Integrability, Controlability and Nonholonomic Property

Rafael Murrieta-Cid

Centro de Investigación en Matemáticas (CIMAT)

murrieta@cimat.mx

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Outline

Vector Fields and Distributions

- Let us assume that the state transition's equation $\dot{x} = f(x, u)$ has the form:

$$\dot{x} = \alpha^1(x)u_1 + \alpha^2(x)u_2 + \cdots + \alpha^m(x)u_m \tag{1}$$

- In that case the state transition's equation can be expressed as:

$$\dot{x} = A(x)u \tag{2}$$

- Let us assume that matrix $A(x)$ is not singular.

Definition

A vector field over a manifold, X, is a function that associates to each element $x \in X$ a vector $\vec{V}(x)$

- An example of a vector field is the velocity field.
- Each vector $\vec{V}(x)$ represents the infinitesimal change of the state with respect to time.

$$\dot{x} = \left[\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}\right] \tag{3}$$

evaluated at the point $x \in X$

Vector Fields and Distributions

- Note that for a fixed control u the state transition's equation $\dot{x} = f(x, u)$ defines a vector field, since $\dot{x}$ is expressed as a function of x.
- Let us assume that the state transition's equation is given by a state space X and a set of inputs (space of controls) $U = \mathbb{R}^m$.
- What we want is to define all the vector fields that can be generated using the available controls (inputs).

Definition

The set of all the vector fields that can be generated using the inputs $u \in U$ is called the distribution, it is denoted as $\Delta(x)$.

$$\Delta(x) = \text{span}\{\alpha^1(x), \alpha^2(x), \ldots, \alpha^m(x)\} \tag{4}$$

Vector Fields and Distributions

- A distribution can be considered as vector space.
- α^i can be interpreted as a vector field.
- Given inputs of the form [0,. . . ,0,1,0,. . . ,0], if $u_i = 1$ and $u_j = 0$ then the state transition's equation yields $\dot{x} = \alpha^i(x)$.
- Each of the inputs of this form can generate a base of the vector field.
- The dimension of the distribution is the number of vectors in its base. That is, the maximun number of linear independent vector fields that can be generated.

Example of the DDR

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{5}$$

- The input $u = [1, 0]$ yields the vector field $\vec{V} = [\cos\theta, \sin\theta, 0]$.
- Using the input $u = [0, 1]$ the vector field $\vec{W} = [0, 0, 1]$ is obtained.
- Any other vector field can be generated using a linear combination of $\vec{V}$ and $\vec{W}$.
- In this case the distribution $\Delta(x)$ has dimension 2, it can be expressed as

$$\Delta(x) = \text{span}\{\vec{V}, \vec{W}\} \tag{6}$$

Lie Bracket

- This operation is denoted by $\left[\vec{V}, \vec{W}\right]$ and it is called Lie bracket.
- The calculation of the Lie bracket is given by:

$$\left[\vec{V}, \vec{W}\right] = D\vec{W} \cdot \vec{V} - D\vec{V} \cdot \vec{W} \tag{7}$$

where

$$D\vec{V} = \begin{pmatrix} \frac{\partial V_1}{\partial x_1} & \frac{\partial V_1}{\partial x_2} & \cdots & \frac{\partial V_1}{\partial x_n} \\ \frac{\partial V_2}{\partial x_1} & \frac{\partial V_2}{\partial x_2} & \cdots & \frac{\partial V_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial V_n}{\partial x_1} & \frac{\partial V_n}{\partial x_2} & \cdots & \frac{\partial V_n}{\partial x_n} \end{pmatrix} \tag{8}$$

and

$$D\vec{W} = \begin{pmatrix} \frac{\partial W_1}{\partial x_1} & \frac{\partial W_1}{\partial x_2} & \cdots & \frac{\partial W_1}{\partial x_n} \\ \frac{\partial W_2}{\partial x_1} & \frac{\partial W_2}{\partial x_2} & \cdots & \frac{\partial W_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial W_n}{\partial x_1} & \frac{\partial W_n}{\partial x_2} & \cdots & \frac{\partial W_n}{\partial x_n} \end{pmatrix} \tag{9}$$

Lie Bracket

- The i-th element of the Lie bracket is given by:

$$\left[\vec{V}, \vec{W}\right]_i = \sum_{j=1}^{n} \left(V_j \frac{\partial W_i}{\partial x_j} - W_j \frac{\partial V_i}{\partial x_j} \right) \tag{10}$$

- Three useful properties are 1) symmetry

$$\left[\vec{V}, \vec{W}\right] = -\left[\vec{W}, \vec{V}\right] \tag{11}$$

- 2) Jacobi identity

$$\left[\left[\vec{V}, \vec{W}\right], \vec{U}\right] + \left[\left[\vec{W}, \vec{U}\right], \vec{V}\right] + \left[\left[\vec{U}, \vec{V}\right], \vec{W}\right] = 0 \tag{12}$$

- 3) bilinearity, for any $a, b \in \mathbb{R}$ and any $\vec{U}, \vec{V}, \vec{W}$.

$$\begin{aligned} \left[a\vec{U} + b\vec{V}, \vec{W}\right] &= a\left[\vec{U}, \vec{W}\right] + b\left[\vec{V}, \vec{W}\right] \\ \left[\vec{U}, a\vec{V} + b\vec{W}\right] &= a\left[\vec{U}, \vec{W}\right] + b\left[\vec{U}, \vec{W}\right] \end{aligned} \tag{13}$$

Lie Bracket

- The Lie bracket allows one to generate vector fields that are not originally in $\Delta(x)$.
- That is, the Lie bracket allows to find velocities that originally are not directly allowed by the state transition equation.
- For instance, in the DDR or the car-like robot the Lie bracket generates a vector field that moves the robot in direction perpendicular to its heading.
- This is done, generating combinations of the vector fields $\vec{V}$ and $\vec{W}$.

Control Lie algebra *CLA*

- *CLA* denotes the control Lie algebra. For a state transition equation of the form
$$\dot{x} = \alpha^1(x)u_1 + \alpha^2(x)u_2 + \cdots + \alpha^m(x)u_m \tag{14}$$
First, consider the set of all vector fields that can be generated by taking Lie brackets $[\alpha^i(x), \alpha^j(x)]$ of vector fileds $\alpha^i(x)$ and $\alpha^j(x)$, for $i \neq j$.
- Next, consider taking Lie brackets of the new vector fields with each other, and with the original vector fields (including nested Lie bracket operations).
- Thus, describing a control Lie algebra requires characterizing all vectors that are obtained under the algebraic closure of the bracket operation.

Integrability and Frobenius Theorem

- In some cases the state transitions equation can be integrable, if so it can be expressed in terms of x and u without using $\dot{x}$.
- But there are cases in which is not possible to integrate $\dot{x} = f(x, u)$ analytically.
- The Frobenius theorem allows one to determine whether or not $\dot{x} = f(x, u)$ is integrable.

Theorem

Frobenius theorem: The state transition equation is integrable if and only if all the vector fields that can be obtained by the Lie bracket operation are contained in $\Delta(x)$. In terms of the dimension, the state transition equation $\dot{x} = f(x, u)$ is integrable if $dim(CLA(\Delta(x)) = dim(\Delta(x))$

- For a proof of the above theorem, see [1, 2]
- A system whose state transition equation is not integrable is called a **nonholonomic** system.

Controllability and Chow Theorem

Definition

The system is locally controllable from x, if the set of reachable points from x, for an admisible trajectory, contains a neighborhood of x.

Definition

The system is small-time locally controllable from x, if the set of reachable points from x, for an admisible trajectory before a given time T, contains a neighborhood of x, for any T.

Theorem

Chow Theorem: A symmetric system without drift is small-time locally controllable from x, if and only if the dimension of the control Lie algebra is equal to the dimension n of the state space X, that is, $dim(CLA(\Delta(x)) = n$.

For a proof of the above theorem see [2].

Controllability

- Small time local controllability (STLC) implies local controllability, converse is not true.
- A Dubins car can only move forward.
- The Dubins car is controllable but it is not STLC.
- An airplane is like a Dubins car but in 3D.

Example

A car-like robot or a DDR.

$$
\begin{aligned}
V &= [\cos\theta, \sin\theta, 0] \\
W &= [0, 0, 1]
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
Z_1 &= \frac{V_1 \partial W_1}{\partial x} - \frac{W_1 \partial V_1}{\partial x} + \frac{V_2 \partial W_1}{\partial y} - \frac{W_2 \partial V_1}{\partial y} + \frac{V_3 \partial W_1}{\partial \theta} - \frac{W_3 \partial V_1}{\partial \theta} \\
Z_1 &= \cos\theta 0 - 0 + \sin\theta 0 - 0 + 0 - 1(-\sin\theta) = \sin\theta \\
Z_2 &= \frac{V_1 \partial W_2}{\partial x} - \frac{W_1 \partial V_2}{\partial x} + \frac{V_2 \partial W_2}{\partial y} - \frac{W_2 \partial V_2}{\partial y} + \frac{V_3 \partial W_2}{\partial \theta} - \frac{W_3 \partial V_2}{\partial \theta} \\
Z_1 &= \cos\theta 0 - 0 + \sin\theta 0 - 0 + 0 - 1\cos\theta) = -\cos\theta \\
Z_3 &= \frac{V_1 \partial W_3}{\partial x} - \frac{W_1 \partial V_3}{\partial x} + \frac{V_2 \partial W_3}{\partial y} - \frac{W_2 \partial V_3}{\partial y} + \frac{V_3 \partial W_3}{\partial \theta} - \frac{W_3 \partial V_3}{\partial \theta} \\
Z_3 &= 0
\end{aligned}
\tag{16}
$$

Example

A car-like robot or a DDR.

One can observe that $\vec{Z}$ is linear independent of $\vec{V}$ and $\vec{W}$
The matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \\ \sin\theta & -\cos\theta & 0 \end{pmatrix} \tag{17}$$

has a determinant $\det(A) \neq 0$ for any (x, y, θ).

This implies that $dim(CLA(\Delta(x)) = 3$, from the Frobenius theorem, the state transition equation is not integrable, hence the system is nonholonomic. Furthermore, from the Chow theorem the system is small-time locally controllable.

Laumond Theorem

Definition

Let X be a set, d_1 and d_2 are two metrics over X, d_1 and d_2 are equivalent if for any $\epsilon > 0$
1) $\exists \delta_1 > 0$ such that $d_1(x, y) < \delta_1 \implies d_2(x, y) < \epsilon$. 2) $\exists \delta_2 > 0$ such that $d_2(x, y) < \delta_2 \implies d_1(x, y) < \epsilon$.

Theorem

If a collision free path exists for a holonomic system then a feasible path also exists for a nonholonomic car-like robot with the same geometry provided that the system fulfil the next properties: 1) The system is symmetric without drift, 2) The system is STLC, 3) The metric to measure the distance between the robot and the obstacles is equivalent (it induces the same topology) as the metric of the shortest paths for the nonholonomic system, and 4) there is any $\epsilon > 0$ *clearance between the robot and the obstacles.*

For a proof of the theorem see [3, 4, 5]

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