RESEARCH STATEMENT

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My major interests lie in Hodge theory, and its related areas. Hodge theory and its arithmetic applications are quite interesting for me both over the field of complex numbers $\mathbb{C}$ and in the l-adic (p-adic) case. Many parts of Hodge theory such as intractions with K-theory, representation theory, modular forms, Calabi-Yau varieties and mirror symmetry are of my interest.

1. RESEARCH PLAN

1.1. **Modularity of Calabi-Yau varieties.** The Shimura-Taniyama-Weil conjecture on modularity of L-function of Elliptic curves proved by A. Wiles, has been generalized over Calabi-Yau varieties in higher dimensions.

Theorem 1.1. *(Shimura-Tanyama-Weil Conjecture - A. Wiles)* [W] *Suppose E is a semi-stable Elliptic curve defined over $\mathbb{Q}$. Then E is modular.*

A Calabi-Yau manifold is a compact complex manifold with trivial canonical bundle. A one dimensional Calabi-Yau manifold is an Elliptic curve. A simply connected Calabi-Yau manifold is a K3 surface.

Conjecture 1.2. *(Modularity Conjecture)* [Y] *Any rigid Calabi-Yau 3-fold X over $\mathbb{Q}$ is modular in the sense that, up to finite Euler factors,*

$$L(H^3_{et}(\bar{X}, \mathbb{Q}), s) = L(f, s), \qquad f \in S_4(\Gamma_0(N))$$

The question arises to which higher dimensional Calabi-Yau varieties are modular. For K3 surfaces the question has been answered positively by Shioda and Inose. This conjecture has been answered in some special cases in low dimension 3. Special properties of Calabi-Yau manifolds and their variations has made this line of research involving many beautiful number theoretic motivations. The question mainly says that the L-function of a Calabi-Yau variety defined over $\mathbb{Q}$ is the L-function of a modular form.

Modularity question of Calabi-Yau manifolds concerns many signeficant interactions between Hodge theory, representation theory and Langlands program.

Key words and phrases. Variation of Hodge structure, p-adic period map, Motivic fundamental group, Algebraic cycles, Modularity of Calabi-Yau manifolds, l-adic local systems.

1.2. **Motivic fundamental group.** The role of the projective line minus three points $X = \mathbb{P} \setminus \{0, 1, \infty\}$ in relation to Galois theory can be traced back to the theorem,

Theorem 1.3. *(Belyi-1979) Every smooth projective algebraic curve defined over* $\bar{\mathbb{Q}}$ *can be realized as a ramified cover of* $\mathbb{P}^1$.

Belyi deduced that the absolute Galois group of $\mathbb{Q}$ acts faithfully on the profinite completion of the fundamental group of X, i.e. the map

$$Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to Aut(\hat{\pi}_1(X(\mathbb{C}, b))$$

Theorem 1.4. [B] *There is an ind-object*

$$\mathcal{O}(\pi_1^{mot}(X, \overrightarrow{1_0}, -\overrightarrow{1_1})) \in Ind(\mathcal{MT}(\mathbb{Z}))$$

whose Betti and de Rham realizations are the affine rings $\mathcal{O}\pi_1^B(X, \overrightarrow{1_0}, -\overrightarrow{1_1}))$ *and* $\mathcal{O}(\pi_1^{dR}(X))$, *respectively.*

there is an exact sequence

$$0 \to I \to \mathbb{Q}[\pi_1^{top}(X(\mathbb{C}), x))] \to \mathbb{Q} \to 0$$

where I is the augmentation ideal. Then one has

$$\mathcal{O}(\pi_1^B(X, x)) = \lim_{N \to \infty} (\mathbb{Q}[\pi_1^{top}(X, x)]/I^{N+1})^\vee$$

when $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ we have

$$\mathcal{O}_1^{dR}(X)) \cong \bigoplus_{n \geq 0} H^1_{dR}(X)^{\otimes n}$$

1.3. **Galois representations and l-adic local systems.** The classical finiteness theorem for abelian varieties generalizing that of G. Faltings has been discussed in the l-adic case by P. Deligne.

Theorem 1.5. [CS] *Let S be a finite set of places of K. There are only finitely many isogeny classes of abelian varieties over K, of a given dimension which have good reductions outside S.*

This theorem mainly states that in an abelian scheme fiberation there exists finitely many isomorphism classes of polarized abelian varieties. In other words, there could be finitely many isomorphism classes of monodromy representations for $(S, s)/\mathbb{C}$ of rank $\leq r$ and weight n. The theorem has already discussed by P. Griffiths concerning $\mathbb{Z}$-polarized variation of Hodge structure over S. In addition to beauty of this theorem and its proof, its generalization for the schemes over finite fields opens more interesting ideas of ramification theory using Swan index of l-adic representations of etale fundamental group and the Deligne-Weil group.

Let $\mathcal{R}_r(X)$ be the set of lisse $\bar{\mathbb{Q}}_l$-Weil sheaves on X of dimension r and up to semi-simplification. For X connected such a sheaf is nothing but an r-dimensional l-adic representation of $W(X)$. A weaker version of the theorem then says the number of classes of irreducible sheaves in $\mathcal{R}_r(X)$ with bounded wild ramification is finite up to twist.

Theorem 1.6. *(P. Deligne) Assume X is smooth separated $/\mathbb{F}_q$ be connected, and $\bar{X}$ be a normal compactification of X with $D = \bar{X} \setminus X$ a normal crossing divisor. Let $\mathcal{R}_r(X, D)$ be the set of representations whose Swan conductor along any smooth curve mapping to $\bar{X}$ is bounded by D. Then the set of irreducible sheaves $V \in \mathcal{R}_r(X, D)$ is finite up to twist by elements of $\mathcal{R}_1(\mathbb{F}_q)$.*

The proof concerns a parametrization of the Frobenius attached to each point of the variety. Studying the irregularity of l-adic representations using the Swan index is another deep construction in the l-adic Hodge theory which also uses local class field theory tools.

I believe that this theorem with its different generalizations is one of the elegant constructions in mathematics both in statement and proof. I think it has the value of working out more and the capacity to generate more knowledge. The project will target possible relations with geometric Langlands program, l-adic perverse sheaves and p-adic Hodge theory.

References

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