

# VARIATION OF MIXED HODGE STRUCTURES AND PRIMITIVE ELEMENTS

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ABSTRACT. We study the asymptotic behaviour of polarization form in the variation of mixed Hodge structure associated to isolated hypersurface singularities. The contribution characterizes the Grothendieck residue as the polarization on the extended fiber in this case. We also provide a discussion on primitive elements to explain conjugation operator in these variations, already existed in the literature.

## INTRODUCTION

One of the important subject of study in Hodge theory and D-modules is the behaviour of the underlying variation of (mixed) Hodge structures in the extensions. We will consider the VMHS associated to isolated hypersurface singularities in the affine space  $\mathbb{C}^{n+1}$ . The mixed Hodge structure would be the Steenbrink limit mixed Hodge structure. Classically there are two equivalent ways to define this MHS. One method which is actually due to J. Steenbrink himself is by applying a spectral sequence argument to the resolution of the singularity in projective fibration followed with Invariant cycle theorem. Another method which is equivalent to the first is to define it by the structure of lattices in the Gauss-Manin system associated to VMHS on the punctured disc. We are mainly interested to the extension of the Gauss-Manin system over the puncture both in the algebraic and analytic case. The extension of the MHS, is then a consequence of the Riemann-Hilbert correspondence. We explain the extension by gluing vector bundles which is called minimal extension in the literature. It mainly involves the fact that, different  $V$ -lattices can be glued with the Brieskorn lattices in the other chart. Classically this is equivalent to the Fourier-Laplace transform of perverse sheaves. the observation made by P. Deligne and M. Saito is the two Hodge filtration on the vanishing cohomology, one defined by the Brieskorn lattices and the second defined by the  $V$ -lattices are opposite. This corresponds to a choice of good basis of the Gauss-Manin system which defines the desired isomorphism. We try to study the polarization in the new chart after the extension. The method we use is to formulate a special presentation of K. Saito higher residue pairing due to the works of F. Pham.

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*Key words and phrases.* Variation of Mixed Hodge structure, Isolated hypersurface singularity, Primitive element.

Suppose  $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$  is an isolated hypersurface singularity at  $0 \in \mathbb{C}^{n+1}$ . Using the Hodge theory of the Brieskorn lattice we show that when  $f$  has isolated critical point, the MHS of the extension fiber is polarized by a sign modification of Grothendieck residue.

### 1. STEENBRINK LIMIT MIXED HODGE STRUCTURE

Suppose we have an isolated singularity holomorphic germ  $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ . By the Milnor fibration Theorem we can always associate a  $C^\infty$ -fiber bundle over a small punctured disc  $T'$ . The associated cohomology bundle  $\mathcal{H} = R^n f_* \mathbb{C}$ , constructed from the middle cohomologies of the fibers will define a variation of mixed Hodge structure on  $T'$ . The Brieskorn lattice is defined by,

$$H'' = f_* \frac{\Omega_{X,0}^{n+1}}{df \wedge d\Omega_{X,0}^{n-1}}$$

The Brieskorn lattice is the stack at 0 of a locally free  $\mathcal{O}_T$ -module  $\mathcal{H}''$  of rank  $\mu$  with  $\mathcal{H}''_T \cong \mathcal{H}$ , and hence  $H'' \subset (i_* \mathcal{H})_0$ , where  $i : T' \hookrightarrow T$ . The Leray residue formula can be used to express the action of  $\partial_t$  in terms of differential forms by

$$\partial_t^{-1}[d\omega] = [df \wedge \omega]$$

In particular,  $\partial_t^{-1}.H'' \subset H''$ , and

$$(1) \quad \frac{H''}{s.H''} \cong \Omega_f := \frac{\Omega_{X,0}^{n+1}}{df \wedge \Omega_{X,0}^n} \cong \frac{\mathbb{C}\{z\}}{(\partial(f))} =: A_f.$$

where  $s = \partial_t^{-1}$ . There exists a  $\mathbb{C}$ -isomorphism

$$(2) \quad \psi = \bigoplus_{-1 < \alpha \leq 0} \psi_\alpha : H_{\mathbb{C}} = \bigoplus_{-1 < \alpha \leq 0} H_{\mathbb{C}}^{e^{-2\pi i \alpha}} \rightarrow \bigoplus_{-1 < \alpha \leq 0} C^\alpha \cong V^{-1}/s.V^{-1}$$

The map  $\psi$  is called the nearby map, and explains content of the Deligne extension of  $\mathcal{H}$ . The monodromy  $M$  on  $H_{\mathbb{C}}$  corresponds to  $\exp(-2\pi i.t\partial_t)$  on  $\bigoplus_{-1 < \alpha \leq 0} C^\alpha$ . The

(Steenbrink) Hodge filtration on  $H^n(X_\infty, \mathbb{C})$  is defined by

$$(3) \quad F^p H(X_\infty)_\lambda = \psi_\alpha^{-1} \partial_t^{n-p} Gr_V^{\alpha+n-p} \mathcal{H}^{(0)}.$$

Set  $\beta = \alpha + n - p$ . Because  $\mathcal{H}^{(0)} \subset V^{-1}$ ,  $Gr_V^\beta = 0$  for  $\beta \leq -1$ . Thus,  $F^p = 0$ ,  $p \geq n$ . Therefore,

$$(4) \quad Gr_V^\beta \Omega_f = Gr_F^p H^n(X_\infty, \mathbb{C}), \quad \Omega_f = \Omega_{X,0}^{n+1}/df \wedge \Omega_{X,0}^n$$

where  $Gr_V^\beta \Omega_f$  is explained as follows,

**Definition 1.1.** *The  $V$ -filtration on  $\Omega_f$  is defined by*

$$(5) \quad V^\alpha \Omega_f = pr(V^\alpha \cap H'')$$

Clearly  $V^\alpha \Omega_f = \bigoplus_{\beta \geq \alpha} \Omega_f^\beta$  and  $\Omega_f \cong \bigoplus Gr_V^\alpha \Omega_f$  hold. We have the following commutative diagram

$$(6) \quad \begin{array}{ccc} Gr_V^\beta \Omega_f & \xrightarrow{t} & Gr_V^{\beta+1} \Omega_f \\ \partial_t^{n-p} \downarrow & & \partial_t^{n-p+1} \downarrow \\ Gr_F^p H_\lambda & \xrightarrow{GrN} & Gr_F^{p-1} H_\lambda \end{array} .$$

## 2. HODGE THEORY AND RESIDUE FORM

The Gauss-Manin system  $\mathcal{G}$  of an isolated hypersurface singularity may be extended over the puncture in the way of gluing two vector bundles on different charts. This can be explained by gluing lattices inside  $\mathcal{G}$  ( $V$ -lattices with Brieskorn lattices). This is called the minimal extension in the literature. In this way we can explain a new fiber over the puncture 0, which is canonically isomorphic to  $\Omega_f$ , cf. [1].

Assume  $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$  is a germ of isolated singularity. Suppose,

$$H^n(X_\infty, \mathbb{C}) = \bigoplus_{p,q,\lambda} I_\lambda^{p,q}$$

is the Deligne-Hodge bigrading, and generalized eigen-spaces of vanishing cohomology, and also  $\lambda = \exp(-2\pi i\alpha)$  with  $\alpha \in (-1, 0]$ . Consider the isomorphism obtained by composing the three maps,

$$(7) \quad \Phi_\lambda^{p,q} : I_\lambda^{p,q} \xrightarrow{\hat{\Phi}_\lambda} Gr_V^{\alpha+n-p} H'' \xrightarrow{pr} Gr_V^\bullet H'' / \partial_t^{-1} H'' \xrightarrow{\cong} \Omega_f$$

where

$$\begin{aligned} \hat{\Phi}_\lambda^{p,q} &:= \partial_t^{p-n} \circ \psi_\alpha | I_\lambda^{p,q} \\ \Phi &= \bigoplus_{p,q,\lambda} \Phi_\lambda^{p,q}, \quad \Phi_\lambda^{p,q} = pr \circ \hat{\Phi}_\lambda^{p,q} \end{aligned}$$

**Definition 2.1.** *(MHS on  $\Omega_f$ ) The mixed Hodge structure on  $\Omega_f$  is defined by using the isomorphism  $\Phi$ . This means that*

$$W_k(\Omega_f) = \Phi W_k H^n(X_\infty, \mathbb{Q}), \quad F^p(\Omega_f) = \Phi F^p H^n(X_\infty, \mathbb{C})$$

and all the data of the Steenbrink MHS on  $H^n(X_\infty, \mathbb{C})$  such as the  $\mathbb{Q}$  or  $\mathbb{R}$ -structure is transformed via the isomorphism  $\Phi$  to that of  $\Omega_f$ . Specifically; in this way we also obtain a conjugation map

$$(8) \quad \bar{\cdot} : \Omega_{f,\mathbb{Q}} \otimes \mathbb{C} \rightarrow \Omega_{f,\mathbb{Q}} \otimes \mathbb{C}, \quad \Omega_{f,\mathbb{Q}} := \Phi H^n(X_\infty, \mathbb{Q})$$

defined from the conjugation on  $H^n(X_\infty, \mathbb{C})$  via this isomorphism.

**Theorem 2.2.** [1] *Assume  $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ , is a holomorphic germ with isolated singularity at 0, with  $f : X \rightarrow T$  the associated Milnor fibration. Embed the Milnor fibration in a projective fibration  $f_Y : Y \rightarrow T$  of degree  $d$  (with  $d$  large enough), by inserting possibly a singular fiber over 0. Then, the isomorphism  $\Phi$  makes the following diagram commutative up to a complex constant;*

$$(9) \quad \begin{array}{ccc} \widehat{Res}_{f,0} : \Omega_f \times \Omega_f & \longrightarrow & \mathbb{C} \\ \downarrow (\Phi^{-1}, \Phi^{-1}) & & \downarrow \times * \\ S : H^n(X_\infty) \times H^n(X_\infty) & \longrightarrow & \mathbb{C} \end{array} \quad * \neq 0$$

where,

$$\widehat{Res}_{f,0} = res_{f,0} (\bullet, \tilde{C} \bullet)$$

and  $\tilde{C}$  is defined relative to the Deligne decomposition of  $\Omega_f$ , via the isomorphism  $\Phi$ . If  $J^{p,q} = \Phi^{-1} I^{p,q}$  is the corresponding subspace of  $\Omega_f$ , then

$$(10) \quad \Omega_f = \bigoplus_{p,q} J^{p,q} \quad \tilde{C}|_{J^{p,q}} = (-1)^{p(d-1)/d}$$

In other words;

$$(11) \quad S(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \times res_{f,0}(\omega, \tilde{C}.\eta), \quad 0 \neq * \in \mathbb{C}$$

**Theorem 2.3.** [1]

*Assume  $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$  is a holomorphic isolated singularity germ. The modified Grothendieck residue provides a polarization for asymptotic fiber  $\Omega_f$ , via the aforementioned isomorphism  $\Phi$ . Moreover, there exists a set of forms  $\{Res_k\}$  giving a graded polarization for  $\Omega_f$ .*

**Remark 2.4.** *Let  $\mathcal{G}$  be the Gauss-Manin system associated to a polarized variation of Hodge structure  $(\mathcal{H}_\mathbb{Q}, F, S)$  of weight  $n$ , with  $S : \mathcal{H}_\mathbb{Q} \otimes \mathcal{H}_\mathbb{Q} \rightarrow \mathbb{Q}(-n)$  the polarization. Then we have the isomorphism*

$$(12) \quad \bigoplus_{k \in \mathbb{Z}} Gr_F^k \mathcal{G} \rightarrow \bigoplus_{k \in \mathbb{Z}} Hom_{\mathcal{O}_X}(Gr_F^{n-k} \mathcal{G}, \mathcal{O}_X)$$

given by (up to a sign factor)  $\lambda \rightarrow S(\lambda, -)$ , for  $\lambda \in Gr_F^k \mathcal{G}$ .

G. Pearlstein and J. Fernandez in [PF] prove a result on a pair of opposite filtrations appearing in the extensions of an admissible VMHS or its asymptotic behaviour. This concept has been studied by M. Saito in another terminology cf. [MS]. The above form polarizes the CVH of G. Pearlstein et. al.

**Theorem 2.5.** [1] *Let  $\mathcal{V}$  be an admissible variation of polarized mixed Hodge structure associated to a holomorphic germ of an isolated hyper-surface singularity. Set*

$$(13) \quad \Psi = \overline{F_\infty^\vee} * W$$

*Then  $\Psi$  extends to a filtration  $\underline{\Psi}$  of  $\mathcal{V}$  by flat sub-bundles, which pairs with the Hodge filtration  $\mathcal{F}$  of  $\mathcal{V}$ , to define a polarized  $\mathbb{C}$ -variation of Hodge structure.*

**Remark 2.6.** [DW] *Consider the map*

$$F : \Omega_X^{n+1} \rightarrow i_* \bigcup_z \text{Hom}(H_n(X, f^{-1}(\eta, \frac{z}{|z|}), \mathbb{Z}), \mathbb{Z}) \cong \oplus_i \mathbb{Z}\Gamma_i, \quad \mathcal{H} := \text{Im}(F)$$

$$\omega \mapsto [z \rightarrow (\Gamma_i \rightarrow \int_{\tilde{\Gamma}_i} e^{-t/z} \omega)],$$

*where  $\Gamma_i$  are the classes of Lefschetz thimbles, and  $\tilde{\Gamma}_i$  is the extension to infinity, [DW]. The vector bundle  $\mathcal{H}$  is exactly the Fourier-Laplace transform of the cohomology bundle  $R^n f_* \mathbb{C}_{S^*} = \cup_t H^n(X_t, \mathbb{C})$ , equipped with a connection with poles of order at most two at  $\infty$ .*

$$(\cup_t H^n(X_t, \mathbb{C}), \nabla) \Leftrightarrow (\mathcal{H}, \nabla')$$

**Corollary 2.7.** *The modified Grothendieck residue*

$$\widehat{\text{Res}}_{f,0} = \text{res}_{f,0}(\bullet, \hat{C}\bullet)$$

*where  $\hat{C}$  is defined relative to the Deligne-Hodge decomposition of  $\Omega_f$ , is the Fourier-Laplace transform of the polarization  $S$  on  $H^n(X_\infty, \mathbb{C})$ .*

### 3. PRIMITIVE ELEMENTS

The extension of an PVMHS's may be explained by the solvability of the Poincare-Birkhoff problem associated to the Gauss-manin systems. One step in solving the Poincare-Birkhoff problem for the Gauss-Manin system of  $f$  is to glue different lattices in the Gauss-Manin vector bundle to obtain a vector bundles over  $\mathbb{C}P(1)$ . In this section we briefly follow C. Sabbah in [SA1] to explain the primitive elements in the variation of MHS of vanishing cohomology.

Assume  $\mathcal{G}$  the associated Gauss-Manin system of the isolated singularity  $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$  and  $\phi_1, \dots, \phi_\mu$ , a frame basis for  $\mathcal{G}$  and  $(\alpha, s_\alpha)$  is the spectral pairs of  $f$ . It is possible to choose the basis in a way that we have the following recursive relations;

$$(14) \quad \phi_{s(i)+k} = \partial_t^{-k} \prod_\alpha (t\partial_t - \alpha)^j \phi_{s(i)}, \quad 1 \leq i \leq r, \quad 1 \leq k \leq k_i.$$

for specific numbers  $0 \leq s(1), \dots, s(r) \leq \mu$ . In this way we reach a set of forms  $\phi_{s(i)}$  indexed by spectral numbers which produce other basis elements by applying

the operators  $t\partial_t - \alpha$  successively. They also describe  $Gr_t^W Gr_\alpha^V \mathcal{G}$  concretely. These forms are called primitive elements relative to the nilpotent operator induced by  $t\partial_t - \alpha$  on  $C_\alpha$ . They provide information about the Jordan blocks structure in  $\mathcal{G}$ . If we denote the Jordan block as

$$B_k := \langle N^j[\omega_{k_0}] \mid j = 0, \dots, \nu_k \rangle,$$

then it holds that;

$$\overline{B_{\alpha,l}} = \begin{cases} B_{1-\alpha, \nu_k-l}, & \alpha \in ]0, 1[ \\ B_{0, \nu_k-l}, & \alpha = 0 \end{cases}$$

cf. [SA2].

**Proposition 3.1.** [SA1], [SA2] *There is a 1-1 correspondence between opposite filtrations on  $H^n(X_\infty)_\mathbb{C}$  and free, rank  $\mu$ ,  $\mathbb{C}[t]$ -submodules  $\mathcal{G}_\infty$  on which the connection is logarithmic where  $\mathcal{G}_0, \mathcal{G}_\infty$  define a trivial vector bundle on  $\mathbb{P}^1$ .*

The submodule  $\mathcal{G}_\infty$  in Proposition is given by;

$$\mathcal{G}_\infty = \mathbb{C}[t]\langle [\omega_0], \dots, \partial_t^{-s_0}[\omega_0], \dots, \partial_t^{-s_r}[\omega_r] \rangle.$$

The primitive elements provide the good bases of the Brieskorn module. They also prove the existence of a solution to The Poincare-Birkhoff problem. In such a basis the matrix of the operator  $t$  has the form;

$$(15) \quad t = A_0 + A_1 \partial_t^{-1}$$

where  $A_0, A_1$  are square matrices of size  $\mu$  and  $A_1$  is a diagonal matrix. It holds (cf. [SA1], [H1]) that, in such a basis the K. Saito higher residue form takes the form

$$(16) \quad K_f(\eta_i, \eta_j) = \delta_{\kappa(i)j} \cdot \partial_t^{-n-1},$$

where  $\delta$  is the Kronecker delta and  $\kappa$  is an involution of the set  $\{1, \dots, \mu\}$  of a specific basis of  $\mathcal{G}$  namely  $\{\eta_i\}_{i=1}^\mu$ .

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