Hamilton-Jacobi-Bellman Equation and Pontryagin Maximum Principle

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Outline

Hamilton-Jacobi-Bellman Equation

Formulation

$$x(t_0) = x_0 \tag{1}$$

$$\dot{x} = f(x, u, t) \tag{2}$$

$$J(t_0, x_0, u(\cdot)) = \int_{t_0}^{t_1} L(x, u, \tau) d\tau + K(x(t_1)) \tag{3}$$

$$V(t_0, x_0) = \inf_{u(\cdot)} J(t_0, x_0, u(\cdot)) \tag{4}$$

Principle of Optimality

$$V(x_0, t_0) = \inf_{u|[t_0, t_0+\Delta t]} \int_{t_0}^{t_0+\Delta t} L(x, u, \tau) d\tau + V(t_o + \Delta t, x_0 + \Delta x) \quad (5)$$

Approximations

$$\int_{t_0}^{t_0+\Delta t} L(x,u,\tau)d\tau \approx L(x_0,u(t_0),t_0)\Delta t \tag{6}$$

$$V(t_o+\Delta t, x_0+\Delta x) \approx V(t_0,x_0) + (\langle \frac{\partial V}{\partial x}, \Delta x\rangle + \frac{\partial V}{\partial t}\Delta t)|_{(t_0,x_0)} \tag{7}$$

Substitution

$$V(t_0, x_0) \approx V(t_0, x_0) + \inf_{u|_{[t_0, t_0+\Delta t]}} (L(x_0, u(t_0), t_0)\Delta t + \langle \frac{\partial V}{\partial x}, \Delta x \rangle + \frac{\partial V}{\partial t} \Delta t) \tag{8}$$

$$0 \approx + \inf_{u|_{[t_0, t_0+\Delta t]}} (L(x_0, u(t_0), t_0)\Delta t + \langle \frac{\partial V}{\partial x}, \Delta x \rangle + \frac{\partial V}{\partial t} \Delta t) \tag{9}$$

HJB

Divide by Δt, and let $\Delta t \to 0$

$$0 = \inf_{u(t_0)} (L(x_0, u(t_0), t_0) + \langle \frac{\partial V}{\partial x}, \dot{x} \rangle + \frac{\partial V}{\partial t}) \tag{10}$$

Rearrange to get HJB:

$$-\frac{\partial V}{\partial t}|_{(t_0, x_0)} = \inf_{u(t_0)} (L(x_0, u(t_0), t_0) + \langle \frac{\partial V}{\partial x}, \dot{x} \rangle|_{(t_0, x_0)}) \tag{11}$$

In short:

$$-\frac{\partial V}{\partial t} = \inf_{u(t)} (L(x_0, u(t_0), t_0) + \langle \frac{\partial V}{\partial x}, \dot{x} \rangle) \tag{12}$$

Pontryagin Maximum Principle

PMP

- The maximum principle can be considered a specialization of the HJB equation, which corresponds to the application of the optimal action $u^*(t)$. This causes the inf to disappear, but along with it, the global properties of the HJB equation also vanish.
- PMP expresses conditions along the optimal trajectory, as opposed to the value function $V(x(t))$ over the whole state space. Therefore, it can at best assure local optimality in the space of possible trajectories.
- In the PMP methodology, the optimal control is function of $\lambda(t) = \nabla V(x(t))$. It is important to note that at moment $u^*(t)$ is chosen, the relation with the state $x(t)$ is lost. That is the reason one denotes $\lambda(t)$ and not $\lambda(x(t))$.
- Later, the optimal motion trajectory of the system is constructed using $u^*(t)$. Therefore, the resulting optimal trajectories are not directly related to the state. However, it is possible to find this relation using a partition of the state space.

Hamiltonian

Let

$$H(x, u, \lambda, t) = L(x, u, t) + \langle \lambda, \dot{x} \rangle \tag{13}$$

$$\mathcal{H}(x, u, t) = H(x, u, \frac{\partial V}{\partial x}, t) \tag{14}$$

In that case, HJB is

$$-\frac{\partial V}{\partial t} = \inf_{u} \mathcal{H}(x, u, t) \tag{15}$$

Adjoint Equation

Suppose $(x^*(t), u^*(t))$ is an optimal trajectory-control.
Define

$$\lambda^*(t) = \frac{\partial V}{\partial x}(t, x^*(t)) \tag{16}$$

$$\dot{\lambda}^*(t) = \frac{d}{dt}\frac{\partial V}{\partial x}(t, x^*(t)) \tag{17}$$

$$\dot{\lambda}^*(t) = \frac{\partial^2 V}{\partial x \partial t}(t, x^*(t)) + \frac{\partial^2 V}{\partial x^2}(t, x^*(t))\dot{x}^* \tag{18}$$

Calculations 1

Recall

$$-\frac{\partial V}{\partial t}(t, x^*) = \mathcal{H}(x^*, u^*, t) \tag{19}$$

$$-\frac{\partial^2 V}{\partial x \partial t}(t, x^*(t)) = \frac{\partial \mathcal{H}}{\partial x}|_{(x^*(t), u^*(t), t)} \tag{20}$$

$$\frac{\partial^2 V}{\partial x \partial t} = -\frac{\partial L}{\partial x} - \frac{\partial^2 V}{\partial^2 x}\dot{x}^* - \frac{\partial f^T}{\partial x}\frac{\partial V}{\partial x} \tag{21}$$

Calculations 2

Substitute Equation 21 in Equation 18 to get

$$\dot{\lambda}^* = -\frac{\partial L}{\partial x} - \frac{\partial f^T}{\partial x}\frac{\partial V}{\partial x} \tag{22}$$

$$\dot{\lambda}^* = -\frac{\partial L}{\partial x} - \frac{\partial f^T}{\partial x}\lambda^* \tag{23}$$

$$\dot{\lambda}^* = -\frac{\partial H}{\partial x}|_{(x^*(t),u^*(t),\lambda^*(t),t)} \tag{24}$$

Pontryagin Maximum Principle

If (x^*, u^*) is an optimal trajectory-control pair, there exists λ^* such that

$$\dot{\lambda}^* = -\frac{\partial H}{\partial x} \tag{25}$$

and

$$H(x^*, u^*, \lambda^*, t) = \min_u H(x^*, u.\lambda^*, t) \tag{26}$$

by definition of H

$$\dot{x}^* = \frac{\partial H}{\partial \lambda} \tag{27}$$

PMP Method

1. Construct the Hamiltonian of the system.
2. Obtain the expression for the optimal control u^* satisfying it. The optimal control is a function of $\nabla V(x)$.
3. Find $\nabla V(x)$ solving the adjoint equation. Use transversality condition.
4. Use the computed control u^* in the integration of the motion equations to find the trajectory followed by the system.

Thanks... Questions?

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