Bayes Filter

Rafael Murrieta-Cid

Centro de Investigación en Matemáticas (CIMAT)

murrieta@cimat.mx

August 2020

Outline

1. Preliminaries
2. Bayes Filter

Preliminaries

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)} = \frac{P(X,Y)}{P(Y)} \quad (1)$$

$$\frac{P(X|Y)}{P(X)} = \frac{P(Y|X)}{P(Y)} \quad (2)$$

Example

$$P(Y|X) = \frac{P(X \cap Y)}{P(X)} = \frac{1/4}{1/2} = \frac{1}{2} \quad (3)$$

$$P(X|Y) = \frac{P(X \cap Y)}{P(Y)} = \frac{1/4}{3/4} = \frac{1}{3} \quad (4)$$

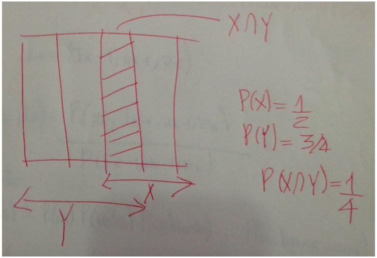

Figure: Geometric interpretation

Bayes Filter

- The goal is to start from the conditional probability equation below.

$$P(x_k|I_k) = \frac{P(x_k, I_k)}{P(I_k)} \tag{5}$$

- and from Equation (5) deduce the Bayes Filter given by

$$P(x_k|I_k) = \frac{\sum_{x_{k-1}} P(x_{k-1}|I_{k-1})P(x_k|x_{k-1}, u_{k-1})P(y_k|x_k)}{\sum_{x_k} \sum_{x_{k-1}} P(x_{k-1}|I_{k-1})P(x_k|x_{k-1}, u_{k-1})P(y_k|x_k)} \tag{6}$$

where x is the state, y the observation, u the control, k denotes the discret time, I_k is the information vector, encoding the observations and actions, $P(x_k|x_{k-1}, u_{k-1})$ corresponds to a motion transition model, and $P(y_k|x_k)$ corresponds to an observation model.

$$I_k = (I_{k-1}, u_{k-1}, y_k) \tag{7}$$

The information vector at time k depends on the information vector at time $k-1$, the control at time $k-1$ and the observation at time k.

Bayes Filter

$$P(x_k|I_k) = \frac{P(x_k, I_{k-1}, u_{k-1}, y_k)}{P(I_{k-1}, u_{k-1}, y_k)} \tag{8}$$

The product probability rule states
$P(a_1, a_2, a_3, \ldots, a_n) = P(a_1)P(a_2|a_1)P(a_3|a_1, a_2)\ldots P(a_n|a_1, a_2, a_3, \ldots, a_{n-1})$
From the product probability rule

$$P(x_k|I_k) = \frac{\cancel{P(I_{k-1})}\,\cancel{P(u_{k-1}|I_{k-1})}P(x_{k-1}|I_{k-1}, u_{k-1})P(y_k|I_{k-1}, u_{k-1}, x_k)}{\cancel{P(I_{k-1})}\,\cancel{P(u_{k-1}|I_{k-1})}P(y_k|I_{k-1}, u_{k-1})}$$

$$P(x_k|I_k) = \frac{P(x_{k-1}|I_{k-1}, u_{k-1})P(y_k|I_{k-1}, u_{k-1}, x_k)}{P(y_k|I_{k-1}, u_{k-1})} \tag{9}$$

If y_k only depends on x_k then $y_k \perp I_{k-1}, u_{k-1}|x_k$, hence $P(y_k|I_{k-1}, u_{k-1}, x_k) = P(y_k|x_k)$

$$P(x_k|I_k) = \frac{P(x_{k-1}|I_{k-1}, u_{k-1})P(y_k|x_k)}{P(y_k|I_{k-1}, u_{k-1})} \tag{10}$$

The Bayes rule states

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{11}$$

From the Bayes rule, working on the denominator of Equation (10)

Bayes Filter

One gets

$$P(y_k|I_{k-1},u_{k-1}) = \frac{P(I_{k-1},u_{k-1}|y_k)P(y_k)}{P(I_{k-1},u_{k-1})} \tag{12}$$

Using the rule of conditional probability $P(A|B) = \frac{P(A,B)}{P(B)}$ for substituting $P(I_{k-1},u_{k-1}|y_k)$, one gets

$$P(y_k|I_{k-1},u_{k-1}) = \frac{P(I_{k-1},u_{k-1},y_k)\cancel{P(y_k)}}{\cancel{P(y_k)}P(I_{k-1},u_{k-1})}$$

$$P(y_k|I_{k-1},u_{k-1}) = \frac{P(I_{k-1},u_{k-1},y_k)}{P(I_{k-1},u_{k-1})} \tag{13}$$

The marginalization rule states

$$P(b) = \sum_a P(b,a) \tag{14}$$

Marginalizing with respect to x_k

$$P(y_k|I_{k-1},u_{k-1}) = \frac{\sum_{x_k} P(I_{k-1},u_{k-1},y_k,x_k)}{P(I_{k-1},u_{k-1})} \tag{15}$$

Applying the product probability rule

$$P(y_k|I_{k-1},u_{k-1}) = \frac{\sum_{x_k} \cancel{P(u_{k-1})}\,\cancel{P(I_{k-1}|u_{k-1})}P(x_k|I_{k-1},u_{k-1})P(y_k|x_k,I_{k-1},u_{k-1})}{\cancel{P(u_{k-1})}\,\cancel{P(I_{k-1}|u_{k-1})}}$$

Bayes Filter

$$P(y_k|I_{k-1}, u_{k-1}) = \sum_{x_k} P(x_k|I_{k-1}, u_{k-1}) P(y_k|x_k, I_{k-1}, u_{k-1}) \tag{16}$$

since $y_k \perp I_{k-1}, u_{k-1}$ then $P(y_k|x_k, I_{k-1}, u_{k-1}) = P(y_k|x_k)$

$$P(y_k|I_{k-1}, u_{k-1}) = \sum_{x_k} P(x_k|I_{k-1}, u_{k-1}) P(y_k|x_k) \tag{17}$$

Substituting Equation (17) in Equation (10) one gets

$$P(x_k|I_k) = \frac{P(x_k|I_{k-1}, u_{k-1}) P(y_k|x_k)}{\sum_{x_k} P(x_k|I_{k-1}, u_{k-1}) P(y_k|x_k)} \tag{18}$$

Considering the term $P(x_k|I_{k-1}, u_{k-1})$ and applying the Bayes rule $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$ one gets

$$P(x_k|I_{k-1}, u_{k-1}) = \frac{P(I_{k-1}, u_{k-1}|x_k) P(x_k)}{P(I_{k-1}, u_{k-1})} \tag{19}$$

Bayes Filter

Using the rule of conditional probability $P(A|B) = \frac{P(A,B)}{P(B)}$

$$P(x_k|I_{k-1}, u_{k-1}) = \frac{P(I_{k-1}, u_{k-1}, x_k)\cancel{P(x_k)}}{\cancel{P(x_k)}P(I_{k-1}, u_{k-1})}$$

$$P(x_k|I_{k-1}, u_{k-1}) = \frac{P(I_{k-1}, u_{k-1}, x_k)}{P(I_{k-1}, u_{k-1})} \tag{20}$$

Marginalizing with respect to x_{k-1}

$$P(x_k|I_{k-1}, u_{k-1}) = \frac{\sum_{x_{k-1}} P(x_k, I_{k-1}, u_{k-1}, x_{k-1})}{P(I_{k-1}, u_{k-1})} \tag{21}$$

Applying the conjoint probability rule as a product of conditional probabilities.

$$P(x_k|I_{k-1}, u_{k-1}) = \frac{\sum_{x_{k-1}} \cancel{P(u_{k-1})}\,\cancel{P(I_{k-1}|u_{k-1})}P(x_{k-1}|I_{k-1}, u_{k-1})P(x_k|x_{k-1}, I_{k-1}, u_{k-1})}{\cancel{P(u_{k-1})}\,\cancel{P(I_{k-1}|u_{k-1})}}$$

Bayes Filter

One gets

$$P(x_k|I_{k-1},u_{k-1}) = \sum_{x_{k-1}} P(x_{k-1}|I_{k-1},u_{k-1})P(x_k|x_{k-1},I_{k-1},u_{k-1}) \quad (22)$$

Since $x_{k-1} \perp u_{k-1}|I_{k-1}$ then $P(x_{k-1}|I_{k-1},u_{k-1}) = P(x_{k-1}|I_{k-1})$
Also $x_k \perp I_{k-1}|x_{k-1},u_{k-1}$ then $P(x_k|x_{k-1},I_{k-1},u_{k-1}) = P(x_k|x_{k-1},u_{k-1})$
Hence

$$P(x_k|I_{k-1},u_{k-1}) = \sum_{x_{k-1}} P(x_{k-1}|I_{k-1})P(x_k|x_{k-1},u_{k-1}) \quad (23)$$

Substituting Equation 23 in Equation 18 one gets the Bayes filter given by:

$$P(x_k|I_k) = \frac{\sum_{x_{k-1}} P(x_{k-1}|I_{k-1})P(x_k|x_{k-1},u_{k-1})P(y_k|x_k)}{\sum_{x_k}\sum_{x_{k-1}} P(x_{k-1}|I_{k-1})P(x_k|x_{k-1},u_{k-1})P(y_k|x_k)} \quad (24)$$

LaValle-06. S. M. LaValle *Planning Algorithms*, Chapter 13, Cambridge University Press, 2006.

Thrun et al.-05. S. Thrun, W. Burgard and D. Fox, *Probabilistic Robotics*, MIT press, 2005.