Modelling an Optimal Control Problem and Application of the Pontryagin Maximum Principle

Rafael Murrieta-Cid

Centro de Investigación en Matemáticas (CIMAT)

April 2020

Outline

Modelling an Optimal Control Problem

Modelling

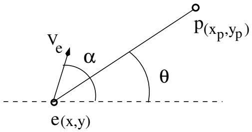

Figure: Evader and pursuer.

$$
\begin{aligned}
x_p &= x + L\cos\theta, \\
y_p &= y + L\sin\theta.
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
\dot{x} &= V_e\cos\alpha, \\
\dot{y} &= V_e\sin\alpha.
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
\dot{x}_p &= \dot{x} - L\sin\theta\dot{\theta}, \\
\dot{y}_p &= \dot{y} + L\cos\theta\dot{\theta}.
\end{aligned} \tag{3}
$$

Substituting Equation 2 in Equation 3

$$
\begin{aligned}
\dot{x}_p &= V_e\cos\alpha - L\sin\theta\dot{\theta}, \\
\dot{y}_p &= V_e\sin\alpha + L\cos\theta\dot{\theta}.
\end{aligned} \tag{4}
$$

Norm of V_p

Obtaining the norm L_2 of the pursuer's velocity vector.

$$V_P^2 = \dot{x}_p^2 + \dot{y}_p^2 \tag{5}$$

$$V_P^2 = V_e^2 + L^2\dot{\theta}^2 + 2LV_e\dot{\theta}(\sin\alpha\cos\theta - \cos\alpha\sin\theta) \tag{6}$$

$$V_P^2 = V_e^2 + L^2\dot{\theta}^2 + 2LV_e\dot{\theta}\sin(\alpha - \theta) \tag{7}$$

$$L^2\dot{\theta}^2 + 2LV_e\sin(\alpha - \theta)\dot{\theta} + V_e^2 - V_P^2 = 0 \tag{8}$$

Solving for $\dot{\theta}$

$$L^2\dot{\theta}^2 + 2LV_e \sin(\alpha - \theta)\dot{\theta} + V_e^2 - V_P^2 = 0 \tag{9}$$

$$\dot{\theta} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{10}$$

$$a = L^2,\ b = 2LV_e \sin(\alpha - \theta),\ c = V_e^2 - V_P^2,\ \rho = \frac{V_e}{V_p}. \tag{11}$$

$$\dot{\theta} = \frac{-2LV_e sin(\alpha - \theta) \pm \sqrt{(2LV_e sin(\alpha - \theta))^2 - 4L^2(V_e^2 - V_P^2)}}{2L^2} \tag{12}$$

After manipulation

$$\dot{\theta} = \frac{V_p}{L}\left(-\rho \sin(\alpha - \theta) \pm \sqrt{1 - \rho^2 \cos^2(\alpha - \theta)} \right) \tag{13}$$

The Optimal Control Problem

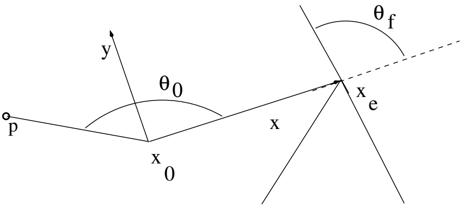

Figure: Reference frame defined with respecto to the evader initial position.

- In formulating the optimal control problem, it is convenient to let x be the independent variable (instead of time t) and to parameterize the control α by x, since the evader will move from the point $x = 0$ to the point $x' = x_e$.
- This leads to a modified system description in which the state of the rod is given by $\zeta = (y, \theta)$, and the system equation is given by:

$$\frac{d\zeta}{dx} = f(\zeta, \alpha) \tag{14}$$

The State Transition Equation

$$\frac{dx}{dt} = V_e \cos\alpha \tag{15}$$

$$\frac{dy}{dt} = V_e \sin\alpha \tag{16}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1} = \tan\alpha = f_1(\zeta,\alpha) \tag{17}$$

$$\frac{d\theta}{dx} = \frac{d\theta}{dt}\left(\frac{dx}{dt}\right)^{-1} = \frac{-\rho\sin(\alpha-\theta) \pm \sqrt{1-\rho^2\cos^2(\alpha-\theta)}}{L\rho\cos\alpha} = f_2(\zeta,\alpha) \tag{18}$$

The Cost Function

$$J = \int_{x_0}^{x_e} \frac{d\theta}{dx} dx = \int_{x_0}^{x_e} f_2(\zeta, \alpha) dx \tag{19}$$

Application of the Pontryagin Maximum Principle

Pontryagin Maximum Principle

Hamiltonian

$$H(x, u, \lambda, t) = L(x, u, t) + \langle \lambda, \dot{x} \rangle \tag{20}$$

If (x^*, u^*) is an optimal trajectory-control pair, there exists λ^* such that

$$\dot{\lambda}^* = -\frac{\partial H}{\partial x} \tag{21}$$

The optimal control is given by

$$u^* = \underset{u}{\operatorname{argmin}}\, H(x^*, u, \lambda^*, t) \tag{22}$$

Application of the Pontryagin Maximum Principle

Hamiltonian

$$H(x,u,\lambda,t) = \frac{d\theta}{dx} + \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} f_1(\zeta,\alpha) \\ f_2(\zeta,\alpha) \end{pmatrix} \tag{23}$$

$$H(x,u,\lambda,t) = f_2(\zeta,\alpha) + \lambda_1 f_1(\zeta,\alpha) + \lambda_2 f_2(\zeta,\alpha) \tag{24}$$

$$H(x,u,\lambda,t) = \lambda_1 f_1(\zeta,\alpha) + (1+\lambda_2) f_2(\zeta,\alpha) \tag{25}$$

$$H(x,u,\lambda,t) = \lambda_1 \tan\alpha + (1+\lambda_2)\frac{-\rho\sin(\alpha-\theta) \pm \sqrt{1-\rho^2\cos^2(\alpha-\theta)}}{L\rho\cos\alpha} \tag{26}$$

Application of the Pontryagin Maximum Principle

Adjoint Equation

$$\dot{\lambda}^* = -\frac{\partial H}{\partial x} \tag{27}$$

$$\begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \begin{pmatrix} -\frac{\partial H}{\partial y} \\ \\ -\frac{\partial H}{\partial \theta} \end{pmatrix} \tag{28}$$

$$\begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -(1+\lambda_2)\frac{\cos(\alpha-\theta)}{L\cos\alpha}\left(1 - \frac{\rho\sin(\alpha-\theta)}{\sqrt{1-\rho^2\cos^2(\alpha-\theta)}}\right) \end{pmatrix} \tag{29}$$

Application of the Pontryagin Maximum Principle

Therefore, λ_1 is constant and the Hamiltonian can be written as:

$$H(x, u, \lambda, t) = K \tan\alpha + (1 + \lambda_2)\frac{-\rho \sin(\alpha - \theta) \pm \sqrt{1 - \rho^2 \cos^2(\alpha - \theta)}}{L\rho \cos\alpha} \tag{30}$$

We seek the control input α that minimises J, subject to the system equation $\dot{\zeta} = f(\zeta, \alpha)$, and subject to the boundary conditions $\theta(0) = \theta, y(0) = y(x_e) = 0$
The optimal control is given by

$$u^* = \operatorname*{argmin}_u H(x^*, u, \lambda^*, t) \tag{31}$$

Shooting Algorithm

Shooting Algorithm

Algorithm OptimalEvaderControl(θ,x_e)

1. Choose initial values for the Lagrange multipliers λ_i
2. Let $j \leftarrow 0$, $\theta(0) \leftarrow \theta$, $y(0) \leftarrow 0$, $x(0) \leftarrow 0$.
3. Choose $\alpha(j)$ that minimizes the Hamiltonian,
   $\alpha^*(j) = \text{argmin}_\alpha H(\zeta^*(j), \alpha, \lambda^*(j), x(j))$
4. Integrate the state equations to determine $\zeta(j+1)$
   $y(j+1) \leftarrow y(j) + f_1(\zeta(j), \alpha(j))\Delta x$
   $\theta(j+1) \leftarrow \theta(j) + f_2(\zeta(j), \alpha(j))\Delta x$
5. Integrate the adjoint equation for λ_2 to determine $\lambda_2(j+1)$
6. $x(j+1) \leftarrow x(j) + \Delta x$
7. If $y(j)$ and $y(j+1)$ have different sign, then the system has crossed the x axis
   If $|x(j+1) - x_e| < \epsilon$ then the optimal trajectory is given by $\alpha(0), \ldots, \alpha(j)$
8. If $y(j)$ and $y(j+1)$ have different sign, but $|x(j+1) - x(j)| > \epsilon$ then we have missed the boundary condition.
   In this case, adjust the initial values for λ and go to step 2.
9. If $y(j)$ and $y(j+1)$ have the same sign, then we have not crossed the x-axis, and we continue to iterate forward: $j \leftarrow j+1$, go to step 3.

Thanks... Questions?

murrieta@cimat.mx

- L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *The Mathematical Theory of Optimal Processes.* JohnWiley, 1962.
- S. M. LaValle, *Planning algorithms*, Cambridge, 2006.
- R. Murrieta-Cid, T. Muppirala, A. Sarmiento, S. Bhattacharya and S. Hutchinson, Surveillances Strategies for a Pursuer with Finite Sensor Range, *Int. J. of Robotics Research*, Vol. 26, No. 3, pages 233-253, 2007.