MOTIVE OF THETA DIVISOR I

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ABSTRACT. It is a well known fact that the Theta divisor on the Jacobian of a non-singular curve is a determinantal variety, i.e. is defined by the zero set of a determinant. A less known is for the generalized Jacobian of a singular curve. We describe that the method of proof of the first statement can be generalized to the case of singular curves. We continue to study the mixed Hodge structure of the generalized Theta divisor using standard techniques in Hodge theory.

INTRODUCTION

Long time, one probably looks for a valuable question in the theory of mixed Hodge structures which involves a rich algebraic geometry. Despite of high lighted questions such as Hodge conjecture or Standard conjectures which look so hard to be solved, perhaps the question of studying the mixed Hodge structure of Theta divisor on the Jacobian of curves is one of the very complicated questions. I got acquainted with this question in a lecture by S. Bloch in Madrid at ICMAT. His lectures on extensions of Hodge structures began almost with this concept. The idea that the method can be generalized to the singular curves reached to me at the same time. After searching a little on the question I found, because of very singularities of Theta, the MHS or what we call here the motive of Theta seems to be so complicated object to describe.

In this note I have tried to interface the idea with my knowledge on algebraic curves which I learnt from the book 'Algebraic groups and class fields', by J. P. Serre. Any way we target to say some thing about the MHS on the cohomologies of generalized Θ-divisors on (compactification of) the generalized Jacobians of Singular curves. Our approach uses modulus on normal projective curves and the theory of Riemann-Roch for singular curves. The aforementioned reference provides an almost complete picture of the theory of generalized Jacobians both over $\mathbb{C}$ and positive characteristic. We work only in char 0 case, i.e. over the complex numbers. Theory of modulus on curves provides a simple way to describe the generalized Jacobians and their relation with the Jacobian in the smooth case as extensions of algebraic groups.

1. singular Curves

The Riemann-Roch theorem in complex dimension 1, can be easily generalized to singular projective curves, using theory of modulus. The proof is a modification of that in smooth case, by careful study of linear series on curves. In order to fix the idea, let X be a complete, irreducible, non-singular curve and let $\mathfrak{m} = \sum n_p.P$ be a modulus on X. let S be the support of $\mathfrak{m}$. Set $X' = (X - S) \cup Q$, where Q is one point, and put $\mathcal{O}'_Q := \cap_{P \to Q} \mathcal{O}_P$. Define the sheaf $\mathcal{O}'$ by setting it to be equal to $\mathcal{O}$ for the points not in S. It is an easy exercise in algebraic geometry to show that the sheaf $\mathcal{O}'$ endows X' with a structure of algebraic curve such that X is its normalization and its singular points are Q. One shows that any singular curve is obtained in this way.

The basic example is $\mathfrak{m} = 2P$ where one gets an ordinary cusp, and another when $\mathfrak{m} = P_1 + P_2$ where one obtains ordinary double point, etc....

The Riemann-Roch theorem for singular curves is written as

$$l'(D) - i'(D) = \deg(D) + 1 - \pi$$

$$l'(D) := \dim H^0(X', L'(D)), \ i'(D) := \dim H^1(X', L'(D))$$

where π is the arithmetic genus. The sheaf $L'(D)$ is the same as $L(D)$ on $X - S$ and equals $\mathcal{O}'_Q$ on S. We have the short exact sequence

$$0 \to \mathcal{O}' \to \mathcal{O} \to \mathcal{O}/\mathcal{O}' \to 0$$

The sheaf $\mathcal{O}/\mathcal{O}'$ is concentrated on S and its annihilator ideal is called the conductor of $\mathcal{O}$ into $\mathcal{O}'$. Thus,

$$\chi(X', \mathcal{O}/\mathcal{O}') = \dim H^0(X', \mathcal{O}/\mathcal{O}') = \sum_{Q \in S} \dim \mathcal{O}_Q/\mathcal{O}'_Q =: \delta$$

One can prove that $\pi = g + \delta$ is equal to the arithmetic genus, and in case the curve is defined by an equation of degree d in a projective space $\pi = 1 + d(d - 3)/2$, $g = 1 - \delta + d(d - 3)/2$, the classical Plucker formulas,[S].

2. Generalized Jacobians

Assume X is a projective irreducible algebraic curve. If $f : X \to \mathcal{J}$ be a rational map the the set of points where f is not regular is a finite set S. When $\mathcal{J}$ is abelian, then $S = \emptyset$, and $f(D) = 0$ if and only if D is the divisor of a rational function. For f regular away from S, there can be find a modulus $\mathfrak{m}$ with support S, such that $f(D) = 0$ iff $D = (h)$ with $h \equiv 1 \mod \mathfrak{m}$ (i.e. $v_P(1 - h) > n_P$, where $\mathfrak{m} = \sum n_P P$), [S], Chap. V.

Theorem 2.1. ([S] *Chap. V, page 88) For every modulus $\mathfrak{m}$, there exists a commutative algebraic group $J_\mathfrak{m}$ and a rational map $\phi_\mathfrak{m} : X \to J_\mathfrak{m}$ which is universal for all the rational $f : X \to G$ regular away from S. This means f factors through $\phi_\mathfrak{m}$ as $f = \theta \circ \phi_\mathfrak{m}$ for a unique rational homomorphism $\theta : J_\mathfrak{m} \to G$.*

The map $\phi_\mathfrak{m}$ defines by extensions to divisor classes a bijection from $C_\mathfrak{m}^0$: the group of divisors of degree zero which are prime to S modulo those of the form (f) where $f \equiv 1 \mod \mathfrak{m}$ to $J_\mathfrak{m}$. One has the following exact sequence

$$0 \to R_\mathfrak{m}/\mathbb{G}_m \to C_\mathfrak{m}^0 \to J \to 0$$

where $R_\mathfrak{m} = \prod_{P \in S} U_P/U_P^{(n_P)}$, and $U_P^{(n)} := \{f; v_P(1 - f) > n\}$. Each of the groups $U/U^{(n)}$ in the product are isomorphic to a product of $\mathbb{G}_m$ with a unipotent group of upper triangular matrices, [S] page 94.

The algebraic group $J_\mathfrak{m}$ is called the generalized Jacobian of the curve X, and the case $\mathfrak{m} = 0$ corresponds to the usual Jacobian J. By standard arguments one shows that for $\mathfrak{m} \geq \mathfrak{m}'$ there is a unique surjective separable homomorphism $F : J_\mathfrak{m} \to J'_\mathfrak{m}$, with connected kernel such that $\phi'_\mathfrak{m} = F \circ \phi_\mathfrak{m}$.

Theorem 2.2. ([S] *page 101)*
$$J_\mathfrak{m} \cong \Omega(-\mathfrak{m})^\vee/H_1(X - S, \mathbb{Z})$$

The Abel theorem on the Jacobian of non-singular curves asserts that $Pic^0(X) \cong J(X)$, where $Pic^0(X)$ stands for group of zero divisors modulo rational equivalence. This isomorphism is given by integration along paths whose boundray is the given 0-cycle. Because $J(X)$ is an abelian group, the universal map $\phi : X \to J(X)$, provides a map $sym^n(X) \to J(X)$ whose fibers are projective spaces. For $n = g$ the genus of the curve X this map is surjective and for $n = g - 1$ its image is the Θ-divisor on $J(X)$.

We consider $X \times J$ with two projections pr_1, pr_2. The universal Poincare line bundle $\wp$ on $X \times J$ is a line bundle s.t. $\wp\mid_{X \times j} \cong \mathcal{O}_X(D(j))$ where $D(j)$ is a divisor class of degree 0 via Abel theorem. Fix a line bundle L_0 a line bundle of degree $g - 1$ on X. Set $\mathcal{L} := pr_1^* L_0 \otimes \wp$. Let $D = x_1 + ... + x_g$ be a divisor of degree g on X. Then $D \times J$ is a divisor on $X \times J$ and $\mathcal{O}(D \times J) = pr_1^* \mathcal{O}_X(D)$. Moreover we have $\mathcal{L}(D) = \mathcal{L} \otimes pr_1^*(\mathcal{O}_X(D))$. The following theorem is well-known.

Theorem 2.3. *The Θ-divisor is a determinantal subvariety of J, i.e is given by the zero set of a determinant function.*

Proof. This can be obtained by applying the Leray cohomological fiber functor $R^\bullet pr_{2,*}$ to the short exact sequence of sheaves on $X \times J$,

$$(1) \qquad\qquad 0 \to \mathcal{L} \to \mathcal{L}(D) \to \mathcal{L}(D)/\mathcal{L} \to 0$$

In the derived functor long exact sequence one obtains

$$(2) \qquad 0 \to pr_{2*}\mathcal{L} \to pr_{2*}\mathcal{L}(D) \to pr_{2*}(\mathcal{L}(D)/\mathcal{L}) \to R^1 pr_{2*}\mathcal{L} \to R^1 pr_{2*}\mathcal{L}(D)$$

We have $pr_{2*}\mathcal{L} = R^1 pr_{2*}\mathcal{L}(D) = 0$ by Riemann-Roch. Thus

$$(3) \qquad\qquad 0 \to pr_{2*}\mathcal{L}(D) \to pr_{2*}(\mathcal{L}(D)/\mathcal{L}) \to R^1 pr_{2*}\mathcal{L} \to 0$$

where the last sheaf has support the Θ-divisor. Therefore the determinant of the first homomorphism defines Θ. $\square$

In case of singular curves the canonical map $\phi_{\mathfrak{m}} : X \to J_{\mathfrak{m}}$ by extension over divisors defines a homomorphism from the group of divisors prime to S onto the group $J_{\mathfrak{m}}$. Its kernel is formed by divisors $\mathfrak{m}$-equivalent to an integral multiple of a fixed point P_0 in X'.

Similar to the non-singular case the map $sym^\pi(X) \to J_{\mathfrak{m}}$ is surjective and the image of $sym^{\pi-1}(X) \to J_{\mathfrak{m}}$ is the locus of generalized Θ-divisor. Thus we may apply the argument in the theorem to $X \times J_{\mathfrak{m}}$ and work with divisors prime to S to obtain

Theorem 2.4. *The generalized $\Theta_{\mathfrak{m}}$ is deteminantal on $J_{\mathfrak{m}}$.*

3. Motive of Theta divisor

The motive associated to the $\Theta_\mathfrak{m}$ is mainly described by the mixed Hodge structure of $H^\pi(J_\mathfrak{m} \setminus \Theta_\mathfrak{m})$. From now on set $J := J_\mathfrak{m}$, $\Theta := \Theta_\mathfrak{m}$. According to what was said, The Θ-divisor is the singular fiber in a degenerate family defined by a determinant function. The mixed Hodge structure in such a family is determined by J. Steenbrink which uses the method of resolution of singularities.

$$(4) \qquad \begin{array}{ccccccc} J_\infty & \longrightarrow & U & \longrightarrow & J & \longleftarrow & D \\ f_\infty \downarrow & & \downarrow f & & \downarrow f & & \downarrow \\ H & \xrightarrow{e} & \Delta^* & \longrightarrow & \Delta & \longleftarrow & 0 \end{array}$$

as in 2.9, where $D = \bigcup_i^m D_i$ is a normal crossing divisor.

Lemma 3.1. [JS3] *The spectral sequence of the double complex*

$$A_D^{pq} := (a_{q+1})_* \Omega_{D^{(q+1)}}^p, \quad D^{(q+1)} = \amalg D_{i_1} \cap ... \cap D_{i_q}$$

with horizontal arrows to be the de Rham differentials, and vertical arrows $\sum_j (-1)^{q+j} \delta_j$, induced by the inclusions $\amalg D_{i_1} \cap ... \cap D_{i_q} \hookrightarrow \amalg D_{i_1} \cap .. \hat{D}_{i_j} .. \cap D_{i_q}$ obtained by possible omitting the indices, on A_D^{pq}, degenerates at E_2, with

$$E_1^{pq} = H^q(D^{(p+1)}, \mathbb{C}) \Rightarrow H^{p+q}(D, \mathbb{C})$$

and computes the cohomologies of D. Moreover, the two filtrations

$$F^p A_{\dot{D}} = \bigoplus_{r \geq p} A_{\dot{D}}^r, \qquad W_q A_{\dot{D}} = \bigoplus_{s \geq -q} A_{\dot{D}}^s$$

induce the Hodge and the weight filtrations on $H^k(D, \mathbb{C})$ for each k, to define a mixed Hodge structure.

Proposition 3.2. [JS2] *The spectral sequence of $B^{pq} := A^{pq}/W_q$ degenerates at E_2 term with*

$$E_1^{-r,q+r} = \bigoplus_{k \geq 0, r} H^{q-r-2k}(D^{(2k+r+1)}, \mathbb{C})(-r-k) \Rightarrow H^q(X_\infty, \mathbb{C})$$

and equips $H^q(X_\infty, \mathbb{C})$ with a mixed Hodge structure.

Thus we assume after suitable blow ups the family is given by $f : J \to \mathbb{C}$, where afar from 0 the fibration looks like fibers of determinant, and $J_0 = f^{-1}(0)$ is a normal crossing divisor with quasi-unipotent monodromies. One can equip $H^m(J_t)$, $H^m(J_0)$, $H_m(J_0)$ with mixed Hodge structures. The formation of the weight and Hodge filtrations is standard in this case, and one may filter the associated double complexes naively in vertical and horizontal directions. The MHS fit in the Clemens-Schmid exact sequence ([SP] page 285, [ITW]) as mixed Hodge structures,

$$... \to H_{2n+2-m}(J_0) \xrightarrow{\alpha} H^m(J_0) \xrightarrow{i_t^*} H^m(J_t) \xrightarrow{N} H^m(J_t) \xrightarrow{\beta} H_{2n-2m}(J_0) \xrightarrow{\alpha} H^{m-2}(J_0) \to ...$$

where α is induced by Poincare duality followed by projection, and β is by inclusion followed by Poincare duality. The monodromy weight filtration on $H^m(J_0)$ can be described using the hypercover structure obtained by intersections of NC divisors. Then the weight filtration of $H^m(J_t)$ can be computed via the induced filtration on $\ker(N) := K_t^m$, and satisfies;

$$Gr_k^W H^m(J_t) \cong \begin{cases} Gr_k K_t^m \oplus Gr_{m-2} K_t^m \oplus ... \oplus Gr_{k-2[k/2]} K_t^m, & k \leq m \\ Gr_{2m-k} H^m(J_t), & k > m \end{cases}$$

The relations between the weight filtrations can be explained by the followings, cf. [ITW];

- i_t^* induces

$$Gr_k H^m(J_0) \cong Gr_k K_t^m$$

- The following sequence is exact

$$0 \to Gr_{m-2} K_t^{m-2} \to Gr_{m-2n-2} H_{2n+2-m}(J_0) \xrightarrow{\alpha} Gr_m H^m(J_0) \to Gr_m K_t^m \to 0$$

Now assume $J \hookrightarrow \bar{J}$ is a projective compactification such that J is Zariski dense in $\bar{J}$. Then we have the long exact sequence

$$... \to H^{m-1}(\bar{J} \setminus J_0) \to H^m(\bar{J}, \bar{J} \setminus J_0) \to H^m(\bar{J}) \to H^m(\bar{J} \setminus J_0) \to ...$$

where the morphisms are of MHS, and the isomorphism

$$H^{2n+2-m}(\bar{J}, \bar{J} \setminus J_0) \cong H_c^{2n+2-m}(J) \cong H^m(J)^\vee$$

as MHS. In our case if we blow up $\bar{J}_{\mathfrak{m}}$ along $\Theta_{\mathfrak{m}}$ then the above method computes the mixed Hodge structure of $H^\pi(\bar{J} \setminus D) = H^\pi(\bar{J}_{\mathfrak{m}} \setminus \Theta_{\mathfrak{m}})$.

Remark 3.3. [ITW] *For non-singular projective curves of genus g the $H^{g-1}(\Theta, \mathbb{Z})$ contains a natural sublattice*

$$P = \ker(H^{g-1}(\Theta, \mathbb{Z}) \to H^{g+1}(J, \mathbb{Z}))$$

where the map is the pushforward by inclusion. Then, as Hodge structures

$$H^{g-1}(\Theta) = P \oplus \Theta.H^{g-3}(J)$$

4. Appendix: Extensions of groups

Let A, B, C be three algebraic groups. A sequence of algebraic homomorphisms $0 \to B \to C \to A \to 0$ is called strictly exact If it is exact in the usual sense, and if all the homomorphisms are separable, In other words if the algebraic structure of B, and A is induced by that of C. This would be equivalent to

$$0 \to \mathfrak{t}_B \to \mathfrak{t}_C \to \mathfrak{t}_A \to 0$$

is exact. A strict exact sequence would be called an extension of A by B. The set of isomorphism classes of all extensions of A by B is denoted $Ext(A, B)$, which is endowed with a composition law as follows. For $f : B \to B'$ and $C \in Ext(A, B)$ then $f_*C \in Ext(A, B')$ may be defined as the quotient of $C \times B'$ by the subgroup formed by $(-b, f(b))$. The element f_*C is the unique extension such that there exists a homomorphism $F : C \to C'$ making a commutative diagram

(5)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \| \\ 0 & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A & \longrightarrow & 0 \end{array}.$$

Similarly one defines the pull back g^*C for $g : A'A$ as the subgroup of $A' \times C$ formed by pairs having the same image in A. If C, C' are two elements in $Ext(A, B)$, then $C \times C'$ can be considered as an element of $Ext(A \times A, B \times B)$, denoting by $d : A \to A \times A$ and $s : B \times B \to B$ the diagonal of A and composition law of B, then $C + C' = d^*s_*(C \times C')$ in $Ext(A, B)$ and , makes it into an additive bi-functor, [S] Chap VII.

Remark 4.1. *([S] page 166) The category $\mathcal{C}_k$ of commutative algebraic groups over the field k is an additive category. It is an Abelian category, when the characteristic of k is 0 and only in this case. However, $\mathcal{C}_k$ contains neither enough injectives nor enough projectives.*

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