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ON THE NON-CLASSICAL INFINITE DIVISIBILITY OF POWER SEMICIRCLE DISTRIBUTIONS

OCTAVIO ARIZMENDI* AND VÍCTOR PÉREZ-ABREU

Abstract. The family of power semicircle distributions defined as normalized real powers of the semicircle density is considered. The marginals of uniform distributions on spheres in high-dimensional Euclidean spaces are included in this family and a boundary case is the classical Gaussian distribution. A review of some results including a genesis and the so-called Poincaré's theorem is presented. The moments of these distributions are related to the super Catalan numbers and their Cauchy transforms in terms of hypergeometric functions are derived. Some members of this class of distributions play the role of the Gaussian distribution with respect to additive convolutions in non-commutative probability, such as the free, the monotone, the anti-monotone and the Boolean convolutions. The infinite divisibility of other power semicircle distributions with respect to these convolutions is studied using simple kurtosis arguments. A connection between kurtosis and the free divisibility indicator is found. It is shown that for the classical Gaussian distribution the free divisibility indicator is strictly less than 2.

1. Introduction

The semicircle or Wigner distribution plays an important role in several fields of mathematics and its applications. In random matrix theory it is the asymptotic spectral measure of the Wigner ensembles of random matrices, including the Gaussian ensembles; see Metha [34], Khorunzhy et al. [27], Wigner [48]. In the context of representations of symmetric groups, it is the limiting distribution of a Markov chain of Young diagrams; see Kerov [24] and Kerov and Vershick [25]. It is known that the semicircle distribution is an infinitely divisible distribution not in the classical but in the free sense, where it plays the role the Gaussian distribution does in classical probability; see Hiai and Petz [19] and Nica and Speicher [39]. Furthermore, the even moments of the semicircle distribution are the Catalan numbers which appear in combinatorics and other unexpected places; see for example Brualdi [11] and Gardner [15].

The semicircle distribution on $(-\sigma,\sigma)$, $\sigma>0$, has a density given by

$$f_0(x\,;\sigma)=\frac{2}{\pi\sigma^2}\sqrt{\sigma^2-x^2}\,1_{(-\sigma,\sigma)}(x).$$

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*Most of this work was done while the author was a SNI-CONACYT research assistant at CIMAT, Guanajuato, Mexico.

We consider distributions constructed from real powers of the semicircle density. For $\theta > -3/2$ and $\sigma > 0$, let

$$f_\theta(x\,;\sigma) = c'_{\theta,\sigma}\left(f_0(x\,;\sigma)\right)^{2\theta+1} = c_{\theta,\sigma}\left(\sigma^2 - x^2\right)^{\theta+1/2} 1_{|x|\leq,\sigma} \tag{1.1}$$

where

$$c'_{\theta,\sigma} = \left(\frac{\pi}{2}\right)^{2\theta+1}\frac{\sigma^{2\theta}}{\sqrt{\pi}}\frac{\Gamma(\theta+2)}{\Gamma(\theta+3/2)} \text{ and } c_{\theta,\sigma} = \frac{1}{\sqrt{\pi}\sigma^2}\frac{\Gamma(\theta+2)}{\Gamma(\theta+3/2)}\ .$$

A distribution with density (1.1) is called a *power semicircle distribution* and is denoted by $PS(\theta, \sigma)$. It is a symmetric compactly supported distribution with shape parameter θ and range parameter σ.

When $d = 2(\theta + 2)$ is an integer, the corresponding power semicircle distribution appears naturally as the distribution of the one-dimensional marginals of the uniform measure on a sphere of radius $\sqrt{d}$ in $\mathbb{R}^d$, as explained, for example, in Kac [22], Kingman [28] and Diaconis and Freedman [12]. On the other hand, $f_\theta(x\,; \sqrt{(\theta+2)/2}\sigma)$ converges, when $\theta \to \infty$, to the classical Gaussian density $(\sqrt{2\pi}\sigma)^{-1}\exp(-x^2/(2\sigma^2))$, a result known as *Poincaré's theorem* and which goes back to the works of Mehler, Maxwell, Borel and Lévy, amongst others; see Diaconis and Freedman [12] and Johnson [21]. As pointed out by Mc Kean [33], Poincaré's theorem explains why one can think of the Wiener measure (all whose marginals are Gaussian) as the uniform distribution on an infinite dimensional spherical surface of radius $\sqrt{\infty}$; a result the second author learned first from professor Gopinath Kallianpur, to whom this paper is dedicated.

Since the power semicircle distributions have compact support, they are not infinitely divisible in the classical sense. However, this family contains all the "Gaussian distributions" with respect to the five additive convolutions of probability measures on $\mathbb{R}$ that are important in non-commutative probability; namely, the commutative, the free, the Boolean, the monotone and the anti-monotone convolutions. These convolutions correspond to the only five independences, as studied in Muraki [37].

The left-boundary case $\theta = -3/2$ is the symmetric Bernoulli distribution on $\{-\sigma, \sigma\}$ playing the role of Gaussian distribution in Boolean convolution (Speicher and Woroudi [45]). For $\theta = -1$ we obtain the arcsine distribution on $(-\sigma, \sigma)$ which plays in monotone convolution the role Gaussian distribution does in classical probability (Franz and Muraki [14]). The case $\theta = 0$ is the semicircle distribution on $(-\sigma, \sigma)$ and the right-boundary case $\theta = \infty$ given by Poincaré's theorem is the classical Gaussian distribution. Other important families of compactly supported distributions which are useful in non-commutative probability and that include the arcsine and semicircle distributions are considered in Kubo, Kuo and Namli [29], [30] and references therein.

The main purpose of this present article is to study the infinite divisibility of other power semicircle distributions with respect to the five additive convolutions in non-commutative probability. In order to do this, we first derive simple necessary conditions based on the kurtosis of a distribution. The use of kurtosis is motivated by the fact that in non-classical infinite divisibility several distributions with bounded support are relevant; see for example Anshelevich [1] and Bozejko and Bryc [10]. We also prove that the Boolean kurtosis bounds the free divisibility indicator that was recently introduced by Belinschi and Nica [5]. For some other simple conditions for infinite divisibility based on the first few cumulants of a distribution see the recent works by Młotkowski [36] for the free case and Hasebe and Saigo [18] for the monotone convolution case.

As one of the main results of this paper, we show that if θ_{g° is the value of the shape parameter of the "Gaussian distribution" $PS(\theta_{g^\circ}, 1)$ with respect to the corresponding convolution $\circ$, then the power semicircle distribution $PS(\theta, 1)$ is not infinitely divisible with respect to the convolution $\circ$, for $\theta < \theta_{g^\circ}$. We also include results and conjectures on the free infinite divisibility of the classical Gaussian distribution, a result recently proved in Belinschi et al. [4]. In particular, we show that the free divisibility indicator of the classical Gaussian distribution is strictly less than 2.

The organization of the paper is as follows. Section 2 presents the main features and properties of power semicircle distributions, including Poincaré's theorem. We also derive their moments and show that they are given in terms of super Catalan numbers. As a consequence the Cauchy transforms of these distributions are derived in terms of hypergeometric functions. Section 3 starts with preliminary material on the analytic approach to free, monotone, anti-monotone and Boolean convolutions and the corresponding infinite divisibility concept with respect to these convolutions. It also derives criteria for infinite divisibility based on the kurtosis of a distribution. The free and the monotone infinite divisibility properties of the symmetric beta distributions considered in Arizmendi et al. [2] are studied. A connection between kurtosis and the free divisibility indicator is also included. Section 4 applies the kurtosis conditions to find the non infinite divisibility of some power semicircle distributions with respect to above non-classical convolutions. We finally include results on the free infinite divisibility of the classical Gaussian distribution.

2. Properties of PSCD: A Review

In this section we present several facts and properties of power semicircle distributions including a recursive representation, Poincaré's theorem, moments and their Cauchy transforms.

2.1. Representations of the classical Gaussian law. The classical Gaussian distribution has the representation of a scale mixture of chi-square distributions with an appropriate power semicircle distributions. This fact is useful to obtain properties of power semicircle distributions.

For positive α, β let $\mathrm{Gam}(\alpha, \beta)$ denote the Gamma distribution with density

$$g_{\alpha,\beta}(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp(-\frac{x}{\beta}),\ x > 0. \tag{2.1}$$

For any $\theta > -3/2$ let $\gamma_{\theta+2} = \chi^2_{2(\theta+2)}$ denote a random variable with chi-square distribution with $2(\theta+2)$ degrees of freedom and independent of the random variable S_θ with power semicircle distribution $PS(\theta, 1)$.

The proof of the following representation theorem of the Gaussian distribution follows easily using the change of variable formula for densities.

Theorem 2.1. *Let $\theta > -3/2$ and let $\gamma_{\theta+2}$ be a random variable with Gamma distribution* $\mathrm{Gam}(\theta+2,2)$ *and independent of the random variable S_θ with power semicircle distribution $PS(\theta,1)$. Then*

$$Z = \sqrt{\gamma_{\theta+2}}S_\theta \tag{2.2}$$

has a standard Gaussian distribution $N(0,1)$.

Proof. We shall use the trivial fact that if V is a nonnegative random variable independent of a symmetric random variable Y, with densities f_V and f_Y respectively, then the density of $X = \sqrt{V}Y$ is given by

$$f_X(x) = |x| \int_{-\infty}^{\infty} \frac{1}{y^2} f_Y(y) f_V(\frac{x^2}{y^2}) dy, \quad x \in \mathbb{R}. \tag{2.3}$$

Using (1.1) and (2.1) in (2.3) we easily obtain

$$\begin{aligned}
f_X(x) &= |x| \int_{-\infty}^{\infty} \frac{1}{y^2} f_{S_\theta}(y) g_{(\theta+2,2)}(\frac{x^2}{y^2}) \mathrm{d}y \\
&= |x| \int_{-1}^{1} \frac{1}{y^2} c'_{\theta,1} (\frac{2}{\pi}\sqrt{1-y^2})^{2\theta+1} \frac{1}{2^{\theta+2}} \frac{(\frac{x^2}{y^2})^{\theta+1}}{\Gamma(\theta+2)} e^{-x^2/(2y^2)} \mathrm{d}y \\
&\overset{(r=1/y)}{=} (\frac{1}{\sqrt{\pi}}) \frac{|x|^{2\theta+3}}{\Gamma(\theta+3/2)} \frac{1}{2^{\theta+1}} \int_1^{\infty} \frac{(\frac{r^2-1}{r^2})^{\theta+\frac{1}{2}} r^{2\theta+4}}{r^2} e^{-x^2r^2/2} \mathrm{d}r \\
&\overset{(t=r^2-1)}{=} (\frac{1}{\sqrt{\pi}}) \frac{|x|^{2\theta+3}}{\Gamma(\theta+3/2)} \frac{1}{2^{\theta+2}} e^{-x^2/2} \int_0^{\infty} t^{\theta+\frac{1}{2}} e^{-x^2 t/2} \mathrm{d}t \\
&= (\frac{1}{\sqrt{2\pi}}) \frac{|x|^{2\theta+3}}{\Gamma(\theta+3/2)} \frac{e^{-x^2/2}}{x^{2\theta+3}} \Gamma(\theta+3/2) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},
\end{aligned}$$

which proves the result. □

Recall that a one-dimensional random variable X is said to be *variance mixture of Gaussian* if the probability distribution of X is of the form $\sqrt{V_\theta}Z$ (in short $X \overset{L}{=} \sqrt{V_\theta}Z$), where Z and V are independent random variables with V being positive *random variance* and Z having the normal distribution with zero-mean and variance one; see [23], [42], [47].

Corollary 2.2. *If $X \overset{L}{=} \sqrt{V}Z$ is a variance mixture of the Gaussian distribution, then for any $\theta > -3/2$, $X \overset{L}{=} \sqrt{V\gamma_{\theta+2}}S_\theta$ where $\gamma_{\theta+2}$ is a random variable with Gamma distribution* $\mathrm{Gam}(\theta+2,2)$, *S_θ has power semicircle distribution $PS(\theta,1)$ and $V, \gamma_{\theta+2}$ and S_θ are independent random variables.*

Power semicircle distributions were considered by Kingman [28], who also defined a convolution of probability measures on $\mathbb{R}_+$. It has been recently proved in Nguyen [38] that the Bessel processes generated by $\gamma_{\theta+2}$ play for Kingman convolution the role Brownian motion does in classical convolution. We learned this result from Geronimo Uribe.

2.2. Recursive representation. Ledoux [31] pointed out that the semicircle distribution is a scale mixture of the arcsine distribution with the uniform distribution. The following proposition is a generalization of this result giving a useful recursive representation for power semicircle distributions.

Proposition 2.3. *Let $\theta \geq -3/2$. Then $S_\theta \stackrel{L}{=} \sqrt{U^{1/(\theta+1)}}S_{\theta-1}$ where U is a random variable with uniform distribution on $(0,1)$ independent of the random variable $S_{\theta-1}$ with power semicircle distribution $PS(\theta-1,1)$. Moreover, $S_\theta^2 \stackrel{L}{=} U^{1/(\theta+1)} S_{\theta-1}^2$.*

Proof. The density of $U^{1/(2(\theta+1))}$ is $f_{U^{1/(2(\theta+1))}}(t) = 2(\theta+1)t^{2\theta+1}$, $0 < t < 1$. Let $V = U^{1/(2(\theta+1))}S_{\theta-1}$ then the density of V is found as follows: For $x \in (-1,1)$,

$$\begin{aligned} f_V(x) &= \int_{-\infty}^{\infty} \frac{1}{|y|} f_{S_{\theta-1}}(y) f_{U^{1/(2(\theta+1))}}(\frac{x}{y})\mathrm{d}y \\ &= 2\frac{\Gamma(\theta+1)}{\sqrt{\pi}\Gamma(\theta+1/2)}(\theta+1)\,|x|^{2\theta+1}\int_x^1 \frac{(\sqrt{1-y^2})^{2\theta-1}}{y^{2\theta+2}}\mathrm{d}y \\ &\stackrel{(y=\sin(u))}{=} 2\frac{\Gamma(\theta+2)}{\sqrt{\pi}\Gamma(\theta+1/2)}\,|x|^{2\theta+1}\int_{\arcsin(x)}^{\pi/2} \frac{\cos(y)^{2\theta}}{\sin(y)^{2\theta+2}}\mathrm{d}y \\ &= \frac{\Gamma(\theta+2)}{\sqrt{\pi}\Gamma(\theta+3/2)}\left(\sqrt{1-x^2}\right)^{2\theta+1} = c'_{\theta,1}(\frac{2}{\pi}\sqrt{1-x^2})^{2\theta+1} = f_\theta(x). \end{aligned}$$

The second statement is a trivial observation. □

As a consequence of the above theorem one can derive representations for some power semicircle distributions as variance mixtures of the arcsine and the semicircular distributions.

Corollary 2.4. *For $\theta = 1,2,...$*

$$S_\theta \stackrel{L}{=} \sqrt{V_\theta}S_0$$

where S_0 has a semicircle distribution on $(-1,1)$ and is independent of the random variable $V_\theta \stackrel{L}{=} \prod_{i=1}^{\theta} U_i^{\frac{1}{i+1}}$, where $U_1,...,U_\theta$ are independent random variables with uniform distribution on $(0,1)$. Moreover, for $\theta = 1,2,...$, $S_\theta \stackrel{L}{=} \sqrt{V_\theta U}S_{-1}$ where U has an uniform distribution on $(0,1)$ and is independent of V_θ.

In fact, any power semicircle distribution $PS(\theta,1), \theta > -1$, is a variance mixture of the arcsine distribution as follows. We provide the proof of this result in Section 2.5 using Cauchy transforms.

Proposition 2.5. *For $\theta > -1$, $S_\theta \stackrel{L}{=} \sqrt{1-U^{1/(1+\theta)}}S_{-1}$ where U has an uniform distribution on $(0,1)$ and is independent of the arcsine random variable S_{-1} on $(-1,1)$.*

2.3. Poincaré's theorem. An important consequence of Theorem 2.1 is the fact that the sequence of random variables (S_n), appropriately scaled, converges in distribution, when n goes to infinite, to the standard classical Gaussian distribution. This result is a Poincaré's type theorem which already appears in the works of Mehler [35] in 1866 and Borel [8] in 1914. One can easily deduce this result from

Theorem 2.1. For another proof see, for example, the Remark in page 387 of Khokhlov [26].

Theorem 2.6. *Let S_n have the power semicircle distribution $PS(n,1)$, for $n = 0,1,2,...$ Then the sequence of random variables $\left\{(\sqrt{2(n+2)}S_n\right\}_{n\geq 1}$ converges in distribution to the standard classical Gaussian distribution.*

Proof. By the law of large numbers, $(n+2)^{-1}\gamma_{n+2}$ converges in probability to $E(\gamma_1) = 2$, where $\gamma_m \sim \mathrm{Gam}(m,2)$. Hence from (2.2) and using Slutsky's theorem we have

$$\sqrt{2(n+2)}S_n = \sqrt{2(n+2)}\,(\gamma_{n+2})^{-1/2}\,Z \Rightarrow_{n\to\infty} Z$$

which gives the proof. □

A modelling feature of Poincaré's theorem is the fact that, for large θ, one can use the power semicircle distribution $PS(\theta, 2(\theta+2))$ as an alternative symmetric model to the Gaussian distribution with the advantage of having finite range. This is specially useful when there is a knowledge of the range of the measurements, as the case of some problems in Metrology; see Lira [32]. This raises the question of the speed of convergence, a problem studied by Stam [46], Diaconis and Freedman [12], Borovkov [9], Johnson [21] and Khokhlov [26], amongst others.

A multivariate version of Poincaré's theorem and its corresponding rate of convergence are considered in [12], [26], [46]. It is an open problem to study the rate of convergence in Poincaré's theorem using Stein's method. See Reinert [41] for a review of this powerful method to study rates of convergence to the Gaussian and other distributions.

2.4. Moments. Using the representation (2.2) and the moments of the Normal and Gamma distributions one can easily obtain the absolute moments and moments of the power semicircle distributions. The latter are given in terms of the *Catalan numbers*

$$C_k = \frac{\binom{2k}{k}}{k+1}$$

which are the even moments of the standard semicircle distribution $PS(0,2)$.

Proposition 2.7. *Let S_θ be a random variable with power semicircle distribution $PS(\theta,\sigma)$, for $\theta > -3/2, \sigma > 0$. Then,*

a) For any $\alpha > 0$

$$E\,|S_\theta|^\alpha = \frac{\sigma^\alpha}{\sqrt{\pi}}\Gamma(\alpha/2+1/2)\frac{\Gamma(\theta+2)}{\Gamma(\theta+2+\alpha/2)}.$$

b) For any integer $k \geq 1, ES_\theta^k = 0$ and

$$ES_\theta^{2k} = \left(\frac{\sigma}{2}\right)^{2k} C_k(k+1)!\frac{\Gamma(\theta+2)}{\Gamma(\theta+2+k)}.$$

c) If θ is an integer

$$ES_\theta^{2k} = \frac{\binom{2k}{k}}{\binom{\theta+k+1}{k}}\left(\frac{\sigma}{2}\right)^{2k}.$$

Proof. It is enough to consider the case $\sigma = 1$. It is well known that for $\alpha > 0$,

$$E\,|Z|^{\alpha} = \frac{2^{\alpha/2}}{\sqrt{\pi}}\Gamma(\alpha/2 + 1/2)$$

and

$$E\gamma_{\theta+2}^{\alpha/2} = 2^{\alpha/2}\frac{\Gamma(\theta + 2 + \alpha/2)}{\Gamma(\theta + 2)}.$$

Hence, using the independence of S_θ and $\gamma_{2\theta+1}$, from (2.2) we obtain

$$E\,|S_\theta|^{\alpha} = \frac{1}{\sqrt{\pi}}\Gamma(\alpha/2 + 1/2)\frac{\Gamma(\theta + 2)}{\Gamma(\theta + 2 + \alpha/2)}.$$

When k is an integer, by symmetry $ES_\theta^k = 0$. On the other hand, taking $\alpha = 2k$ we have $\Gamma(\alpha/2 + 1/2) = \sqrt{\pi}2^{-2k}(2k)!/k!$. Thus,

$$ES_\theta^{2k} = \frac{1}{2^k}C_k(k+1)!\frac{\Gamma(\theta + 2)}{\Gamma(\theta + 2 + k)},$$

which gives (b). Finally, we observe that

$$ES_\theta^{2k} = \frac{1}{2^k}C_k(k+1)!\frac{(\theta + 1)!}{(\theta + 1 + k)!} = \frac{1}{2^k}\frac{\binom{2k}{k}}{\binom{\theta+k+1}{k}}$$

which proves (c). $\square$

For given $\theta > -3/2$, the corresponding standard distribution (zero-mean and vairance-one) is obtained when $\sigma^2 = 2\Gamma(\theta + 3)/\Gamma(\theta + 2)$. In particular, when θ is an integer, the corresponding standard distribution is given when $\sigma^2 = 2(\theta + 2)$.

On the other hand, when θ is an integer, the even moments $C_k^\theta = \binom{2k}{k}/\binom{\theta+k+1}{k}$ of $PS(\theta, 2(\theta + 2))$ are a kind of *generalized Catalan numbers.* Indeed, they are different from the so-called *super Catalan numbers* by a factor $\binom{2(\theta+1)}{\theta+1}$ and $(\theta + 2)(\theta+3)\cdots(2\theta+1)C_k^\theta$ is an integer multiple of $\theta!$. We refer to Gessel [16] or Hilton and Pedersen [20] for the study of this kind of generalized Catalan numbers.

2.5. Cauchy transform. The Cauchy transform plays an important role in the study of different convolutions of probability measures in non-commutative probability and their related infinitely divisible aspects. For a Borel probability measure μ on $\mathbb{R}$, its *Cauchy transform* is defined as

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z - t}\mu(\mathrm{d}t), \qquad z \in \mathbb{C}^+,$$

where $\mathbb{C}^+ = \{\zeta \in \mathbb{C} : \mathrm{Im}(\zeta) > 0\}$.

For power semicircle distributions, their Cauchy transforms are given in terms of the Gauss hypergeometric function

$$\mathcal{F}(a, b; c, z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!}z^k \tag{2.4}$$

where for a complex ζ and a non integer k we use the Pochhammer symbol $(\zeta)_k$ to denote the expression $(\zeta)_k = \zeta(\zeta + 1)\cdots(\zeta + k - 1)$.

Proposition 2.8. *The Cauchy transform G_θ of $f_\theta(x\,;\sigma)$ is given by*

$$G_\theta(z) = z^{-1}\mathcal{F}(\frac{1}{2}, 1; \theta+2; \sigma^2 z^{-2}). \tag{2.5}$$

Proof. We follow the proof using the moment generating function, as done in [19] for the semicircle distribution. From Proposition 2.7(b) we have

$$\begin{aligned} G_\theta(z) &= z^{-1}\sum_{k=0}^{\infty} z^{-2k} EX^{2k} \\ &= z^{-1}\sum_{k=0}^{\infty} z^{-2k}\left(\frac{\sigma}{2}\right)^{2k}\frac{(2k)!}{k!}\frac{\Gamma(\theta+2)}{\Gamma(\theta+2+k)}. \end{aligned} \tag{2.6}$$

Using in (2.4) the expressions $(\theta+2)_k = \Gamma(\theta+2+k)/\Gamma(\theta+2)$, $(1/2)_k = (2k)!/2^{2k}$ and $(1)_k = k!$, from (2.6) we obtain (2.5). □

An alternative integral representation for the Cauchy transform follows from the integral expression of the hypergeometric function; see Gradshteyn and Ryzhil [17, eq. 9.111].

Corollary 2.9. *For $\theta > -1$*

$$G_\theta(z) = (\theta+1)\int_0^1 (1-t)^\theta (z^2 - t\sigma^2)^{-1/2}\mathrm{d}t.$$

Example 2.10. There is an explicit formula for the Cauchy transform of the power semicircle distributions in the following special cases:

a) For the arcsine distribution on $(-\sigma, \sigma)$

$$G_{-1}(z) = (z^2 - \sigma^2)^{-1/2}.$$

b) For the uniform distribution on $(-\sigma, \sigma)$

$$G_{-1/2}(z) = \frac{1}{2}\ln\frac{z+\sigma}{z-\sigma}.$$

c) For the semicircle distribution on $(-\sigma, \sigma)$

$$G_0(z) = \frac{2}{\sigma^2}(z - (z^2 - \sigma^2)^{1/2}).$$

Proof of Proposition 2.5. We have to prove that if U has uniform distribution on $(0,1)$ and is independent of the arcsine random variable S_{-1} on $(-1,1)$, then, for $\theta > -1$, $\sqrt{1 - U^{1/(\theta+1)}}S_{-1}$ follows a power semicircle distribution $PS(\theta, 1)$. From Corollary 2.9, for $\theta > -1$ and $\sigma = 1$, we obtain

$$G_\theta(z) = (\theta+1)\int_0^1 (1-t)^\theta (z^2 - t)^{-1/2}\mathrm{d}t.$$

Example (a) above says that $(z^2 - t)^{-1/2}$ is the Cauchy transform of the arcsine density $\frac{1}{\pi}(t - x^2)^{-1/2}$ on $(-\sqrt{t}, \sqrt{t})$. Hence, using change of variables we have

$$G_\theta(z) = \int_{-1}^1 \frac{1}{z-x} g_\theta(x)\mathrm{d}x$$

where

$$g_\theta(x) = (\theta+1)\int_{x^2}^{1}(1-t)^\theta \frac{1}{\pi}(t-x^2)^{-1/2}\mathrm{d}t.$$

Observe that $(\theta+1)(1-t)^\theta$ is the density of the random variable $1-U^{1/(\theta+1)}$ with U uniformly distributed on $(0,1)$. Thus, if S_{-1} has arcsine distribution on $(-1,1)$ and is independent of U, a straightforward change of variables similar to the one used in the proof of Proposition 2.3 shows that $g_\theta(x)$ is the density of the random variable $X=(1-U^{1/(\theta+1)})^{1/2}S_{-1}$. The uniqueness of the Cauchy transform gives that X has distribution $PS(\theta,1)$, for $\theta>-1$. □

The following interesting relation between the power of the Cauchy transform of the semicircle distributions and the generalized Cauchy transform of a power semicircle distribution was recently proved in Demni [13]. For any $z\in\mathbb{C}^+$

$$\int_{-2}^{2}\frac{1}{(z-x)^\lambda}(4-x^2)^{\lambda-1/2}dx = d_\lambda\left(\int_{-2}^{2}\frac{1}{2\pi(z-x)}\sqrt{4-x^2}dx\right)^\lambda$$

for a constant $d_\lambda>0$ and $\lambda>0$.

3. Infinite Divisibility

Recall that the *classical convolution* of two probability measures μ_1,μ_2 on $\mathbb{R}$ is defined as the probability measure $\mu_1*\mu_2$ on $\mathbb{R}$ such that

$$\mathcal{C}_{\mu_1*\mu_2}(t)=\mathcal{C}_{\mu_1}(t)+\mathcal{C}_{\mu_2}(t),\quad t\in\mathbb{R},$$

where $\mathcal{C}_\mu(t)=\log\widehat{\mu}(t)$ with $\widehat{\mu}(t)$ the characteristic function of μ. The classical cumulants associated to this convolution are defined as the coefficients $c_n=c_n(\mu)$ in the series expansion

$$\mathcal{C}_\mu(t)=\sum_{n=1}^{\infty}\frac{c_n}{n!}t^n.$$

The relation between the classical cumulants and the moments $m_n=m_n(\mu)$ is related to the partitions $P(n)$ of $\{1,...,n\}$, that is

$$m_n=\sum_{\pi\in P(n)}\prod_{V\in\pi}c_{|V|}$$

3.1. Convolutions and non-classical infinite divisibility.

The *reciprocal of the Cauchy transform* is the function $F_\mu(z):\mathbb{C}_+\to\mathbb{C}_+$ given by $F_\mu(z)=1/G_\mu(z)$.

(a) Free convolution and free cumulants

It was proved in Bercovici and Voiculescu [7] that there are positive numbers η and M such that F_μ has a right inverse F_μ^{-1} defined on the region

$$\Gamma_{\eta,M}:=\{z\in\mathbb{C};|\mathrm{Re}(z)|<\eta\,\mathrm{Im}(z),\quad \mathrm{Im}(z)>M\}.$$

The *Voiculescu transform of* μ is defined by

$$\varphi_\mu(z)=F_\mu^{-1}(z)-z,$$

on any region of the form $\Gamma_{\eta,M}$ where F_μ^{-1} is defined; see [7]. The *free cumulant transform* or *R-transform* is a variant of φ_μ defined as

$$\mathcal{C}_\mu^{\boxplus}(z) = z\phi_\mu(\frac{1}{z}) = zF_\mu^{-1}\left(\frac{1}{z}\right) - 1,$$

for z in a domain $D_\mu \subset \mathbb{C}_-$ such that $1/z \in \Gamma_{\eta,M}$ where F_μ^{-1} is defined.

The *free additive convolution* of two probability measures μ_1, μ_2 on $\mathbb{R}$ is the probability measure $\mu_1 \boxplus \mu_2$ on $\mathbb{R}$ such that $\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z)$ or equivalently

$$\mathcal{C}^{\boxplus}_{\mu_1 \boxplus \mu_2}(z) = \mathcal{C}^{\boxplus}_{\mu_1}(z) + \mathcal{C}^{\boxplus}_{\mu_2}(z)$$

for $z \in D_{\mu_1} \cap D_{\mu_2}$.

Free cumulants were introduced by Speicher [44]. They are the coefficients $k_n = k_n(\mu)$ is the series expansion

$$\mathcal{C}_\mu^{\boxplus}(z) = 1 + \sum_{n=1}^{\infty} k_n z^n.$$

The relation between the free cumulants and the moments is related to the combinatorics of the lattice of non-crossing partitions $NC(n)$, namely,

$$m_n = \sum_{\pi \in NC(n)} \prod_{V \in \pi} k_{|V|}.$$

(b) Boolean convolution and Boolean cumulants

The *Boolean convolution* of two probability measures μ_1, μ_2 on $\mathbb{R}$ is defined as the probability measure $\mu_1 \uplus \mu_2$ on $\mathbb{R}$ such that the transform $K_\mu(z) = z - F_\mu(z)$, (usually called the *self energy)*, satisfies

$$K_{\mu_1 \uplus \mu_2}(z) = K_{\mu_1}(z) + K_{\mu_2}(z), \quad z \in \mathbb{C}_+,$$

see [45]. *Boolean cumulants* are defined as the coefficients $h_n = h_n(\mu)$ in the series

$$K_\mu(z) = 1 + \sum_{n=1}^{\infty} h_n z^n.$$

A relation between moments and Boolean cumulants is described in terms of the combinatorics of the lattice of interval partitions $I(n)$, namely,

$$m_n = \sum_{\pi \in I(n)} \prod_{V \in \pi} h_{|V|}.$$

(c) Monotone convolution and monotone cumulants

The *monotone convolution* of two probability measures μ_1, μ_2 on $\mathbb{R}$ is defined as the probability measure $\mu_1 \rhd \mu_2$ on $\mathbb{R}$ such that

$$F_{\mu_1 \rhd \mu_2}(z) = F_{\mu_1}(F_{\mu_2}(z)), \quad z \in \mathbb{C}_+,$$

and similarly, the *anti-monotone convolution* $\mu_1 \lhd \mu_2$ is defined as the probability measure on $\mathbb{R}$ such that $F_{\mu_1 \lhd \mu_2}(z) = F_{\mu_2}(F_{\mu_1}(z))$, for $z \in \mathbb{C}_+$; see [14].

Recently, Hasebe and Saigo [18] have defined the notion of *monotone cumulants* $(r_n)_{n \geq 1}$ which satisfy that $r_n(\mu \rhd \mu) = 2r_n(\mu)$.

3.2. Connections between kurtosis and infinite divisibility. Similar to the definition of infinite divisibility with respect to classical convolution ([43]), it is said that a probability measure μ is *infinitely divisible with respect to the convolution* $\circ$ if for every positive integer n there exists a probability measure μ_n such that

$$\mu = \underbrace{\mu_n \circ \mu_n \circ \cdots \circ \mu_n}_{n \text{ times}}.$$

The kurtosis of a distribution is useful to derive a simple necessary conditions for infinitely divisible with respect to the classical, free, monotone and anti-monotone convolutions. It is known that for the Boolean convolution all distributions on $\mathbb{R}$ are infinitely divisible.

The *classical kurtosis* of a probability measure μ with finite fourth moment is defined as

$$Kurt(\mu) = \frac{c_4(\mu)}{(c_2(\mu))^2} = \frac{\widetilde{m}_4(\mu)}{(\widetilde{m}_2(\mu))^2} - 3,$$

where $c_2(\mu)$ and $c_4(\mu)$ are the second and fourth classical cumulants, and $\widetilde{m}_2(\mu)$ and $\widetilde{m}_4(\mu)$ the second and fourth moments around the mean. It is always true that $Kurt(\mu) \geq -2$.

In Steutel and Van Harn [47] a necessary condition for the classical infinite divisibility of a distribution based on the first fourth classical cumulants is presented. Below we present a condition based on the kurtosis.

Proposition 3.1. *Let μ be a probability measure on $\mathbb{R}$ with finite fourth moment. If μ is infinitely divisible in the classical sense then $Kurt(\mu) \geq 0$.*

Proof. It is well known that if $Y = X_1 + \cdots + X_n$ is the sum of n identical random variables which are independent in the classical sense, all with the same distribution as X, then $nKurt[Y] = Kurt[X]$. (This is only the fact that classical cumulants are additive with respect to classical convolution). In other words,

$$Kurt(\mu) = nKurt(\underbrace{\mu * \cdots * \mu}_{n \text{ times}}).$$

Suppose that μ is infinitely divisible in the classical sense and $Kurt(\mu) = \alpha < 0$. Let μ_n be such that $\underbrace{\mu_n * \cdots * \mu_n}_{n \text{ times}} = \mu$.Since $Kurt(\mu_n) = nKurt(\mu) = n\alpha$, we can choose n large enough such that $n\alpha < -2$, which is a contradiction since $Kurt \geq -2$. $\square$

The free kurtosis is defined similarly using the free cumulants instead of the classical cumulants. That is, the *free kurtosis* of a probability measure μ is defined as

$$Kurt^{\boxplus}(\mu) = \frac{k_4(\mu)}{(k_2(\mu))^2} = \frac{\widetilde{m}_4(\mu)}{(\widetilde{m}_2(\mu))^2} - 2 = Kurt(\mu) + 1$$

where $(k_n(\mu))_{n\geq 0}$ is the sequence of free cumulants.

Let Λ denote the Bercovici-Pata bijection [6] between classical and free infinitely divisible distributions. Since this bijection preserves cumulants, we have

$$Kurt^{\boxplus}(\Lambda(\mu)) = Kurt(\mu).$$

Since $k_4(\mu \boxplus \cdots \boxplus \mu) = nk_4(\mu)$ and $k_2(\mu \boxplus \cdots \boxplus \mu) = nk_2(\mu)$ we have similarly as for the classical case that if $Y = X_1 + \cdots + X_n$ is the sum of n non-commutative random variables which are independent in the free sense, all with the same spectral distribution as X, then

$$nKurt^{\boxplus}[Y] = Kurt^{\boxplus}[X].$$

Thus, using similar arguments as for Proposition 12, we have that if the probability measure μ on $\mathbb{R}$ is infinitely divisible in the free sense then $Kurt^{\boxplus}(\mu) \geq 0$. The following result is a criterion for a measure to be free infinitely divisible in terms of the classical kurtosis.

Proposition 3.2. *Let μ be a probability measure on $\mathbb{R}$ with finite fourth moment. If μ is infinitely divisible in the free sense then $Kurt(\mu) \geq -1$.*

Proof. Let μ be free infinitely divisible. Since $Kurt^{\boxplus}(\mu) = Kurt(\mu) + 1$ and $Kurt^{\boxplus}(\mu) \geq 0$, we get the result. □

The *monotone kurtosis* of a zero-mean distribution μ is defined as

$$Kurt^{\rhd}(\mu) = \frac{2m_4(\mu) - 3m_2(\mu)^2}{2(m_2(\mu))^2} = Kurt(\mu) + 1.5.$$

Recently Hasebe and Saigo [18] defined the monotone cumulants $(r_n)_{n\geq 0}$. Hence, the monotone kurtosis defined in this paper can be regarded as

$$Kurt^{\rhd}(\mu) = \frac{r_4(\mu)}{(r_2(\mu))^2}.$$

The following result gives a necessary condition in terms of kurtosis for a measure with zero-mean to be infinitely divisible with respect to monotone convolution.

Proposition 3.3. *Let μ be a probability measure on $\mathbb{R}$ with zero-mean and finite fourth moment. If μ is infinitely divisible with respect to monotone convolution then $Kurt(\mu) \geq -1.5$.*

Proof. It is enough to prove that $Kurt^{\rhd} \geq 0$. For zero-mean measures the following identities hold

$$m_4(\mu \rhd \nu) = m_4(\nu) + m_4(\mu) + 3m_2(\mu)m_2(\nu) \tag{3.1}$$

$$m_2(\mu \rhd \nu) = m_2(\mu) + m_2(\nu). \tag{3.2}$$

Hence, using (3.1) and (3.2) we have

$$\begin{aligned} Kurt^{\rhd}(\mu \rhd \mu) &= \frac{2m_4(\mu \rhd \mu) - 3m_2(\mu \rhd \mu)^2}{2(m_2(\mu \rhd \mu))^2} \\ &= \frac{2(2m_4(\mu) + 3m_2(\mu)^2) - 3(2m_2(\mu))^2}{2(2m_2(\mu))^2} \\ &= \frac{4m_4(\mu) - 6m_2(\mu)^2}{8m_2(\mu)^2} = \frac{1}{2}Kurt^{\rhd}(\mu). \end{aligned}$$

Suppose now that μ is infinitely divisible with respect to monotone convolution and $\alpha = Kurt^{\rhd} < 0$. Using repeatedly the fact that $Kurt^{\rhd}(\mu) = 2Kurt^{\rhd}(\mu \rhd \mu)$, we obtain

$$Kurt^{\rhd}(\mu) = 2^n Kurt^{\rhd}(\underbrace{\mu \rhd \cdots \rhd \mu}_{2^n \text{ times}}).$$

Let μ_n be such that $\underbrace{\mu_n \rhd \cdots \rhd \mu_n}_{2^n \text{ times}} = \mu$. Hence $Kurt^{\rhd}(\mu_n) = 2^n Kurt^{\rhd}(\mu) = 2^n \alpha$. Again, choosing n large enough so that $2^n \alpha < -2$ we get a contradiction since for all n, $Kurt^{\rhd}(\mu_n) = Kurt(\mu_n) + 1.5 > Kurt(\mu_n) \geq -2$. □

Example 3.4. Symmetric Beta Distribution. We illustrate the above criteria in the case of the family of symmetric beta distributions. Recall that for $\alpha, \beta > 0$, a probability measure has a symmetric beta distribution $BS(\alpha, \beta)$, if it is absolutely continuous with density function

$$g(x) = \frac{1}{2B(\alpha,\beta)} |x|^{\alpha-1} (1 - |x|)^{\beta-1}, \quad |x| < 1.$$

It was shown in [2], that the symmetric beta distribution $BS(1/2, 3/2)$ is free infinitely divisible. Therefore, a natural question is whether other members of this family are infinitely divisible in the free sense. If μ is a symmetric beta distribution $BS(\alpha, \beta)$, the kurtosis of μ is given by

$$Kurt(\mu) = \frac{(\alpha+2)(\alpha+3)(\alpha+\beta)(\alpha+\beta+1)}{\alpha(\alpha+1)(\alpha+\beta+2)(\alpha+\beta+3)} - 3.$$

Thus, from Proposition 3.2 we have that if μ is free infinitely divisible the following inequality must hold

$$(\alpha+2)(\alpha+3)(\alpha+\beta)(\alpha+\beta+1) \geq 2\alpha(\alpha+1)(\alpha+\beta+2)(\alpha+\beta+3). \tag{3.3}$$

In particular, when $\beta = \alpha$ we have that if $\alpha > 7/25$ then $BS(\alpha, \alpha)$ is not $\boxplus$-infinitely divisible. Taking $\alpha = 1/2$, we have that $BS(1/2, \beta)$ is not $\boxplus$-infinitely divisible for $\beta < 1/2$. Observe that $BS(1/2, 3/2)$ satisfies the inequality (3.3).
In a similar way, from Proposition 3.3 we have that if μ is monotone infinitely divisible the following inequality must hold

$$(\alpha+2)(\alpha+3)(\alpha+\beta)(\alpha+\beta+1) \geq \frac{3}{2}\alpha(\alpha+1)(\alpha+\beta+2)(\alpha+\beta+3).$$

Using similar ideas we define, for a probability measure μ on $\mathbb{R}$ with zero-mean and fourth moment, the *Boolean kurtosis* as

$$Kurt^{\uplus}(\mu) = \frac{h_4(\mu)}{(h_2(\mu))^2} = Kurt^{\boxplus}(\mu) + 1 = Kurt(\mu) + 2. \tag{3.4}$$

We might expect to obtain a similar criterion as above for Boolean infinite divisibility, but since any measure is infinitely divisible with respect to Boolean convolution, this would only lead to the fact that kurtosis is greater than -2.

Instead of this we shall end this section with the study of a connection between the Boolean kurtosis and the following remarkable and useful family of homomorphisms introduced in Belinschi and Nica [5]. Let $\mathcal{P}$ be the class of all Borel probability measures on the real line $\mathbb{R}$. For $\mu \in \mathcal{P}$ and every $t \geq 0$ consider the transformation $\mathbb{B}_t : \mathcal{P} \to \mathcal{P}$

$$\mathbb{B}_t(\mu) = \left(\mu^{\boxplus(1+t)}\right)^{\uplus 1/(1+t)} \tag{3.5}$$

and the $\boxplus$-divisibility indicator

$$\phi(\mu) := \sup\{t \in [0,\infty] \mid \mu \in \mathbb{B}_t(\mathcal{P})\}.$$

The relation between $\mathbb{B}_t(\mu)$, the $\boxplus$-divisibility indicator and free infinite divisibility is given in the following proposition due to [5]. Let $\mathbb{B}(\mu)$ denote the Boolean Bercovici-Pata bijection that sends a distribution μ into a free infinitely divisible distribution $\mathbb{B}(\mu)$.

Proposition 3.5. *Let $\mu \in \mathcal{P}$. The following statements hold*

a) $\mathbb{B}_1(\mu) = \mathbb{B}(\mu), \forall \mu \in \mathcal{P}$.

b) μ is infinitely divisible with respect to $\boxplus$ if and only if $\phi(\mu) \geq 1$.

c) There exists a free infinitely divisible distribution ν such that $\mu = \mathbb{B}(\nu)$ if and only if $\phi(\mu) \geq 2$.

One useful property that allows us to calculate $\phi(\mu)$ for some probability measures is the following: For each $t > 0$

$$\phi(\mathbb{B}_t(\mu)) = \phi(\mu) + t. \tag{3.6}$$

We can prove a similar relation to (3.6) using Boolean kurtosis instead of ϕ.

Proposition 3.6. *Let μ be a probability measure on $\mathbb{R}$ having zero-mean and fourth moment. Then for each $t > 0$*

$$Kurt^{\uplus}(\mathbb{B}_t(\mu)) = Kurt^{\uplus}(\mu) + t. \tag{3.7}$$

Proof. Since Boolean cumulants are additive with respect to the Boolean convolution, we have that for each $t > 0$ $Kurt^{\uplus}(\mu^{\uplus t}) = \frac{1}{t} Kurt^{\uplus}(\mu)$ and similarly

$$Kurt^{\boxplus}(\mu^{\boxplus(1+t)}) = \frac{1}{(1+t)} Kurt^{\boxplus}(\mu).$$

Hence, using (3.5) and (3.4) we obtain

$$\begin{aligned} Kurt^{\uplus}(\mathbb{B}_t(\mu)) &= Kurt^{\uplus}\left(\left(\mu^{\boxplus(1+t)}\right)^{\uplus 1/(1+t)}\right) \\ &= (1+t) Kurt^{\uplus}\left(\mu^{\boxplus(1+t)}\right) \\ &= (1+t)\left(Kurt^{\boxplus}\left(\mu^{\boxplus(1+t)}\right) + 1\right) \\ &= (1+t)\left(\frac{1}{1+t} Kurt^{\boxplus}(\mu) + 1\right) \\ &= Kurt^{\uplus}(\mu) + t. \end{aligned}$$

□

Using (3.7) and Propositions 3.2 and 3.5, we obtain that the Boolean kurtosis is an upper bound for the free divisibility indicator.

Theorem 3.7. *Let μ be a probability measure on $\mathbb{R}$ with zero-mean and fourth moment. Then*

$$Kurt^{\uplus}(\mu) \geq \phi(\mu) \tag{3.8}$$

with equality if and only if $\mu = \mathbb{B}_t(PS(-3/2, \sigma^2)$ for some $t > 0$. Moreover

$$\psi(\mu) = \psi_t(\mu) = Kurt^{\uplus}(\mathbb{B}_t(\mu)) - \phi(\mathbb{B}_t(\mu)) \geq 0 \tag{3.9}$$

depends only on μ.

Proof. Using (3.6) and (3.7) we have that for each $t > 0$

$$Kurt^{\uplus}(\mu) - \phi(\mu) = Kurt^{\uplus}(\mathbb{B}_t(\mu)) - \phi(\mathbb{B}_t(\mu)).$$

Hence $\psi(\mu) = \psi_t(\mu)$ is independent of t. We would like to prove that $\psi(\mu) \geq 0$. If $\phi(\mu) = 1$, by Proposition 3.5 μ is infinitely divisible with respect to $\boxplus$ and then by Proposition 3.2 we have $Kurt^{\uplus}(\mu) \geq 1$. Hence $\psi(\mu) \geq 0$. If $\phi(\mu) < 1$, let $\nu = \mathbb{B}_{1-\phi(\mu)}(\mu)$. Then by (3.6), $\phi(\nu) = 1$ and hence $\psi(\mu) = \psi(\mathbb{B}_{1-\phi(\mu)}(\mu)) = \psi(v) \geq 0$. If $\phi(\mu) > 1$ we have $\mu = \mathbb{B}_{\phi(\mu)-1}(\nu)$ for some probability measure ν on $\mathbb{R}$. Using (3.6) we have $\phi(\mu) = \phi((\mathbb{B}_{\phi(\mu)-1}(\nu)) = \phi(v) + \phi(\mu) - 1$ and hence $\phi(v) = 1$. This gives $0 \leq \psi(v) = \psi(\mathbb{B}_{\phi(\mu)-1}(\nu) = \psi(\mu)$. Then (3.8) is proved.

On the other hand, it has been shown in [5] that $\phi(PS(-3/2, \sigma^2)) = 0$ and it is well known that $Kurt(PS(-3/2, \sigma^2)) = -2$, hence $Kurt^{\uplus}(PS(-3/2, \sigma^2)) = 0$. Thus, equality in (3.8) holds for $\mu = PS(-3/2, \sigma^2)$ and by (3.9) it also hods for $\mathbb{B}_t(PS(-3/2, \sigma^2)$, for $t > 0$.

Suppose now that $Kurt^{\uplus}(\mu) = \phi(\mu) = p$ for some $p > 0$. From [5, Remark 5.5] we have that there exists ν such that $\mu = \mathbb{B}_p(\nu)$. Using again (3.6), we have that

$$Kurt^{\uplus}(\nu) = Kurt^{\uplus}(\mu) - p = 0. \tag{3.10}$$

This means that $\mu = \mathbb{B}_p(\nu)$ and $Kurt(\nu) = Kurt^{\uplus}(\nu) - 2 = -2$, which can only happen if ν is the symmetric Bernoulli distribution $PS(-3/2, \sigma^2)$. □

Remark 3.8. a) The semicircle and the arcsine distributions satisfy the equality in (3.8). This follows since $\mathbb{B}_{1/2}(PS(-3/2, 1)) = PS(-1, 1)$ and $\mathbb{B}_1(PS(-3/2, 1)) = PS(0, 1)$; see [5].

b) If μ is the classical Gaussian distribution, μ does not satisfy the equality in (3.8). This follows since there does not exist $t > 0$ such that $\mu = \mathbb{B}_t(PS(-3/2, \sigma^2)$, a fact that can be easily proved using (4.13) in Example 4.5 in [5].

4. Infinite Divisibility of PSCD

We now use the kurtosis conditions of the last section to study the infinite divisibility of the power semicircle distributions with respect to different convolutions. We recall that the symmetric Bernoulli distribution $PS(-3/2, \sigma)$ is the Gaussian distribution with respect to $\uplus$; the arcsine distribution $PS(-1, \sigma)$ is the Gaussian distribution with respect to $\rhd$ and $\lhd$; the semicircle distribution $PS(0, \sigma)$ is the free Gaussian and $PS(\infty, \sigma)$ is the classical Gaussian distribution.

We first prove that for $\theta < 0$, the power semicircle distributions are not free infinitely divisible.

Corollary 4.1. *The power semicircle distribution $PS(\theta, \sigma)$ is not infinitely divisible in the free sense if $\theta < 0$.*

Proof. It is enough to consider the case $\sigma = 1$. From Proposition 2.7 we have that

$$ES_\theta^{2k} = \frac{1}{2^{2k}} \frac{(2k)!}{(k)!} \frac{\Gamma(\theta+2)}{\Gamma(\theta+2+k)}.$$

Then, the kurtosis of S_θ is obtained as

$$Kurt(S_\theta) = \frac{ES_\theta^4}{(ES_\theta^2)^2} - 3 = -\frac{3}{(\theta+3)}.$$

Hence for $\theta < 0$, $Kurt(S_\theta) < -1$ and the result follows by Proposition 3.2. □

For $\theta < -1$ the power semicircle distributions are not infinitely divisible in the monotone sense. The result is an immediate consequence of Proposition 3.3.

Corollary 4.2. *The power semicircle distribution $PS(\theta, \sigma)$ is not infinitely divisible in the monotone sense if $\theta < -1$.*

Proof. Follows since $Kurt(S_\theta) = -\frac{3}{(\theta+3)} < -1.5$ for $\theta < -1$. □

In summary we have the following general result. In the theorem below $\circ$ stands for any of the convolutions classical $\star$, free $\boxplus$, monotone $\rhd$ or Boolean $\uplus$.

Theorem 4.3. *Let θ_{g° be the value of the Gaussian distribution $PS(\theta_{g^\circ}, \sigma)$ with respect to the corresponding convolution $\circ$. The power semicircle distribution $PS(\theta, \sigma)$ is not infinitely divisible with respect to the convolution $\circ$ for $\theta < \theta_{g^\circ}$.*

Remark 4.4. We conjecture that for $\theta > 0$, the distribution $P(\theta, 1)$ is free infinitely divisible. This conjecture is based on testing with MATLAB the positive definiteness of a large number of free cumulants of $P(\theta, 1)$.

Remark 4.5. If the above conjecture were true, by Poincaré's theorem the classical Gaussian distribution would be free infinitely divisible. This fact is also supported by using MATLAB for testing the positive definiteness of the free cumulants of the Gaussian distribution. This conjecture has recently been proved to be true in [4]. Moreover, we conjecture that the classical Gaussian distribution is a free multiplicative convolution of the semicircle distribution in the sense of [3], [40].

Remark 4.6. By Theorem 3.7 and since $Kurt(S_\theta) = -\frac{3}{(\theta+3)} < 0$ for all θ, even if $PS(\theta, \sigma)$ were free infinitely divisible, it would not belong to $\mathbb{B}_t(\mathcal{P})$ for $t \geq 2$. Namely, $PS(\theta, \sigma)$ would not be the image of a free infinitely divisible measure under the Boolean Bercovici-Pata bijection $\mathbb{B}$.

We conclude with the following strict bound for the free divisibility indicator of the classical Gaussian distribution. Its proof follows from Theorem 3.7 and Remarks 3.8 (b) and 4.5.

Theorem 4.7. *Let μ be the classical Gaussian distribution. Then $1 \leq \phi(\mu) < 2$ and does not exist a free infinitely divisible distribution ν on $\mathbb{R}$ such that $\mu = \mathbb{B}(\nu)$.*

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Octavio Arizmend: Departament of Mathematics and Statistics, Queen's University, Jeffery Hall, Kingstom, ON K7L 3N6, Canada
E-mail address: octarech@gmail.com

Víctor Pérez-Abreu: Centro de Investigación en Matemáticas, A.C., Apdo. Postal 402, Guanajuato, 36000 Gto. Mexico
E-mail address: pabreu@cimat.mx