Hodge Theory of Isolated Hypersurface Singularities

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Isolated hypersurface singularities

- Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ a holomorphic germ with isolated singularity. It gives the Milnor fibration (C^∞),

$$f : X \to T$$

- Provides 1-parameter degenerating family, $T' = T \setminus 0$

$$\begin{array}{ccccc} X_\infty & \longrightarrow & U & \longrightarrow & X \\ {\scriptstyle f_\infty}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ H & \xrightarrow{e} & T' & \longrightarrow & T \end{array} \qquad (1)$$

- Monodromy transformation $M : H_\mathbb{Z} \to H_\mathbb{Z}, H_\mathbb{Z} = H^n(X_{s_0}, \mathbb{Z})$, where $s_0 \in T'$ and $\dim H^n(X_{s_0}, \mathbb{Q}) = \mu$ (the Milnor number of f).

Monodromy theorem

Theorem

(Monodromy theorem) The eigenvalues of M are $m-th$ root of unity, for a suitable integer m, i.e if $M = M_s.M_u$ is Jordan decomposition, then

$$M_s^m = 1.$$

If l is the largest integer that for some p, $H^{i,k-i}(X_t, \mathbb{C}) \neq 0$, if $p \leq i \leq p+l$. Then

$$(M_u - 1)^l = 0$$

and hence

$$(M^m - 1)^l = 0,$$

for $l \leq min(k, 2n-k) + 1$.

Compactification of Milnor fibration

- Assume $f : X \to T$ is a Milnor fibration of an isolated hypersurface singularity at $0 \in \mathbb{C}^{n+1}$.
- Can assume f is a polynomial of sufficiently high degree, say $d = \deg f$.
- Can embed the fibration into a projective one

$$f_Y : \mathbb{P}^{n+1} \times T \to \mathbb{C}$$

  defined via a homogeneous polynomial f_Y.
- Obtain a locally trivial C^∞-fibration $f_Y : Y' \to T'$ with,

$$F(z_0, ..., z_{n+1}) = z_{n+1}^d f(z_0/z_{n+1}, ..., z_n/z_{n+1}),$$

$$Y = \{(z, t) \in \mathbb{P}^{n+1}(\mathbb{C}) \times T \mid f_Y = F(z) - tz_{n+1}^d = 0\}.$$

- Set $Y_\infty = Y' \times_{T'} H$.

Nilpotent Orbit Theorem-Limit mixed Hodge structure

- $N_Y = \log(M_{Y,u})$ where $M_Y = M_{Y,s}.M_{Y,u}$ is the Jordan decomposition;
- N gives a Weight filtration $W_{[\bullet]}$ on $H_{\mathbb{Z}}$ defined by

(a) $N : W_k \to W_{k-2}$

(b) $N^k : Gr^W_{m+k} \to Gr^W_{m-k}$ is an isomorphism.

- $s := \exp(\sqrt{-1}.t)$ where $Im(t) > 0$;
- By the nilpotent orbit theorem (W. Schmid), the limit

$$\lim_{Im(t)\to\infty} \exp(-tN)F^p_s =: F^p_\infty$$

exists, and $(H^n(Y_\infty, \mathbb{Q}), W_k, F^p_\infty)$ is a MHS.

- If Y is a projective variety of dimension n, Then

$$H^n(Y,\mathbb{C}) = \begin{cases} P^n(Y,\mathbb{C}), & n \text{ odd} \\ P^n(Y,\mathbb{C}) \oplus \omega^{n/2}, & n \textit{ even} \end{cases}$$

where $P^n = \ker(. \wedge \omega : H^n \to H^n)$, ω Kahler form.

- We set (I_Y^{coh} cup product)

$$S_Y = (-1)^{n(n-1)/2} I_Y^{coh}.$$

Theorem

(W. Schmid) $S_Y, N_Y, W_\bullet, F_\infty^\bullet$ give a polarized mixed Hodge structure on $P^n(Y_\infty)$. It is invariant w.r.t $M_{Y,s}$.

Steenbrink limit mixed Hodge structure-First definition

Theorem

If f is a polynomial of sufficiently high degree s.t the properties above are satisfied. Then the mapping $i^* : P^n(Y_\infty) \to H^n(X_\infty)$ *is surjective and the kernel is* $\ker(i^*) = \ker(M_Y - id)$. *Moreover, there exists a unique MHS on* $H^n(X_\infty)$ *namely Steenbrink MHS, which makes the following short exact sequence an exact sequence of mixed Hodge structures*

$$0 \to \ker(M_Y - id) \to P^n(Y_\infty) \xrightarrow{i^*} H^n(X_\infty) \to 0.$$

The MHS's are invariant w.r.t the semi-simple part of the monodromy. The Steenbrink LMHS is polarized by

$$S(a,b) = \begin{cases} S_Y(i^*a, i^*b) & a,b \in H_{\neq 1} \\ S_Y(i^*a, N_Y i^*b) & a,b \in H_1 \end{cases}$$

Cohomology bundle-Vanishing cohomology

- The cohomology bundle (rank μ-local system of MHS)

$$H := \bigcup_{t \in T'} H^n(X_t, \mathbb{C})$$

- Holomorphic integrable (Gauss-Manin) connection;

$$\partial_t : \mathcal{H} \to \mathcal{H}, \qquad \mathcal{H} \cong (\mathcal{O}_{T'})^{\mu}$$

- Define

$$H^n(X_\infty, \mathbb{C})_\lambda = \ker(M_s - \lambda) \subset H^n(X_\infty, \mathbb{C})$$

Elementary section-Deligne extension

- Elementary sections;

$$s(A, \alpha)(t) = t^{\alpha} exp[\log(t).\frac{-N}{2\pi i}]\ A_{\alpha}(t),$$

$$A_{\alpha}(t) \in H^n(X_t, \mathbb{C})_{\lambda},\ e^{-2\pi i\alpha} = \lambda,$$

- Define a map,

$$\psi_{\alpha} : H^n(X_{\infty}, \mathbb{C}) \to (i_* \mathcal{H})_0,$$

$$\psi_{\alpha}(A) := i_* s(A, \alpha).$$

- It gives the isomorphism

$$\psi_{\alpha} : H^n(X_{\infty}, \mathbb{C})_{\lambda} \to C^{\alpha} \subset \mathcal{G}_0,$$

$$-N/2\pi i = \psi_{\alpha}^{-1} \circ (t\partial_t - \alpha) \circ \psi_{\alpha}$$

Nearby map

- Builds up the isomorphism;

$$\psi = \bigoplus_{-1<\alpha\leq 0} \psi_\alpha : H_{\mathbb{C}} = \bigoplus_{-1<\alpha\leq 0} H_{\mathbb{C}}^{e^{-2\pi i\alpha}} \to \bigoplus_{-1<\alpha\leq 0} C^\alpha$$

- Monodromy M on $H_{\mathbb{C}}$ corresponds to $\exp(-2\pi i.t\partial_t)$.
- ψ is called Deligne nearby map.

Kashiwara-Malgrange V-filtration

- **Definition:** Gauss-Manin system

$$\mathcal{G} = \bigoplus_{-1<\alpha\leq 0} \mathbb{C}\{t\}[t^{-1}]C^{\alpha}.$$

- **Definition:** V-filtration is defined on $\mathcal{G}$ by

$$V^{\alpha} = \sum_{\beta\geq\alpha} \mathbb{C}\{t\}C^{\beta}, \qquad (V^{>\alpha} = \sum_{\beta>\alpha} \mathbb{C}\{t\}C^{\beta})$$

(a) $t.V^{\alpha} \subset V^{\alpha+1}$,
(b) $\partial_t.V^{\alpha} \subset V^{\alpha-1}$,
(c) $t^i\partial_t^j V^{\alpha} \subset V^{\alpha}$ for all $i > j$,
(d) The operator $t\partial_t - \alpha$ is nilpotent on Gr_V^{α}.

- $V^{\alpha}, \ V^{>\alpha}$ are $\mathbb{C}\{t\}$-modules of rank μ.

Gauss-Manin system

- We have,

$$\mathcal{G} = \mathbb{C}\{t\}[\partial_t] \bigoplus \bigoplus_{\lambda} \bigoplus_{j=1}^{m_\lambda} \frac{\mathbb{C}\{t\}[\partial_t]}{\mathbb{C}\{t\}[\partial_t](t\partial_t - \alpha_\lambda)^{n_{\lambda,j}}}.$$

- In case of isolated hypersurface singularities, $\partial_t : \mathcal{G} \to \mathcal{G}$ is invertible.
- The Gauss-Manin connection $\partial_t : \mathcal{G} \to \mathcal{G}$ of isolated hypersurface singularities, has an extension to the whole disc T that has a regular singularity at 0, i.e. has a pole of order at most 1 at 0.

- **Definition:** E. Brieskorn defines the $\mathcal{O}_T$-modules (X Milnor ball)

$$H'' = f_*\left(\frac{\Omega_X^{n+1}}{df \wedge d\Omega_X^{n-1}}\right)$$

$$H' = f_*\left(\frac{df \wedge \Omega_X^{n}}{df \wedge d\Omega_X^{n-1}}\right)$$

- They have rank μ, such that

$$H'|_{T'} = H''|_{T'} = \mathcal{H}.$$

- We have canonical isomorphisms,

$$\Omega_f = \frac{\Omega^{n+1}}{df \wedge \Omega^n} = \frac{H''}{\partial_t^{-1}.H''} = \frac{V^{-1}}{t.V^{-1}} = H^n(X_\infty) \tag{2}$$

Steenbrink Limit mixed Hodge structure-Second definition

- Hodge filtrations on $H^n(X_\infty, \mathbb{C})$,

$$F^p_{St}H^n(X_\infty, \mathbb{C})_\lambda = \psi_\alpha^{-1}\left(\frac{V^\alpha \cap \partial_t^{n-p} H_0''}{V^{>\alpha}}\right), \qquad \alpha \in (-1, 0],$$

$$F^p_{Va}H^n(X_\infty, \mathbb{C})_\lambda = \psi_\alpha^{-1}\left(\frac{V^\alpha \cap t^{-(n-p)} H_0''}{V^{>\alpha}}\right), \qquad \alpha \in (-1, 0]$$

Knowing that $V^{-1} \supset H_0''$, and $0 = F^{n+1} = F^{n+1}_{Va}$.

- With the weight filtration W define mixed Hodge structures on $H^n(X_\infty, \mathbb{C})$.

Theorem

The Hodge filtration F_{St} is the Steenbrink limit Hodge filtration.

Theorem

(A. Varchenko) $(F^\bullet, W_\bullet)$ *and* $(F^\bullet_{Va}, W_\bullet)$ *are different on* $H^n(X_\infty, \mathbb{C})$ *in general, However* $F^p_{Va}Gr^W_l = F^p_{St}Gr^W_l$.

Theorem

(E. Brieskorn) Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ *is a holomorphic map with isolated singularity, inducing the Milnor fibration* $f : X' \to T'$. *Then we have the following isomorphisms*

$$\mathcal{G} = R^n f_* \mathbb{C} \otimes \mathcal{O}_{T'} = R^n f_* \Omega_{X'/T'} = \frac{\Omega^{n+1}[t, t^{-1}]}{(d - tdf \wedge)\Omega^{n+1}[t, t^{-1}]}$$

where t *is a variable.*

Extenstion of Gauss-Manin system-algebraic case

- The Brieskorn Lattice is defined by,

$$\mathcal{G}_0 := image(\Omega^{n+1}[t] \to \mathcal{G}) = \frac{\Omega^{n+1}[t]}{(td - df\wedge)\Omega^{n+1}[t]}$$

- Set $\mathcal{G}_p := \tau^p \mathcal{G}_0, \tau = t^{-1}$.
- There are isomorphisms given by multiplication by t^p.

$$\frac{\mathcal{G}_p \cap V^\alpha}{\mathcal{G}_{p-1} \cap V^\alpha + \mathcal{G}_p \cap V^{>\alpha}} \cong \frac{V^{\alpha+p} \cap \mathcal{G}_0}{V^\alpha \cap \mathcal{G}_{-1} + V^{>\alpha} \cap \mathcal{G}_0}$$

- The gluing is done via the isomorphisms,

$$Gr_F^{n-p}(H_\lambda) \cong Gr_{\alpha+p}^V(H''/\tau^{-1}.H'')$$

Extension of Gauss-Manin system

Theorem

The identity

$$\frac{\mathcal{H}^{(0)}}{\tau^{-1}.\mathcal{H}^{(0)}} = \frac{\Omega^{n+1}}{df \wedge \Omega^n} = \Omega_f$$

defines the extension fiber of the Gauss-Manin system of the isolated singularity $f : \mathbb{C}^{n+1} \to \mathbb{C}$.

Good basis

Theorem

(M. Saito) Assume $\{(\alpha_i, d_i)\}$ is the spectrum of a germ of isolated singularity $f : \mathbb{C}^{n+1} \to \mathbb{C}$. There exists elements $s_i \in C^{\alpha_i}$ with the properties

(1) $s_1, ..., s_\mu$ *project onto a* $\mathbb{C}$*-basis of* $\bigoplus_{-1<\alpha<n} Gr_V^\alpha H''/Gr_V^\alpha \partial_t^{-1} H''$.

(2) $s_{\mu+1} := 0$; *there exists* $\nu : \{1, ..., \mu\} \to \{1, ..., \mu, \mu+1\}$ *with* $(t - (\alpha_i + 1)\partial_t^{-1})s_i = s_{\nu(i)}$

(3) *There exists an involution* $\kappa : \{1, ..., \mu\} \to \{1, ..., \mu\}$ *with* $\kappa = \mu + 1 - i$ *if* $\alpha_i \neq \frac{1}{2}(n-1)$ *and* $\kappa(i) = \mu + 1 - i$ *or* $\kappa(i) = i$ *if* $\alpha_i = \frac{1}{2}(n-1)$, *and*

$$P_S(s_i, s_j) = \pm\delta_{(\mu+1-i)j}.\partial_t^{-1-n}$$

.

where P is the Saito higher residue pairing.

Opposite filtrations

Theorem

(M. Saito) The filtration

$$U^p C^\alpha := C^\alpha \cap V^{\alpha+p} H''$$

is opposite to the filtration Hodge filtration F on $\mathcal{G}$.

Two filtrations F and U on H are called opposite, if

$$Gr_p^F Gr_U^q H = 0, \qquad \text{for } p \neq q$$

The two filtrations F^p and

$$U'_q := U^{n-q} = \psi^{-1}\{\oplus_\alpha C^\alpha \cap V^{\alpha+n-q} H''\} = \psi^{-1}\{\oplus_\alpha Gr_V^\alpha [V^{\alpha+n-q} H'']\}$$

are two opposite filtrations on $H^n(X_\infty, \mathbb{C})$.

Extension of mixed Hodge structure

- The Deligne bigrading,

$$H^n(X_\infty, \mathbb{C}) = \bigoplus_{p,q,\lambda} I_\lambda^{p,q}$$

- Define,

$$\Phi_\lambda^{p,q} : I_\lambda^{p,q} \xrightarrow{\hat{\Phi}_\lambda} Gr_V^{\alpha+n-p} H'' \xrightarrow{pr} Gr_V^{\bullet}(H''/\partial_t^{-1} H'') \xrightarrow{\cong} \Omega_f \tag{3}$$

where

$$\hat{\Phi}_\lambda^{p,q} := \partial_t^{p-n} \circ \psi_\alpha | I_\lambda^{p,q}$$

$$\Phi = \bigoplus_{p,q,\lambda} \Phi_\lambda^{p,q}, \qquad \Phi_\lambda^{p,q} = pr \circ \hat{\Phi}_\lambda^{p,q}$$

ψ_α is the nearby isomorphism.

Mixed Hodge structure on Ω_f

Theorem

The map Φ is a well-defined $\mathbb{C}$-linear isomorphism.

Definition

The mixed Hodge structure on Ω_f is defined by using the isomorphism Φ. This means that

$$W_k(\Omega_f) = \Phi W_k H^n(X_\infty, \mathbb{Q}), \qquad F^p(\Omega_f) = \Phi F^p H^n(X_\infty, \mathbb{C})$$

and the data of the Steenbrink MHS on $H^n(X_\infty, \mathbb{C})$ such as the $\mathbb{Q}$ or $\mathbb{R}$-structure is transformed via the isomorphism Φ to that of Ω_f. Specifically; in this way we also obtain a conjugation map

$$\bar{\cdot} : \Omega_{f,\mathbb{Q}} \otimes \mathbb{C} \to \Omega_{f,\mathbb{Q}} \otimes \mathbb{C}, \qquad \Omega_{f,\mathbb{Q}} := \Phi H^n(X_\infty, \mathbb{Q}) \tag{4}$$

defined from the conjugation on $H^n(X_\infty, \mathbb{C})$ via this map.

Main result

Theorem: Assume $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$, is a holomorphic germ with isolated singularity at 0, with $f:X\to T$ the associated Milnor fibration. Embed the Milnor fibration in a projective fibration $f_Y:Y\to T$ of degree d (with d large enough), by inserting possibly a singular fiber over 0. Then, the isomorphism Φ makes the following diagram commutative up to a complex constant;

$$\begin{array}{ccc} \widehat{Res}_{f,0}:\Omega_f\times\Omega_f & \longrightarrow & \mathbb{C} \\ \downarrow{\scriptstyle(\Phi^{-1},\Phi^{-1})} & & \downarrow{\scriptstyle\times *} \\ S:H^n(X_\infty)\times H^n(X_\infty) & \longrightarrow & \mathbb{C} \end{array} \qquad *\neq 0 \qquad (5)$$

where,

$$\widehat{Res}_{f,0}=\text{res}_{f,0}\,(\bullet,\tilde{C}\,\bullet)$$

Main result

and $\tilde{C}$ is defined relative to the Deligne decomposition of Ω_f, via the isomorphism Φ. If $J^{p,q} = \Phi^{-1} I^{p,q}$ is the corresponding subspace of Ω_f, then

$$\Omega_f = \bigoplus_{p,q} J^{p,q} \qquad \tilde{C}|_{J^{p,q}} = (-1)^p \tag{6}$$

In other words;

$$S(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \times \ \mathrm{res}_{f,0}(\omega, \tilde{C}.\eta), \qquad 0 \neq * \in \mathbb{C} \tag{7}$$

Example-Quasihomogeneous fibrations

- Take $f = 0$ with f to be quasi-homogeneous in weighted degrees $(w_1, ..., w_n)$, with the unique Milnor fiber $X_\infty = f^{-1}(1)$.
- MHS on $H^n(X_\infty, \mathbb{C})$: the Hodge filtration given by the degree of forms in the weighted projective space, and the weight filtration as

$$0 = W_{n-1} \subset W_n \subset W_{n+1} = H^n(X_\infty, \mathbb{C})$$

- Assume $\{\phi_1, ..., \phi_\mu\}$ be a basis for Ω_f, we consider the corresponding Leray residues

$$\eta_i = c_i.Res_{f=1}(\frac{\phi_i}{(f-1)^{l(i)}})$$

Here $c_i \in \mathbb{C}$ is a normalizing constant.

- Then,

$$S(\eta_i, \eta_j) = * \times res_{f,0}(\phi_i, \tilde{C}\phi_j)$$

- The isomorphism Φ is as follows,

$$\Phi^{-1} : [z^i dz] \longmapsto c_i.[res_{f=1}(z^i dz/(f-1)^{[l(i)]})]$$

with $c_i \in \mathbb{C}$, and z^i in the basis mentioned above.

Example

- For instance if $f = x^3 + y^4$, then as basis for Jacobi ring, we choose

$$z^i : \; 1, \; y, \; x, \; y^2, \; xy, \; xy^2$$

- The basis correspond to top forms with degrees

$$l(i) = \sum \alpha_i(\omega_i + 1).$$

$$l(i) : \; 7/12, \; 10/12, \; 11/12, \; 13/12, \; 14/12, \; 17/12$$

- The Hodge filtration is defined via

$$F^p := \mathbb{C}.\{\omega; p - 1 < l(\omega) < p\}.$$

Example

- The above basis projects onto a basis
$$\bigoplus_{-1<\alpha=l(i)-1<n} Gr_\alpha^V H'' \twoheadrightarrow Gr_V \Omega_f$$
- $h^{1,0} = h^{0,1} = 3$. Therefore, because Φ is an isomorphism.

$$< 1.\omega,\ y.\omega,\ x.\omega >= \Omega_f^{0,1}, \qquad < y^2.\omega,\ xy.\omega,\ xy^2.\omega >= \Omega_f^{1,0}$$

where $\omega = dx \wedge dy$, and the Hodge structure is pure, because $Gr_2^W H^n(X_\infty) = 0$.

$$\overline{< 1.dx \wedge dy,\ y.dx \wedge dy,\ x.dx \wedge dy >} = \\ < c_1.xy^2.dx \wedge dy,\ xy.dx \wedge dy,\ y^2.dx \wedge dy >$$

Graded polarizations on primitive subspaces-Consequences

Theorem

Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic isolated singularity germ. The modified Grothendieck residue provides a polarization for the extended fiber Ω_f, via the aforementioned isomorphism Φ. Moreover, there exists a unique set of forms $\{\widehat{Res}_k\}$ polarizing the primitive subspaces of $Gr_k^W \Omega_f$ providing a graded polarization for Ω_f.

$$\widehat{Res}_k = \widehat{Res} \circ (id \otimes \mathfrak{f}^k) : PGr_k^W \Omega_f \otimes_{\mathbb{C}} PGr_k^W \Omega_f \to \mathbb{C}, \tag{8}$$

Corollary

The polarization S of $H^n(X_\infty)$ will always define a polarization of Ω_f, via the isomorphism Φ. In other words S is also a polarization in the extension, i.e. of Ω_f.

Summary of results

Theorem

Assume $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic hypersurface germ with isolated singularity at $0 \in \mathbb{C}^{n+1}$. The variation of mixed Hodge structure is polarized. This VMHS can be extended to the puncture with the extended fiber isomorphic to Ω_f, and it is polarized. The Hodge filtration on the new fiber Ω_f correspond to an opposite Hodge filtration on $H^n(X_\infty, \mathbb{C})$.

Riemann-Hodge bilinear relations for Grothendieck pairing on Ω_f- new result

Corollary: Assume the holomorphic isolated singularity Milnor fibration $f : X \to T$ can be embedded in a projective fibration of degree d with $d >> 0$. Suppose $\mathfrak{f}$ is the corresponding map to N on $H^n(X_\infty)$, via the isomorphism Φ. Define

$$P_l = PGr_l^W := \ker(\mathfrak{f}^{l+1} : Gr_l^W \Omega_f \to Gr_{-l-2}^W \Omega_f)$$

Going to W-graded pieces;

$$\widehat{Res}_l : PGr_l^W \Omega_f \otimes_{\mathbb{C}} PGr_l^W \Omega_f \to \mathbb{C} \tag{9}$$

is non-degenerate and according to Lefschetz decomposition

$$Gr_l^W \Omega_f = \bigoplus_r \mathfrak{f}^r P_{l-2r}$$

we obtain a set of non-degenerate bilinear forms,

$$\widehat{Res}_l = \widehat{Res} \circ (id \otimes \mathfrak{f}^l) : PGr_l^W \Omega_f \otimes_{\mathbb{C}} PGr_l^W \Omega_f \to \mathbb{C}, \qquad (10)$$

$$\widehat{Res}_l = res_{f,0}\, (id \otimes \tilde{C}.\, \mathfrak{f}^l) \qquad (11)$$

such that the corresponding hermitian form associated to these bilinear forms is positive definite. In other words,

- $\widehat{Res}_l(x, y) = 0, \qquad x \in P_r,\ y \in P_s,\ r \neq s$
- If $x \neq 0$ in P_l,

$$Const \times res_{f,0}\, (C_l x, \tilde{C}.\, \mathfrak{f}^l.\bar{x}) > 0$$

where C_l is the corresponding Weil operator.

Real splitting of a MHS

MHS (H, F, W) and $\mathfrak{g} = \mathfrak{gl}(H) = End_{\mathbb{C}}(H)$.

$$\mathfrak{g}^{-1,-1} = \{X;\ X(I^{p,q}) \subset \bigoplus_{r \leq p-1,\ s \leq q-1} I^{r,s}\}.$$

Theorem

(P. Deligne) Given a mixed Hodge structure (W, F), there exists a unique $\delta \in \mathfrak{g}_{\mathbb{R}}^{-1,-1}(W, F)$ s.t. $(W, e^{-i\delta}.F)$ is a mixed Hodge structure which splits over $\mathbb{R}$.

Theorem

The bigrading $J_1^{p,q}$ defined by $J_1^{p,q} := e^{-i.\delta}.J^{p,q}$ is split over $\mathbb{R}$. The operator $\tilde{C}_1 = Ad(e^{-i.\delta}).\tilde{C} : \Omega_f \to \Omega_f$ defines a real splitting MHS on Ω_f.

- This says if $\Omega_{f,1} = \oplus_{p<q} J_1^{p,q}$ then

$$\Omega_f = \Omega_{f,1} \oplus \overline{\Omega_{f,1}} \oplus \bigoplus_p J_1^{p,p}, \qquad \overline{J_1^{p,p}} = J_1^{p,p}$$

- The relation (7) is valid when the operator $\tilde{C}$ is replaced with $\tilde{C}_1$;

$$S(\Phi^{-1}(\omega), \Phi^{-1}(\eta)) = * \times \ \mathrm{res}_{f,0}(\omega, \tilde{C}_1.\eta), \qquad 0 \neq * \in \mathbb{C}$$

and this equality is defined over $\mathbb{R}$.

Hodge index for Grothendieck pairing-new result

Theorem

The signature associated to the modified Grothendieck pairing $\widehat{Res}_{f,0}$ associated to an isolated hypersurface singularity germ f; is equal to the signature of the polarization form associated to the MHS of the vanishing cohomology given by

$$\sigma = \sum_{p+q=n+2} (-1)^q h_1^{pq} + 2 \sum_{p+q \geq n+3} (-1)^q h_1^{pq} + \sum (-1)^q h_{\neq 1}^{pq} \tag{12}$$

where $h_1 = \dim H^n(X_\infty)_1, h_{\neq 1} = \dim H^n(X_\infty)_{\neq 1}$ are the corresponding Hodge numbers. This signature is 0 when the fibers have odd dimensions.

Asymptotic Hodge theory

Theorem

(P. Deligne) Let $\mathcal{V} \to \triangle^{*n}$ be a variation of pure polarized Hodge structure of weight k, for which the associated limiting mixed Hodge structure is Hodge-Tate. Then the Hodge filtration $\mathcal{F}$ pairs with the shifted monodromy weight filtration $\mathcal{W}[-k]$, of $\mathcal{V}$, to define a Hodge-Tate variation over a neighborhood of 0 in $\triangle^{*n}$.

The form $\widehat{Res}$ polarizes the complex variation of HS studied by G. Pearlstein-Fernandez in case of isolated hypersurface singularities.

Application to asymptotic Hodge theory-new result

Theorem

(G. Pearlstein-J. Fernandez) Let $\mathcal{V}$ be a variation of mixed Hodge structure, and

$$\mathcal{V} = \bigoplus_{p,q} I^{p,q}$$

denotes the C^∞-decomposition of $\mathcal{V}$ to the sum of C^∞-subbundles, defined by point-wise application of Deligne theorem. Then the Hodge filtration $\mathcal{F}$ of $\mathcal{V}$ pairs with the increasing filtration

$$\bar{U}_q = \sum_k \bar{\mathcal{F}}^{k-q} \cap \mathcal{W}_k \tag{13}$$

to define an un-polarized $\mathbb{C}$VHS.

Theorem

Let $\mathcal{V}$ be an admissible variation of polarized mixed Hodge structure associated to a holomorphic germ of an isolated hyper-surface singularity. Set

$$U' = \overline{F_{\infty}^{\vee}} * W. \tag{14}$$

Then U' extends to a filtration $\underline{U'}$ of $\mathcal{V}$ by flat sub-bundles, which pairs with the limit Hodge filtration $\mathcal{F}$ of $\mathcal{V}$, to define a polarized $\mathbb{C}$-variation of Hodge structure, on a neighborhood of the origin.

Corollary

The mixed Hodge structure on the extended fiber Ω_f, can be identified with

$$\Phi(U' = \overline{F_{\infty}^{\vee}} * W)$$

Extensions of Hodge structure

- Suppose $f : X \to T$ a degenerate family of of curves having isolated singularity.
- Suppose that

$$J^1(H^1_s) = H^1_{s,\mathbb{Z}} \setminus H^1_{s,\mathbb{C}}/F^0 H^1_{s,\mathbb{C}}$$

$$J(\mathcal{H}) = \bigcup_{s \in S^*} J^1(H_s)$$

here we have assumed the Hodge structures have weight -1, and $\dim(S) = 1$.

Extensions of Hodge structure-new result

- On S^* we have an extension of integral local classes

$$0 \to \mathcal{H}_s \to \mathcal{J}_s \to \mathbb{Z}_s \to 0$$

- On the Gauss-Manin systems we get

$$0 \to M \to N \to \mathbb{Q}^H_{S^*}[n] \to 0$$

- The extended Jacobian simply is

$$X_0 = J^1(\Omega_f) = \Omega_{f,\mathbb{Z}} \setminus \Omega_f / F^0\Omega_f$$

Theorem

The extension of a degenerate 1-parameter holomorphic family of Θ-divisors polarizing the Jacobian of curves in a projective fibration, is a Θ-divisor polarizing the extended Jacobian.

- At the level of local systems,

$$\begin{array}{cccccc} \kappa: & \mathcal{H} & \otimes & \mathcal{H} & \to & \mathbb{C} \\ & \downarrow & & \downarrow & & \\ \kappa_J: & J & \otimes & J & \to & \mathbb{C} \\ & \downarrow & & \downarrow & & \\ \times: & \mathbb{Q} & \otimes & \mathbb{Q} & \to & \mathbb{C} \end{array} \tag{15}$$

- At the level of Gauss-Manin systems,

$$\begin{array}{cccccc} K: & \mathcal{G} & \otimes & \mathcal{G} & \to & \mathbb{C}[t,t^{-1}] \\ & \downarrow & & \downarrow & & \\ K_J: & N & \otimes & N & \to & \mathbb{C}[t,t^{-1}] \\ & \downarrow & & \downarrow & & \\ \times: & \mathbb{Q}^H_S & \otimes & \mathbb{Q}^H_S & \to & \mathbb{C}[t,t^{-1}] \end{array} \tag{16}$$

Hypersurface rings

- A hyper-surface ring is a ring of the form $R := P/(f)$, where P is an arbitrary ring and f a non-zero divisor.
- Localizing we may assume P is a local ring of dimension $n+1$.
- $P = \mathbb{C}\{x_0, ..., x_n\}$ and f a holomorphic germ, or $P = \mathbb{C}[x_0, ..., x_n]$ and then f would be a polynomial.
- We assume $0 \in \mathbb{C}^{n+1}$ is the only singularity of f.
- the R-modules have a minimal resolution that is eventually 2-periodic.

Hochster Theta Invariant

Definition

(Hochster Theta pairing) The theta pairing of two R-modules M and N over a hyper-surface ring $R/(f)$ is

$$\Theta(M,N) := l(Tor_{2k}^R(M,N)) - l(Tor_{2k+1}^R(M,N)), \qquad k >> 0$$

Hochster theta pairing is additive on short exact sequences,

Theorem

(Moore-Piepmeyer-Spiroff-Walker) If f is homogeneous with isolated singularity at 0*, and n odd the restriction of the pairing* $(-1)^{(n+1)/2}\Theta$ *to*

$$im(ch^{\frac{n-1}{2}}) : K(X)_{\mathbb{Q}}/\alpha \to \frac{H^{(n-1)/2}(X,\mathbb{C})}{\mathbb{C}.\gamma^{\frac{n-1}{2}}}$$

is positive definite. i.e. $(-1)^{(n+1)/2}\Theta(v,v) \geq 0$ *with equality holding if and only if* $v = 0$*. In this way* θ *is semi-definite on*

Positivity of Hochster theta pairing/$\mathbb{C}$-new result

Theorem

Let S be an isolated hypersurface singularity of dimension n over $\mathbb{C}$. *If n is odd, then* $(-1)^{(n+1)/2}\Theta$ *is positive semi-definite on* $G(R)_{\mathbb{Q}}$, *i.e* $(-1)^{(n+1)/2}\Theta(M, M) \geq 0$.