ON A STACK STRUCTURE FOR PERIOD AND MUMFORD-TATE DOMAINS

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Abstract. We compare some of the structure theorems for Mumford-Tate domains with those of toric stacks. Specifically we mention an orbifold structure on MT-domains and provide a partial uniformization for them.

Introduction

In [R] we showed a toric stack structure on Mumford-Tate domains via a toric partial compactification of these spaces. In this short note we try to apply some uniformization results on toric stacks stated in [FMN], to higher structure on Hodge domains. In this way we obtain similar uniformizations for MT-domains. As a consequence it follows that MT-domains are smooth Deligne-Mumford stacks.

Toric stacks are natural generalization of toric varieties as the action groupoid of Deligen-Mumford tori as a Picard stack. The stacky structure of toric varieties carries much more information than their scheme structure, as it carries the information on stabilizers of each point also. This has been shown in several examples in this text.

A simple way to define a toric stack is by the definition of a stacky fan. Toric stacks are Deligen-Mumford. Such stacks are also smooth as Deligne-Mumford stacks and their boundary points have codimension 1 as a sub-stack.

Toric stacks satisfy a simple uniformization structure as a global quotient via the structure of their Chow group or equivalently their Picard group of line bundles. We explain this by root sheaves or root of line bundles associated to rays in their fan.

In [R] we showed that the toroidal compactification of Mumford-Tate domains satisfy a toric stack structure using the definition by stacky fans. In this text we show that MT-domains of Hodge structure are smooth DM-stacks and in this way satisfy the same uniformization theorem as general smooth DM-stacks. The basic definitions on toric stacks is taken from [FMN], and on MT-domains from [GGK].

1. Toric Stacks

In this section we briefly mention some basic results on toric stacks as smooth Deligne-Mumford stacks. The main reference is [FMN].

Let S be a base scheme. We define a Picard stack $\mathcal{G}$ over S to be a groupoid over S. A morphism of such stacks is one who preserves the corresponding operations via 2-equivalence. The neutral element is a couple (e, ϵ) where $e : S \to \mathcal{G}$ is a section and

$\epsilon : e.e \rightarrow\Rightarrow e$. A neutral element is unique up to a unique isomorphism (equivalence by 2-arrows). The action of a Picard S-stack on an S-stack is a morphism $\mathcal{G} \times_S X \to X$ with two arrows $\eta_x : e \times x \Rightarrow x,\ (g_1.g_2) \times x \Rightarrow x$.

Example 1.1. *Let $w_0, ..., w_n \in \mathbb{N}^n_{>0}$, and $\phi : \mathbb{Z}^{n+1} \to \mathbb{Z}$ be $(a_0, ..., a_n) \mapsto \sum w_i a_i$ The the associated Picard stack is $(\mathbb{C}^*)^n \times B\mu_d$, where $d = gcd(w_0, .., w_n)$.*

A Deligne-Mumford torus is a Picard stack over $Spec(\mathbb{C})$ which is obtained as a quotient $[T_L/G_N]$, with $\phi : L \to N$ is a morphism of finitely generated abelian groups such that $\ker(\phi)$ is free and $coker(\phi)$ is finite. Any Deligne-Mumford (DM) torus is isomorphic as Picard stack to $T \times BG$, where T is a torus and G is a finite abelian group. Then, a smooth toric Deligne-Mumford stack is a smooth separated DM-stack X together with an open immerssion of a Deligne-Mumford torus $\imath : \mathcal{T} \hookrightarrow X$ with dense image such that the action of $\mathcal{T}$ on itself extends to an action $\mathcal{T} \times X \to X$. In this case a morphism is a morphism of stacks which extends a morphism of Deligne-Mumford tori. A toric orbifold is a toric DM-stack with generically trivial stabilizer. A toric DM-stack is a toric orbifold iff its DM-torus is an ordinary torus, [FMN].

Theorem 1.2. [FMN] *Let X be a smooth toric DM-stack, with course moduli $\bar{X}$. Let Σ be a fan of X in $N_{\mathbb{Q}}$. Assume that the rays of σ generate $N_{\mathbb{Q}}$. Then there exists a stacky fan such that X is isomorphic to the smooth DM-stack associated to the stacky fan. Moreover, if X has trivial generic stabilizer the the stacky fan is unique.*

Definition 1.3. *A morphism $r : Y \to X$ of algebraic stacks is a gerbe over X if it is locally surjective and all objects in each fiber are isomorphic. It is the same to say $Y \times_X X' \cong BG \times_X X'$, with $X' \to X$ faithfully flat and locally of finite presentation, for a G-gerbe.*

A gerbe on a topological space is stack G of groupoids over X which is locally non-empty. A canonical example is the gerbe of principal G-bundles with a fixed structure group. The trivial G-bundle fullfils the non-emptiness condition, and the fact that they are locally isomorphic tells the gerbe condition.

Let (Σ, N, β) be a stacky fan. Denote by v_i the unique generator of $\rho_i \cap (N/N_{tor})$. Denote by β_{rig} the composition of β with $N \to N/N_{tor}$. There exists a unique a_i such that $\beta_{rig}(e_i) = a_i v_i$. Then $X^{rig} = (\Sigma, \beta_{rig}, N/N_{tor})$. moreover there exists a unique group homomorphism $\beta_{can} : \mathbb{Z}^n \to N/N_{tor}$ which makes the following diagram commutative,

$$\begin{array}{ccc} \mathbb{Z}^n & \xrightarrow{\beta} & N \\ {\scriptstyle (diag(a_1,...,a_n)}\downarrow & & \downarrow \\ \mathbb{Z}^n & \xrightarrow[\beta_{rig}]{} & N/N_{tor} \end{array} \tag{1}$$

Then $X^{can} = (\Sigma, \beta_{can}, N/N_{tor})$.
Let X be a toric stack, then the structure morphism $X \to \bar{X}$ into its coarse moduli space factors canonically via the toric morphisms $X \to X^{\mathrm{rig}} \to X^{\mathrm{can}} \to \bar{X}$ where $X \to X^{\mathrm{rig}}$ is an abelian gerbe over X^{rig}, and $X^{\mathrm{rig}} \to X^{\mathrm{can}}$ is a fiber product of roots of toric divisors, and $X^{\mathrm{rig}} \to X$ is the minimal orbifold having $\bar{X}$ as a coarse moduli space.

The canonical Deligne-Mumford stack is universal for dominant codimension preserving morphisms from orbifolds into the coarse moduli, in the sense of stacks terminology. The rigidification $r : X \to X^{rig}$ by a central subgroup G of the generic stabilizer is the stackification of the pre-stack with the same objects as X but the automorphism group of each object x is replaced by $Aut(x)/G$. X^{rig} is an orbifold having the same coarse moduli as X.

Example 1.4. *The stack $\mathbb{P}(w) \cong [\mathbb{C}^{n+1} \setminus 0/\mathbb{C}^*]$ is a toric DM-stack with deligne-Mumford torus $[\mathbb{C}^{n+1}/\mathbb{C}^*] \cong \mathbb{C}^n \times B\mu_d$. It is canonical if $gcd(w_0, ..., \hat{w}_i, ..., w_n) = 1$. It is an orbifold if $gcd(w_0, ..., w_n) = 1$.*

2. Structure of toric stacks

This section contains standard uniformization theorem for toric stacks, taken from [FMN].

A coherent sheaf on a DM stack $[Z/G]$ is a G-equivariant sheaf on Z, i.e a coherent sheaf L together with an isomorphism $\phi_g : L \to g^*L$ for all g such that $\phi_{hg} = h^*\phi_g \circ \phi_h$. Let Z be a subvariety of $\mathbb{C}^n$ of codimension equal or higher than 2. let G be an abelian group scheme over $\mathbb{C}$ that acts on Z such that $[Z/G]$ is a Deligne-Mumford stack. A line bundle on $[Z/G]$ is given by a character χ of G. Hence the data of an invertible sheaf L with a global section s on $[Z/G]$ give a morphism of groupoids between $[Z/G]$ and $[\mathbb{A}^1/\mathbb{C}^*]$. Explicitly this morphism is given by $(s, \chi) : Z \times G \to \mathbb{A} \times \mathbb{C}^*$ and $s : Z \to \mathbb{A}^1$.

Definition 2.1. *let b be a positive integer. We denote by $\sqrt[a]{L/X}$ the fiber product*

$$\begin{tikzcd}
\sqrt[a]{L/X} \arrow[r] \arrow[d] & B\mathbb{C}^* \arrow[d, "\wedge b"] \\
X \arrow[r, "L"'] & B\mathbb{C}^*
\end{tikzcd} \tag{2}$$

where $\wedge b$ is the map that sends an invertible sheaf to its b-th power. More explicitly an object of ${}^{a}\!\sqrt{L/X}$ over $f : S \to X$ is a couple (M, ϕ) where M is an invertable sheaf on S and $\phi : M^{\otimes b} \xrightarrow{\cong} f^*L$ is an isomorphism.

Let X be a smooth algebraic stack. Assume $D = (D_1, ..., D_n)$ are n effective Cartier divisors and $a = (a_1, ..., a_n) \in \mathbb{N}^n_{>0}$. Then the a-th root of (X, D) is the fiber product

$$\begin{tikzcd} \sqrt[a]{D/X} \arrow[r] \arrow[d] & {[\mathbb{A}^n/(\mathbb{C}^*)^n]} \arrow[d, "\wedge a"] \\ X \arrow[r, "L"'] & {[\mathbb{A}^n/(\mathbb{C}^*)^n]} \end{tikzcd} \tag{3}$$

$\wedge a : [\mathbb{A}^n/(\mathbb{C}^*)^n] \to [\mathbb{A}^n/(\mathbb{C}^*)^n]$ is the stack morphism $x_i \to x_i^{a_i}$ and $\lambda_i \to \lambda_i^{a_i}$, where x_i, λ_i are coordinates in $\mathbb{A}^n$ and $(\mathbb{C}^*)^n$ respectively.

Proposition 2.2. [FMN] *let X be toric Deligne-Mumford stack with Deligne-Mumford torus isomorphic to $T \times BG$. Given $G = \prod_{j=1}^{l} \mu_{b_j}$. There exists L_j in $Pic(X^{rig})$ such that X is isomorphic as a G-gerbe over X^{rig} to*

$$\sqrt[b_1]{L_1/X^{rig}} \times ... \times \sqrt[b_l]{L_l/X^{rig}} \tag{4}$$

where the classes $(L_1, ..., L_l) \in \prod_{j=1}^{l} Pic(X^{rig})/b_j Pic(X^{rig})$ is unique. Moreover the reduced closed substack $X \setminus \mathcal{T}$ is a simple normal crossing divisor.

Let X be a smooth toric Deligne-Mumford stack with DM-torus isomorphic to $T \times BG$, with $\bar{X}$ as coarse moduli. denote by n the number of rays in the fan of $\bar{X}$. Then there exists a unique $(a_1, ..., a_n) \in \mathbb{N}^n_{>}$ such that

$$X^{\mathrm{rig}} \cong \sqrt[a_1]{D_1/X} \times_X ... \times_X \sqrt[a_n]{D_n/X}$$

where D_i is the divisor corresponding to the ray ρ_i.

Proposition 2.3. [FMN] *Let Z be a subvariety of $\mathbb{C}^n$ and G be an abelian group scheme over $\mathbb{C}$ that acts on Z such that $[Z/G]$ is a Deligne-Mumford stack. let $(L, s) := ((L_1, s_1), ..., (L_k, s_k))$ be k invertible sheaves with global sections over the quotient stack $[Z/G]$. Denote by $\chi = (\chi_1, ..., \chi_k)$the representations associated to the invertible sheaves L. let $d = (d_1, ..., d_k) \in \mathbb{N}^k_{>0}$. Then $\sqrt[d]{(L, s)/[Z/G]}$ is isomorphic to $\tilde{Z}/\tilde{G}$, where*

$$\begin{tikzcd} \tilde{Z} \arrow[r] \arrow[d] & \mathbb{A}^k \arrow[d, "\wedge d"] & \tilde{G} \arrow[r] \arrow[d, "\phi"'] & \mathbb{G}_m^k \arrow[d, "\wedge d"] \\ Z \arrow[r, "s"'] & \mathbb{A}^k & G \arrow[r, "\chi"'] & \mathbb{G}_m^k \end{tikzcd} \tag{5}$$

The action of $\tilde{G}$ on $\tilde{Z}$ is given by

$$(g, (\lambda_1, ..., \lambda_k)).(x_1, ..., x_k) = (g.z, (\lambda_1 x_1, ..., \lambda_k x_k)) \tag{6}$$

With these setting $\sqrt[d]{L/[Z/G]} \cong [Z/\tilde{G}]$.

Example 2.4. [FMN] *Consider the toric line $\mathbb{P}^1$ with homogeneous coordinate x_1, x_2. Let $D_i = (\mathcal{O}(1), x_i)$. Let $a_1, a_2 \in \mathbb{N}_{>0}$ with $d = [a_1, a_2],\ m = (a_1, a_2)$ be the gcd and lmm. The Picard group of the root stack $\sqrt[(a_1,a_2)]{(D_1, D_2)/\mathbb{P}^1}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2$. Then*

$$\sqrt[(a_1,a_2)]{(D_1, D_2)/\mathbb{P}^1} \cong [\mathbb{C}^2 \setminus 0/(\mathbb{C}^* \times \mu_d)] \tag{7}$$

with action, $(\lambda, t), (x_1, x_2)) \mapsto (\lambda^{m/a_1} t^{k_2} x_1, \lambda^{m/a_2} t^{-k_1} x_2)$, where

$$\frac{k_1}{a} + \frac{k_2}{b} = \frac{1}{m} \tag{8}$$

3. Structure theorem for Mumford-Tate domains

In this section we apply the structure theorems of section 2 to the stacky structure on period and Mumford-Tate domains. We begin by some generalities on Hodge domains, which can be found in [GGK].

Definition 3.1. *The Mumford-Tate group of a variation of Hodge structure $\Phi : S \to \Gamma \setminus D$ is defined to be the MT-group $M_{\Phi(\eta)}$ of a generic point $\eta \in S$.*

By choosing a base point $s_0 \in S$ there would be a monodromy representation

$$\rho : \pi_1(S, s_0) \to \Gamma \tag{9}$$

By replacing S with a finite cover if necessary one may assume

$$\Gamma \subset G(\mathbb{Z}) \cap M_\Phi \tag{10}$$

Because M_Φ is a reductive Lie group, we have the almost product decomposition

$$M_\Phi = M_1 \times ... \times M_l \times T \tag{11}$$

Denoting by D_i the $M_i(\mathbb{R})$-orbit of $\phi(s_0)$ we have a splitting of the Mumford-Tate domain

$$D_{M_\Phi} = D_1 \times ... \times D_l \tag{12}$$

As a structure theorem the monodromy group Γ decomposes as an almost direct product

$$\Gamma = \Gamma_1 \times ... \times \Gamma_l \tag{13}$$

where $\Gamma_i(\mathbb{Q}) = M_i$.

Theorem 3.2. *The variation of Hodge structure is given by*

$$\Phi : S \to \Gamma_1 \setminus D_1 \times ... \times \Gamma_k \setminus D_k \times D' \subset \Gamma \setminus D$$

where $D' = D_{k+1} \times ... \times D_l$, *and* $\Gamma_i(\mathbb{Q}) = M_i$.

Assume $\phi : \mathbb{U} \to M$ be a Hodge structure, with the associated Mumford-Tate domain $D_M = M(\mathbb{R}).\phi$. Set $\mathfrak{m} = Lie(M)$. The boundary component associated to $\mathbb{Q}_{\geq 0}\langle N_1, ..., N_r\rangle \subset \mathfrak{m}$ is

$$B_\sigma := \tilde{B}_\sigma / e^{\langle \sigma \rangle_{\mathbb{C}}} \tag{14}$$

where

$$\tilde{B}_\sigma := \{F^\bullet \in \check{D} \mid Ad(e^\sigma).F^\bullet \text{ is a nilpotent orbit}\} \tag{15}$$

Kato-Usui define a compactification of D_M as follow,

$$D_{M,\sigma} := \amalg_{\sigma \in \Sigma} \ \{ \ Z \subset \check{D_M} \mid \ Z \ \text{ is a } \sigma - \text{nilpotent orbit}\} = \amalg_{\sigma \in \Sigma} \ B(\sigma)$$

This always contain $B(\{0\}) = D_M$. In particular $D_{M,\sigma} = D_{M,\text{faces of } \sigma}$. Let Σ be a maximal fan in $\mathfrak{m} := Lie(M)$. Consider the injective map $\beta = e : \log \gamma \mapsto \gamma$ restricted to $\Sigma = \mathbb{Z}\langle N_1, ..., N_r\rangle$. The map e is an isomorphism onto its image $\mathfrak{m}_{\mathbb{Z}}$ a maximal lattice in $\mathfrak{m}$. The triple $(D_M, e : \Sigma_{\mathbb{Z}} \to \mathfrak{m}_{\mathbb{Z}}, \Sigma)$ is a toric fantastack. On the lattices e is an isomorphism, mentioning that the generic stabilizer is trivial.

Theorem 3.3. *The triple* $(D_M, \mathrm{e} : \Sigma_{\mathbb{Z}} \to \mathfrak{m}_{\mathbb{Z}}, \Sigma)$ *is a smooth toric Deligne-Mumford stack.*

Proof. This follows from Theorem 1.2 and Theorem 7.1 in [R]. ☐

Theorem 3.4. *The stacky structure of* D_M *decomposes as*

$$D_{M_\Phi} = D_1 \times ... \times D_l \tag{16}$$

where

$$D \cong \sqrt[b_1]{L_1/X} \times ... \times \sqrt[b_l]{L_l/X} \cong \sqrt[d]{L/Z}$$

$$D_i \cong \sqrt[b_{1i}]{L_{1i}/X} \times ... \times \sqrt[b_{li}]{L_{li}/X} \cong \sqrt[d]{L_i/Z_i}$$

where $b_{ij} \in \mathbb{N}_{>0}$. *Moreover* D_{M_Φ} *is rigid.*

Proof. The decomposition in (16) follows from Theorems 2.2 and 2.3 and the uniqueness of splitting in (12) and the toric stacks structure theorems. It is rigid because the map $e : \Sigma_{\mathbb{Z}} \to \mathfrak{m}_{\mathbb{Z}}$ is an isomorphism onto its image. ☐

References

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