequence such that $d\eta^{i-1} = \omega^i(M)$.

$$0 \rightarrow \Omega_f^{2i-1} / \Omega_f^{2i-2} \rightarrow \Omega_f^{2i} / (df \wedge \Omega^{2i-1}) \rightarrow \Omega_f^{2i} / \Omega_f^{2i-1} \rightarrow 0$$

If the number of variables $n+1$ is even, then a top degree form sits in the Brieskorn module

$$\mathcal{H}_f^{(0)} = \Omega^n / (df \wedge d\Omega^{n-1})$$

a free $\mathbb{C}[[t]]$ -module of rank μ . The higher residue pairing

$$K : \mathcal{H}_f^{(0)} \times \mathcal{H}_f^{(0)} \rightarrow \mathbb{C}\{\{\partial_t^{-1}\}\}$$

of K. Saito can be seen as the de Rham realization of the Seifert form of the singularity, [BVS].

Proposition 5.1. [PV], [CW] *The Hochster Θ -pairing of two maximal Cohen-Macaulay modules M, N is given up to a sign by the local residue of their chern classes as elements in Ω_f . That is*

$$\Theta(M, N) = (-1)^{n(n-1)/2} res_{f,0}(ch(M), ch(N))$$

only for M, N maximal Cohen-Macaulay.

Remark 5.2. *There always exists a change of coordinates such that the natural map*

$$i^* : H^n(Y_\infty, \mathbb{C}) \rightarrow H^n(X_\infty, \mathbb{C})$$

becomes surjective. We can regard the chern character as an element of $H^n(X_\infty, \mathbb{C})$ via the map Φ in section 3. In this way functoriality of chern classes implies that a chern class corresponded to $ch(M)$ is always in the image of another chern class in projective space, and it must be a Hodge class, i.e of type (p, p) in the pure HS of $H^n(Y_\infty, \mathbb{C})$.

6. POSITIVITY OF THETA PAIRING-MAIN RESULT

The following theorem was conjectured in [MPSW].

Theorem 6.1. *Let S be an isolated hypersurface singularity of dimension n . If n is odd, then $(-1)^{(n+1)/2}\Theta$ is positive semi-definite on $G(R)_\mathbb{Q}$.*

Proof. By additivity of Θ on each variable, we may replace M, N by maximal Cohen-Macaulay modules. According to this, determination of the sign of Θ amounts to understanding how the image of chern classes look like in the MHS of Ω_f . By theorem 3.2 it amounts to the same things for the image in $H^n(X_\infty)$ under the isomorphism Φ . The following diagram is commutative by the functorial properties of chern character.

$$(22) \quad \begin{array}{ccc} K'_0(Y_0) & \xrightarrow{\Phi_Y^{-1} \circ ch_Y} & H^n(Y_\infty) \\ i^* \downarrow & & \downarrow i^* \\ K'_0(X_0) & \xrightarrow{\Phi_X^{-1} \circ ch_X} & H^n(X_\infty). \end{array}$$

We are assuming that i^* is surjective. By what was said, the chern class we are concerned with, is a Hodge cycle. The commutativity of the above diagram allows us to replace the pre-image of the chern character for X , with similar cycle upstairs. Because the polarization form S_X was defined via that of S_Y . Thus, if

$$H^n(Y_\infty) = \bigoplus_{p+q=n} H^{p,q}$$

be the Hodge decomposition, the only non trivial contribution in the cup product will be for the $H^{n/2, n/2}$, and the polarization form is evidently definite on this subspace (Hodge cycles). Note that here the corresponding chern class should lie in $H_{\neq 1}^n$. Because the map N_Y is of type $(-1, -1)$ for the Hodge structure of $H^n(Y_\infty)$ and the polarization $S_Y(H^{n/2, n/2}, H^{n/2-1, n/2-1}) = 0$ for obvious reasons. In this way one only needs to prove the positivity statement for Hochster Θ when the chern character is in $H_{Y, \neq 1}$, and this is the content of Theorem 2.8. \square

7. APPLICATION TO SERRE MULTIPLICITY CONJECTURE

The intersection multiplicity of finitely generated modules over a regular ring has been defined by J. P. Serre, as

$$(23) \quad \chi_R(M, N) := \sum_{i=0}^{\dim R} (-1)^i l(\text{Tor}_i^R(M, N))$$

where R is a regular ring, and M, N are finitely generated R -modules. He conjectured that this invariant is 0 in non-proper intersections, and positive when the intersection is proper, i.e $\dim M + \dim N = \dim R$. This conjecture is still open for positivity.

Theorems 6.1 and 2.3 together recover a very special case of Serre multiplicity conjecture, for intersections at the isolated singularity (proved in general by O. Gabber).

Theorem 7.1. *The Serre intersection multiplicity $\chi(M, N)$ is always non-negative for proper intersections at 0, where M and N are regular $R = P/(f)$ -modules essentially of finite type over \mathbb{C} .*

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