A REMARK FOR AUTOMORPHIC FORMS ON PERIOD DOMAINS

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Abstract. In this article we mention the fact that the Hodge metric on Mumford-Tate domains is the same as the Peterson inner product of the automorphic forms on these spaces. This fact was known in other area literature, however not appeared in Hodge texts.

Introduction

This article is a translation of the work of Leon A. Takhtajan, P. Zograf and C. Meneses presented in [3], and also lectured in the second World Mexican mathematicians conference taken place at CIMAT, Dec. 2014, but in much more basic language for Hodge theory literature. Thus I apologize if the result is probably known, but omitted in Hodge texts. We follow [1] and [2] for the basic definitions in Hodge theory. Thus we prevent of mentioning this in the text for connectivity of the statements.

For a Hodge structure (V, ϕ) of weight n let $h^{p,q}$ be the Hodge numbers. A period domain D is the set of polarized Hodge structures (V, Q, ϕ) with the given Hodge numbers $h^{p,q}$. The compact dual $\check{D}$ is the set of filtrations $F^\bullet$ of $V_{\mathbb{C}}$ with $\dim(F^p) = \sum_{i \geq p} h^{p,q}$, and satisfying $Q(F^p, F^{n-p+1}) = 0$. The group $G_{\mathbb{R}} = Aut(V_{\mathbb{R}}, Q)$ is a real simple Lie group that acts transitively on D. The isotropy group H of a reference polarized Hodge structure (V, Q, ϕ) preserves a direct sum of definite Hermitian forms, and therefore is a compact subgroup of $G_{\mathbb{R}}$ that contains a compact maximal torus T. Then one has

$$D = \{\phi : S^1 \to G_{\mathbb{R}} \ ; \ \phi = g^{-1}\phi_0 g\} \tag{1}$$

It follows that $H = Z_{\phi_0}(G_{\mathbb{R}})$, the centralizer of $\phi_0(S^1)$. An easy exercise in linear algebra shows

$$H \cong \begin{cases} U(h^{2m+1}) \times ... \times U(h^{m+1,m}) & n = 2m+1 \\ U(h^{2m}) \times ... \times U(h^{m+1,m-1}) \times \mathcal{O}(h^{m,m}) & n = 2m \end{cases} \tag{2}$$

The group $G_{\mathbb{C}}$ is a complex simple Lie group that acts transitively on $\check{D}$. The subgroup P that stabilizes a $F_0^\bullet$ is a parabolic subgroup with $H = G_{\mathbb{R}} \cap P$. The case $n = 1$ is classical and one knows that $D = H_g$ the Siegel generalized upper half space $= \{Z \in M_{g\times g} : Z =^t Z,\ Im(Z) > 0\}$

The Lie algebra $\mathfrak{g}$ of the simple Lie group G is a $\mathbb{Q}$-linear subspace of $End(V)$, and the form Q induces on $\mathfrak{g}$ a non-degenerate symmetric bilinear form

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \tag{3}$$

which upto scale is just the Cartan-Killing form $\mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y))$. For each point $\phi \in D$

$$\mathrm{Ad}\phi : \mathbb{U}(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}}, B) \tag{4}$$

is a Hodge structure of weight 0 on $\mathfrak{g}$. This Hodge structure is polarized by B.

Associated to each nilpotent transformation $N \in \mathfrak{g}$ one defines a limit mixed Hodge structure. The local system $\mathfrak{g} \to \Delta^*$ is then equipped with the monodromy $T = e^{\mathrm{ad}N}$ and Hodge filtration defined with respect to the multi-valued basis of $\mathfrak{g}$ by $e^{\log(t)\frac{N}{2\pi i}}F^\bullet$. This has a limit MHS $(\mathfrak{g}, F^\bullet, W(N)_\bullet)$.

The polarizing form gives perfect pairings

$$B_k : Gr_k^{W(N)}\mathfrak{g} \times Gr_{-k}^{W(N)}\mathfrak{g} \to \mathbb{Q}, \qquad B_k(u, v) = B(v, N^k v) \tag{5}$$

where $B_k, (B_{-k})$ are defined via the hard Lefschetz isomorphism $N^k : Gr_{-k}^{W(N)}\mathfrak{g} \cong Gr_k^{W(N)}\mathfrak{g}$.

Let D be a period domain for a PHS (V, Q, ϕ) of weight n and set $\Gamma_{\mathbb{Z}} = Aut(V_{\mathbb{Z}}, \mathbb{Q})$. In the tangent bundle TD there is a homogeneous sub-bundle W whose fiber at ϕ is $W_\phi = \mathfrak{g}_\phi^{-1,1} = \{\psi \in T_\phi D : \psi(F_\phi^p) \subset F_\phi^{p+1}\}$ defined by, namely infinitesimal period relations (IPR).

A variation of Hodge structure (VHS) is given by a locally liftable holomorphic map

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\Phi}} & D \\ \downarrow & & \downarrow \\ S & \xrightarrow[\Phi]{} & \Gamma_{\mathbb{Z}} \setminus D \end{array} \tag{6}$$

where the left vertical map is the universal covering, and the infinitesimal period relation says $\tilde{\Phi}_* : T\tilde{S} \to W$. $\Gamma := \Phi_*(\pi_1(S, s_0)) \subset G_{\mathbb{Z}}$ is called the monodromy group.

1. Hodge bundles and Automorphic presentations

Over $\check{D}$ there are $G_{\mathbb{C}}$-homogeneous vector bundles

$$F^p \to \check{D} \tag{7}$$

whose fiber at a given point $F^\bullet$ is F^p. Over $D \subset \check{D}$ we have

$$V^{p,q} = F^p / F^{p+1} \tag{8}$$

These are homogeneous vector bundles for the action of $G_{\mathbb{R}}$. They are hemitian vector bundles with $G_{\mathbb{R}}$-invariant Hermitian metric given in each fiber by the polarization form.

Suppose for the moment $\dim(V) = 2$ and $n = 1$. The equivalence classes of polarized Hodge structures of weight 1 can be identified with $Sl_2(\mathbb{Z})\backslash H$, with H to be the upper half plane. More generally for geometric reasons one wishes to consider congruence subgroups $\Gamma \subset Sl_2(\mathbb{Z})$ and the quotient spaces

$$M_\Gamma = \Gamma \setminus H \tag{9}$$

M_Γ is a Riemann surfaces. It is not compact but has only cusps. Let $H^{n,0} := (H^{1,0})^{\otimes n}$.

Definition 1.1. *A holomorphic automorphic function of weight n is given by a holomorphic section $\psi \in \Gamma(\Gamma \setminus H, H^{n,0})$ that is finite on the cusps.*

Such an automorphic form can be written as

$$\psi(\tau) = f(\tau) d\tau^{n/2} \tag{10}$$

where f is holomorphic on H and satisfies

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^n f(\tau) \tag{11}$$

Around a cusp one sets $q = e^{2\pi i \tau}$ and expands in Laurent series

$$f(q) = \sum_n a_n q^n \tag{12}$$

The finiteness condition at the cusps are then $a_n = 0$ for $n < 0$. If we consider the Deligne extension $\underline{H_e^{n,0}}$ the the modular forms are sections of $H_e^{n,0} \to \Gamma \setminus H$ that extend to $H_e^{n,0} \to \overline{\Gamma \setminus H}$. A modular form is a cusp form if $a_0 = 0$, i.e it vanishes at the cusps. This condition is equivalent to $\int_{\Gamma \setminus H} \parallel \psi \parallel d\mu < \infty$.

2. Peterson Inner product and Hodge Polarization

The reader may regard this section as a translation of the facts in [3] in a very basic language. We first mention the definition of Peterson inner product on automorphic forms [4].

Let $f, \ g$ be two cusp forms of weight n on H the upper half plane. One proves easily that the measure

$$\mu(f, g) = f(z)\overline{g(z)} y^{n-2} dx dy, \qquad x = Re(z), \ y = Im(z) \tag{13}$$

is invariant by $\Gamma = PSl_2(\mathbb{Z})$ and it is a bounded measure on the space $\Gamma \setminus H$. By putting

$$\langle f, g \rangle = \int_{\Gamma \setminus H} \mu(f, g) \tag{14}$$

one obtains a hermitian scalar product on the space of modular forms which is positive and non-degenerate. One can check that

$$\langle T(k)f, g\rangle = \langle f, T(k)g\rangle \tag{15}$$

where $T(k)$ is the k-th Hecke operator defined by

$$T(k).f = k^{n-1} \sum_{a \geq 0,\ 0 \leq b < d}^{ad=n} d^{-n} f(\frac{az+b}{d}) \tag{16}$$

We continue with our example for $M_\Gamma = \Gamma \setminus H$ with the Hodge bundle $H^{n,0}$ and $\mathfrak{g}$. Lets define the sheaf as $\mathcal{G} := \Gamma \setminus (H \times \mathfrak{g})$. Lets denote the hermitian metric h_{M_Γ} on M_Γ induced from the metric on the upper half plane H. It induces a metric $h_\mathcal{G}$ on $\mathcal{G}$. We use the hyperbolic metric descended from H to define Hodge $*$-operator on $H^{n,0}$ and $\mathcal{G}$. Then the Laplace operator and the harmonic forms can be defined in the same fashion as in Hodge theory.

When $H^{n,0}$ or $\mathcal{G}$ are equipped with a monodromy representation ρ. Then we are involved with the sections of $A^{p,q}(H, M_\Gamma)$ and $A^{p,q}(H, \mathcal{G})$, in which the automorphic forms are defined via

$$f(\gamma.z)\gamma'(z)^p\overline{\gamma'(z)^q} = Ad(\rho)f(z) \tag{17}$$

The the Hodge inner product is

$$\langle f, g\rangle = \int_H f(z) \wedge *g(z)dxdy = \int_{\Gamma\setminus H} tr(f(z) \wedge g(z)^*)y^{p+q-2}dxdy, \qquad f^* = \bar{f}^t \tag{18}$$

Proposition 2.1. *The Hodge polarization of the Hodge bundles $H^{n,0}$ or $\mathcal{G} := \Gamma \setminus (H \times \mathfrak{g})$ is given by the Peterson Inner product. More specifically the Cartan-Killing form on $\mathfrak{g}$ is induced from the Peterson-inner product.*

The proof goes trivially from the discussion in sections 1 and 2.

Corollary 2.2. *The Peterson inner product induces the Cartan-Killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ on the fibers of $\mathcal{G}$. Moreover there are graded forms $\langle ., .\rangle_k : Gr_k^{W(N)}\mathfrak{g} \times$*

$Gr_{-k}^{W(N)}\mathfrak{g} \to \mathbb{Q}$, $\langle .,. \rangle_k := \langle \mathfrak{l}^k u, v\rangle = \langle v, \mathfrak{l}^k v\rangle$ *when the fibration is involved with the monodromy* $T = e^{ad(N)}$. *Moreover the action of the Hecke operators induce self adjoint linear transformations on* $\mathfrak{g}$.

Proof. The corollary follows from the uniqueness of the polarization form of (mixed) Hodge structures and the explanations of (3), (4) and (5). The last part is a consequence of (15). □

References

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