Basic Concepts: Stability, Controlabiity and Observability

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August 2020

Outline

Preliminaries and Stability

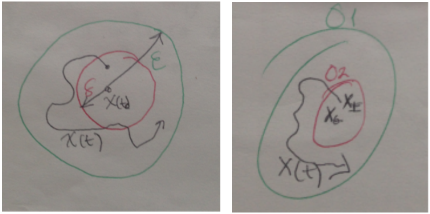
- Stability addresses properties of a vector field at a given point.
- Let X be a smooth manifold over which a vector field is defined.
- X might be the configuration or state space.
- The given point x_G can be interpreted in motion planning as the goal.
- Let $f(\cdot)$ be a velocity vector field, which yields a velocity vector $\dot{x} = f(x)$.
- For a dynamical system of the form $\dot{x} = f(x, u)$, the control u can be fixed by designing a feedback plan or feedback motion policy $\Pi : x \to u$.
- This design process yields $\dot{x} = f(x, \Pi(x))$. The process of designing a stable feedback plan is referred to in control literature as feedback stabilization.

Stability

- An equilibrium point x for $\dot{x} = f(x)$ is such that $f(\dot{x}) = 0$.
- Stability for x at the equilibrium point implies:

$$||x(t_0)|| < \delta \rightarrow ||x(t)|| < \epsilon \tag{1}$$

- An equilibrium point $x_G \in X$ is called Lyapunov stable if for any open set O_1 of x_G there exists another open neighborhood O_2 of x_G such that $x_I \in O_2$ implies that $x(t) \in O_1 \forall\ t > 0$.



(a) δ and ϵ

(b) Open sets O_1 and O_2

Figure: Stability

Stability

- Stability is weak in the sense that it does not imply that $x(t)$ converges to x_G when t tends to infinity. The state only lies around x_G.
- Corvengency requieres a stronger notion called asymptotic stability.
- A point x_G is an equilibrium point asymptotically stable of f if:
  1. It is an equilibrium point of f in the sense of Lyapunov.
  2. There exists an open set O around x_G such that for any point $x_I \in O$, $x(t)$ converges to x_G when t tends to infinity.
- For $X = \mathbb{R}^n$, the second condition can be expressed as follows:
- There exists $\delta > 0$ such that for any $x_I \in X$ with $||x_I - x_G|| < \delta$ the state $x(t)$ converges to x_G when t tends to infinity.
- Asymptotic stability does not imply nothing about how much time takes the convergency.
- If x_G is asymptotically stable and there exist some $m > 0$ and some $\alpha > 0$ such that

$$||x(t) - x_G|| \leq me^{-\alpha t}||x_I - x_G|| \tag{2}$$

  then x_G is called exponentially stable.

Summary Stability

Definition

Stability for x at the equilibrium point:

$$||x(t_0)|| < \delta \rightarrow ||x(t)|| < \epsilon \tag{3}$$

for $t > t_0$

Definition

Asymptotic stability: $||x(t)|| \rightarrow 0$ as $t \rightarrow \infty$
Define a function

$$V : X \rightarrow \mathbb{R} \tag{4}$$

such that $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$ for $x = 0$, $V(x)$ is called a candidate Lyapunov function.

Theorem

$x(0)$ is an equilibrium point if there exists a Lyapunov function V such that

$$\frac{dV}{dt} \leq 0 \tag{5}$$

For a proof of the above theorem, see [1, 2].

Example

Consider the following differential equation with solution x on $\mathbb{R}$

$$\dot{x} = -x. \tag{6}$$

Considering that x^2 is always positive around the origin it is a natural candidate to be a Lyapunov function to help us study x. So let $V(x) = x^2$ on $\mathbb{R}$. Then,

$$\dot{V}(x) = V'(x)\dot{x} = 2x \cdot (-x) = -2x^2 < 0. \tag{7}$$

This shows that the above differential equation, x, is asymptotically stable about the origin.

Controllability

Four versions of nonlinear controllability.

Let V be a neighborhood of a point $x \in \mathcal{M}$. Let $R^V(x, T)$ indicate the set of reachable points at time T by trajectories remaining inside V and satisfying $\dot{x} = f(x, u)$ where $u \in \Omega$, a subset of $\mathbb{R}^l$, $x \in \mathcal{M}$, a C^∞ connected manifold of dimension m, and let

$$R^V(x, \leq T) = \bigcup_{0<t<T} R^V(x, t) \tag{8}$$

Definition

The system is **controllable** from x if, for any $x_{goal} \in \mathcal{M}$, there exists a $T > 0$ such that $x_{goal} \in R^{\mathcal{M}}(x, \leq T)$

Definition

The system is **accessible** from x if $R^{\mathcal{M}}(x, \leq T)$ contains a full-dimensional subset of $\mathcal{M}$ for some $T > 0$.

Controllability

Definition

The system is **small-time locally accessible** (STLA) from x if $R^V(x, \leq T)$ contains a full-dimensional subset of $\mathcal{M}$ $\forall$ neighborhoods V and $\forall$ $T > 0$.

Definition

The system is **small-time locally controllable** (STLC) from x if $R^V(x, \leq T)$ contains a neighborhood of x $\forall$ neighborhoods V and $\forall$ $T > 0$.

The phrase "small-time" indicates that the property holds for any time $T > 0$, and the phrase "locally" indicates that the property holds for arbitrarily small (but full-dimensional) "room for maneuver" around the initial state.

Controllability

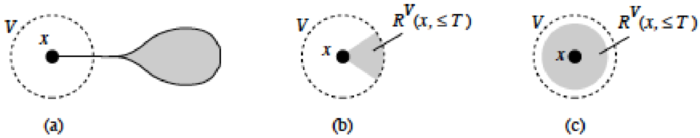


Figure: accesible, STLA, STLC, image taken from [4]

Reachable spaces for three systems on $\mathbb{R}^2$. (a) This system is accessible from x, but neither small-time locally accessible nor small-time locally controllable. The reachable set is two-dimensional, but not while confined to the neighborhood V. (b) This system is small-time locally accessible from x, but not small-time locally controllable. The reachable set without leaving V does not contain a neighborhood of x. (c) This system is small-time locally controllable from x.

Observability

Considering a system Σ,

$$\begin{aligned} \dot{\mathbf{x}} = & \quad f(\mathbf{x}, u), \\ \mathbf{y} = & \quad g(\mathbf{x}), \end{aligned} \tag{9}$$

where $u \in \Omega$, a subset of $\mathbb{R}^l$, $\mathbf{x} \in M$, a C^∞ connected manifold of dimension m, $\mathbf{y} \in \mathbb{R}^p$, f and g are C^∞ functions, and assume the trajectories of Σ to satisfy the initial condition $\mathbf{x}(t^0) = \mathbf{x}^0$. The input-output map of the pair $(\Sigma, \mathbf{x}^0)$, the indistinguishable property between states $\mathbf{x}^0$ and $\mathbf{x}^1$, and the observability of a system Σ, are defined as follows.

Definition

Let $(u(t), [t^0, t^1])$ be the admissible input that gives rise to a solution $(\mathbf{x}(t), [t^0, t^1])$ of $\dot{\mathbf{x}} = f(\mathbf{x}, u(t))$ satisfying the initial condition. This, in turn, defines an output $(\mathbf{y}(t), [t^0, t^1])$ by $\mathbf{y}(t) = g(\mathbf{x}(t))$. Then, the *input-output map of* Σ *at* $\mathbf{x}^0$ is denoted by

$$\Sigma_{\mathbf{x}^0} : (u(t), [t^0, t^1]) \mapsto (\mathbf{y}(t), [t^0, t^1]). \tag{10}$$

Observability

Definition

A pair of states $\mathbf{x}^0$ and $\mathbf{x}^1$ are *indistinguishable*, denoted $\mathbf{x}^0 I \mathbf{x}^1$, if $(\Sigma, \mathbf{x}^0)$ and $(\Sigma, \mathbf{x}^1)$ realize the same input-output map, i.e., for every admissible input $(u(t), [t^0, t^1])$.

$$\Sigma_{\mathbf{x}^0} : (u(t), [t^0, t^1]) = \Sigma_{\mathbf{x}^1} : (u(t), [t^0, t^1]). \tag{11}$$

Definition

Σ is said to *be observable at* $\mathbf{x}^0$ if $I(\mathbf{x}^0) = \{\mathbf{x}^0\}$ and Σ is *observable*, if $I(\mathbf{x}) = \{\mathbf{x}\}$ for every $\mathbf{x} \in M$. $I(\cdot)$ returns the set of indistinguishable elements of the argument.

A note

Because of its nice properties, many times, it is assumed that the state transition equation $\dot{x} = f(x, u)$ of the dynamical system is a Lipschitz function.

Definition

A state transition equation is called Lipschitz continuous, if there exists a positive real constant c such that, for all states x and x',

$$||f(x, u) - f(x', u)|| \leq c||x - x'|| \tag{12}$$

This still is a broad class of systems.

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