Random Matrices and Eigenvalues Process

Free Probability, Random Matrices and Infinite Divisibility

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Plan of the Lecture

1. Review Lecture I
   1.1 Asymptotic spectral distributions of random matrices.
   1.2 Free asymptotics and free convolution of measures.
2. Free and Matrix-valued Brownian Motions.
3. Dyson Brownian Motion and its Measure-valued Process.
4. Functional Limit Theorems for Traces
   4.1 Law of large numbers.
   4.2 Central limit theorem.
5. Wishart process and Marchenko-Pastur Law.
6. Towards Lecture 3

I. Ensembles of Gaussian random matrices

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- Density of eigenvalues of $\lambda_{n,1}, \ldots, \lambda_{n,n}$ of Z_n:

$$f_{\lambda_{n,1},\ldots,\lambda_{n,n}}(x_1,\ldots,x_n) = k_n \left[\prod_{j=1}^{n} \exp\left(-\frac{1}{4t}x_j^2\right)\right] \left[\prod_{j<k} |x_j - x_k|\right].$$

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- Nondiagonal RM: eigenvalues are strongly dependent due to Vandermont determinant: $x = (x_1, \dots, x_n) \in \mathbb{C}^n$

$$\Delta(x) = \det\left(\left\{x_j^{k-1}\right\}_{j,k=1}^{n}\right) = \prod_{j<k} (x_j - x_k).$$

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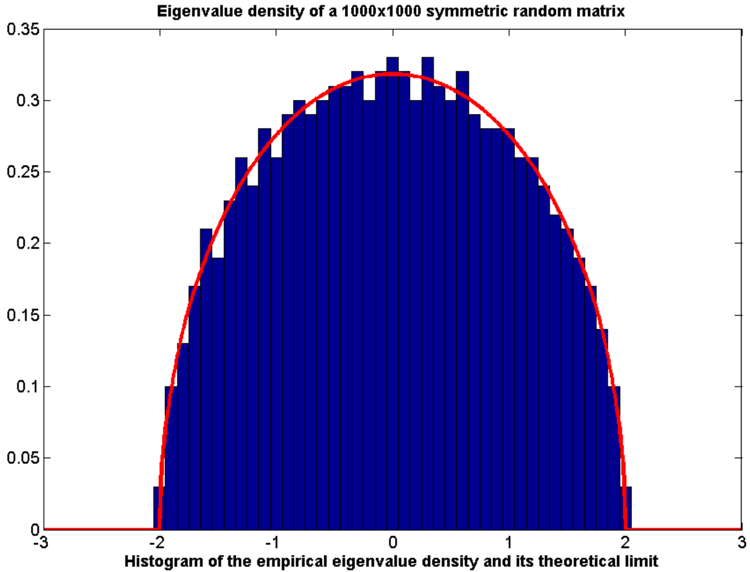
- **Asymptotic spectral distribution (ASD):** $\widehat{F^t}_n$ converges, as $n \longrightarrow \infty$, to **semicircle distribution on** $(-2\sqrt{t}, 2\sqrt{t})$

$$w_t(x) = \frac{1}{2\pi}\sqrt{4t - x^2}, \quad |x| \leq 2\sqrt{t}.$$

I. Simulation of Wigner law



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Theorem

$t > 0$. $\forall\ f \in C_b(\mathbb{R})$ *and* $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\int f(x)\mathrm{d}\widehat{F^t}_n(x) - \int f(x)\mathrm{w}_t(\mathrm{d}x)\right| > \varepsilon\right) = 0.$$

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- **Universality**. Law holds for **Wigner random matrices**:

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$\{Z_{j,k}\}_{j\le k}, \{Y_j\}_{j\ge 1}$ independent sequences of i.i.d. r.v.
$\mathbb{E}Z_{1,2} = \mathbb{E}Y_1 = 0, \mathbb{E}Z_{1,2}^2 = 1.$

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- Convergence of extreme eigenvalues as $n \to \infty$

$$\mathbb{P}(\lambda_{n,n} \to 2\sqrt{t}) = \mathbb{P}(\lambda_{n,1} \to -2\sqrt{t}) = 1.$$

Idea of a proof of Wigner theorem

- Basic observation

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- *Catalan numbers*

$$C_k = \frac{1}{k+1}\binom{2k}{k},\ k \geq 1.$$

I. Review: Asymptotically free random matrices

- For an ensemble of Hermitian random matrices $\mathbf{X} = (X_n)_{n\geq 1}$ define "expectation" τ as the linear functional τ, $(\tau(\mathbf{I}) = 1)$

$$\tau(\mathbf{X}) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\mathrm{tr}(X_n)\right].$$

- Hermitian ensembles $\mathbf{X}_1$ & $\mathbf{X}_2$ are *asymptotically free* (AF) if $\forall\ r \in Z_+$ & polynomials $p_i(\cdot)$, $q_i(\cdot)$, $1 \leq i \leq r$ with

$$\tau(p_i(\mathbf{X}_1)) = \tau(q_i(\mathbf{X}_2)) = 0,$$

we have

$$\tau(p_1(\mathbf{X}_1)q_1(\mathbf{X}_2)...p_r(\mathbf{X}_1)q_r(\mathbf{X}_2)) = 0.$$

- Examples:
  - If $\mathbf{X}_1$ and $\mathbf{X}_2$ are *independent Wigner ensembles*, they are AF.
  - If $\mathbf{X}_1$ and $\mathbf{X}_2$ are *independent GOE*(t), they are AF.

I Review: classical and free convolutions

- Fourier transform of probability measure μ on $\mathbb{R}$

$$\widehat{\mu}(s) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sx} \mu(\mathrm{d}x), \quad s \in S.$$

- Cauchy transform of μ

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - x} \mu(\mathrm{d}x), \quad z \in \mathbb{C}/\mathbb{R}.$$

- Classical cumulant transform

$$c_\mu(s) = \log \widehat{\mu}(s), \quad s \in \mathbb{R}.$$

- Free cumulant transform

$$C_\mu(z) = zG_\mu^{-1}(z) - 1, \quad z \in \Gamma_\mu$$

I. Review: classical and free convolutions

- Classical convolution $\mu_1 * \mu_2$ is defined by

$$c_{\mu_1 * \mu_2}(s) = c_{\mu_1}(s) + c_{\mu_2}(s).$$

- If X_1 & X_2 are classical independent random variables with distributions μ_1 & μ_2, $X_1 + X_2$ has distribution $\mu_1 * \mu_2$.
- Free convolution $\mu_1 \boxplus \mu_2$ is defined by

$$C_{\mu_1 \boxplus \mu_2}(z) = C_{\mu_1}(z) + C_{\mu_2}(z), \quad z \in \Gamma_{\mu_1} \cap \Gamma_{\mu_2}.$$

- If $\mathbf{X}_1$ & $\mathbf{X}_2$ AF ensembles of random matrices with ASD μ_1 and μ_2, then the ASD of $\mathbf{X}_1 + \mathbf{X}_2$ is $\mu_1 \boxplus \mu_2$.

I. Free convolution of Wigners

Towards the free Brownian motion

- *Semicircle distribution* w_{m,σ^2} *on* $(m-2\sigma, m+2\sigma)$ *centered at* m

$$w_{m,\sigma^2}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-(x-m)^2}1_{[m-2\sigma,m+2\sigma]}(x).$$

- Free cumulant transform:

$$C_{\mathrm{w}_{m,\sigma^2}}(z) = mz + \sigma^2 z^2.$$

- $\boxplus$*-convolution of Wigner distributions is a Wigner distribution*:

$$\mathrm{w}_{m_1,\sigma_1^2} \boxplus \mathrm{w}_{m_2,\sigma_2^2} = \mathrm{w}_{m_1+m_2,\sigma_1^2+\sigma_2^2}.$$

- Of special interest: free Brownian motion

$$\mathrm{w}_t = \mathrm{w}_{0,t},\ t \geq 0.$$

II. Free Brownian motion

- *Law of Free Brownian motion*

$$\mathrm{w}_t(\mathrm{d}x) = \frac{1}{2\pi t}\sqrt{4t^2 - x}1_{[-2\sqrt{t},2\sqrt{t}]}(x)\mathrm{d}x.$$

- Free cumulant transform

$$C_{\mathrm{w}_t}(z) = tz^2.$$

- In law:
  - $\mathrm{w}_0 = \delta_0$
  - "Stationary increments": distribution w_{t-s} depends on $t - s$.
  - "Independent increments": $0 < t_1 < t_2$

$$\mathrm{w}_{t_2-t_1} \boxplus \mathrm{w}_{t_1} = \mathrm{w}_{t_2}.$$

- Realization for free Brownian motion?

II. Matrix Brownian motion

- $n \times n$ symmetric matrix valued Brownian motion

$$B_n(t) = (b_{ij}(t)), t \geq 0,$$

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- $(\{B_n(t)\}_{t \geq 0})_{n \geq 1}$ is realization of free Brownian motion.

III. Dyson-Brownian process

- *Fix* $n > 0$, and consider $n \times n$ Hermitiam matrix Brownian motion $B_n(t) = (b_{ij}(t)), t \geq 0$,
  $\mathrm{Re}(b_{ij}(t)) \sim \mathrm{Im}(b_{ij}(t)) \sim N(0, t(1+\delta_{ij})/\,(2n))$,
  $\mathrm{Re}(b_{ij}(t)), \mathrm{Im}(b_{ij}(t)), 1 \leq i \leq j \leq n$ independent.
- $(\lambda_{n,1}(t), \cdots, \lambda_{n,n}(t)), t \geq 0$, eigenvalues process of $B_n(t/n)$.

Theorem

(Dyson, 1962) Consider Hermitian matrix Brownian motion. There exist n independent 1-dimensional standard Brownian motions $b_1^{(n)}(t), \dots, b_n^{(n)}(t)$ such that if $\lambda_{n,1}(0) < \cdots < \lambda_{n,n}(0)$ a.s.

$$\lambda_{n,i}(t) = \lambda_{n,i}(0) + \frac{1}{\sqrt{n}} b_i^{(n)}(t) + \frac{1}{n} \sum_{j \neq i} \int_0^t \frac{1}{\lambda_{n,j}(s) - \lambda_{n,i}(s)} \mathrm{d}s,$$

- **Brownian part** plus **noncollinding part.**
- $\mathbb{R}^d$-valued SDE with **non smooth** drift.
- For now on we consider Hermitian matrix Brownian motion.

III. The associated measure valued processes

Cabanal-Duvillard and Guionnet(01), PA and Tudor (07)

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- Notation: If f is μ-integrable function

$$\langle \mu, f \rangle = \int f(x)\mu(\mathrm{d}x).$$

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$$\mu_t^{(n)} = \widehat{F^t}_n(x) = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_{n,j}(t)}, \quad t \geq 0.$$

- Notation: If f is μ-integrable function

$$\langle \mu, f \rangle = \int f(x)\mu(\mathrm{d}x).$$

- Functional Law of large numbers: $\forall\ f \in C_b(\mathbb{R})$

$$\mathbb{P}\left(\lim_{n\to\infty} \sup_{0\leq t\leq T} \left| \left\langle \mu_t^{(n)}, f \right\rangle - \langle \mathrm{w}_t, f \rangle \right| = 0\right) = 1.$$

III. The associated measure valued processes

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- $\forall T > 0$

$$\sup_{0\leq t\leq T} \lambda_{n,n}(t) \xrightarrow[n\to\infty]{a.s.} 2\sqrt{T}, \quad \inf_{0\leq t\leq T} \lambda_{n,1}(t) \xrightarrow[n\to\infty]{a.s.} -2\sqrt{T}.$$

III. The associated measure valued processes

Cabanal-Duvillard and Guionnet (01), PA and Tudor (07)

Notation: $C\left(\mathbb{R}_+, \mathcal{P}(\mathbb{R})\right)$ continuous functions from $\mathbb{R}_+ \to \mathcal{P}(\mathbb{R})$, with topology of uniform convergence on compact intervals of $\mathbb{R}_+$.

Theorem

If $\mu_0^{(n)} \to \delta_0$, the family $\left(\mu_t^{(n)}\right)_{t\geq 0}$ of measure valued-processes converges weakly in $C\left(\mathbb{R}_+, \mathcal{P}(\mathbb{R})\right)$ to unique continuous probability-measure valued function such that $\forall\ f \in C_b^2(\mathbb{R})$

$$\langle \mu_t, f \rangle = f(0) + \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y).$$

Moreover, $\mu_t = \mathrm{w}_t$, $t \geq 0$.

III. Key tools for the proof

- $m_r(t)$ r-moment of w_t and Cauchy transform $G_t = -G_{\mathrm{w}_t}$.
- *For each* $r \geq 2$ *and* $t > 0$

$$m_r(t) = \frac{r}{2} \sum_{j=0}^{r-2} \int_0^t m_{r-2-j}(s) m_j(s) \mathrm{d}s.$$

- $(\mathrm{w}_t)_{t \geq 0}$ is characterized by its Cauchy transforms being unique solution of

$$\frac{\partial G_t(z)}{\partial t} = G_t(z) \frac{\partial G_t(z)}{\partial z}, \quad t > 0$$
$$G_0(z) = -\frac{1}{z}, \quad z \in \mathbb{C}^+,$$

$G_t(z) \in \mathbb{C}^+$ for $z \in \mathbb{C}^+$ & $\lim_{\eta \to \infty} \eta \, |G_t(i\eta)| < \infty \; \forall \; t > 0$.

III. Asymptotic Fluctuations

Smooth vs. non smooth interacting SDE

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- Interacting SDEs with **non smooth drift coefficient** arise naturally in the study of eigenvalue processes of matrix-valued stochastic processes [Bru (89), Rogers & Shi (93), Chan (97), Konig & O´Connell (01), Katori & Tanemura (04)].

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Central limit theorem for Dyson measure valued process

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- Israelson (01), Bender (09): $Y_n(t)$ converges weakly to a Gaussian process in the dual of a nuclear Fréchet space.
- PA & Tudor (07): Propagation of chaos & fluctuations of traces processes $(\{M_{n,p}(t)\}_{t\geq 0}, n \geq 1), p \geq 0$, given by the semimartingales

$$M_{n,p}(t) = \frac{1}{n}\mathrm{tr}([B_n(t)]^p) = \int_{\mathbb{R}} x^p \mu_t^{(n)}(\mathrm{d}x) = \frac{1}{n}\sum_{j=1}^{n}[\lambda_{n,j}(t)]^p$$

and fluctuations of moments processes

$$V_{n,p}(t) = \int x^p Y_t^{(n)}(\mathrm{d}x) = n\left(M_{n,p}(t) - m_p(t)\right).$$

IV. Asymptotics for traces processes

Almost sure and k mean convergence

- The martingales, $p \geq 0$ & $n \geq 1$,

$$X_{n,p}(t) = \frac{1}{n^{3/2}} \sum_{j=1}^{n} \int_0^t [\lambda_{n,j}(s)]^p \, \mathrm{d}b_j^{(n)}(s), \, t \geq 0,$$

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$$M_{n,r}(t) = M_{n,r}(0) + rX_{n,r-1}(t) + \frac{r}{2} \sum_{j=0}^{r-2} \int_0^t M_{n,r-2-j}(s) M_{n,j}(s) \mathrm{d}s$$

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- Under conditions on $M_{n,2p}^{2k}(0)$,

$$\sup_{0 \leq t \leq T} |M_{n,2p}(t) - m_{2p}(t)| \xrightarrow{a.s.} 0 \text{ as } n \to \infty,$$

$$\mathbb{E} \sup_{0 \leq t \leq T} |M_{n,2p}(t) - m_{2p}(t)|^{2k} \longrightarrow 0 \text{ as } n \to \infty.$$

IV. Asymptotics of associated martingales

PA & Tudor (07)

- $$R_{n,p}(t) = nX_{n,p}(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int_0^t [\lambda_{n,j}(s)]^p \, \mathrm{d}b_j^{(n)}(s), \quad t \geq 0, p \geq 0.$$

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- $R_{n,p}$ converges weakly in $C(\mathbb{R}_+, \mathbb{R})$, as $n \to \infty$, to a centered Gaussian martingale R_p with covariance function
$$E\left(R_p(s)R_p(t)\right) = \frac{C_p}{p+1}\left(s \wedge t\right)^{p+1}$$
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- Limiting process R_p is $\frac{p+1}{2}$-self-similar Gaussian process with independent increments
$$R_p(t) = C_p^{\frac{1}{2}} \int_0^t s^{\frac{p}{2}} \mathrm{d}b_s,$$

IV. CLT for traces processes

Israelson (01), PA & Tudor (07)

- Under assumptions on $V_{n,p}^{2k}(0)$, $V_{n,p}$ converges weakly in $C(\mathbb{R}_+, \mathbb{R})$ to centered Gaussian process Z_p satisfying $Z_0 = 0$,

$$Z_p(t) + p\int_0^t Z_p(s)\mathrm{d}s = V_p^{(0)} + \frac{p}{2}\int_0^t \{2\left[m_{p-2}(s) + m_{p-3}(s)Z_1(s) + \ldots + m_1(s)Z_{p-3}(s)\right] + Z_{p-2}(s)\}\,\mathrm{d}s + pR_{p-1}(t).$$

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- Alternative expression for Z_p:

$$Z_p(t) = a_{p-1}(t) - p\int_0^t e^{-p(t-s)}a_{p-1}(s)\mathrm{d}s.$$

$$a_p(t) = V_p^{(0)} + \frac{p+1}{2}\int_0^t \{2\,[m_{p-1}(s) + m_{p-2}(s)Z_1(s) + \ldots + m_1(s)Z_{p-2}(s)] + Z_{p-1}(s)\}\,ds + (p+1)R_p(t),$$

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- $\exists$ measurable deterministic Volterra kernel K_p such that

$$Z_p(t) = \int_0^t K_p(t,s)db_s.$$

V. Wishart process

- $m, n \geq 1$, $\{B_{m,n}(t)\}_{t\geq 0} = \left\{ \left(b^{j,k}_{m,n}(t) \right)_{1\leq j\leq m, 1\leq k\leq n} \right\}_{t\geq 0}$, $\left\{ \mathrm{Re}\left(b^{j,k}_{m,n}(t) \right) \right\}_{t\geq 0}$ & $\left\{ \mathrm{Im}\left(b^{j,k}_{m,n}(t) \right) \right\}_{t\geq 0}$ independent unidimensional Brownian motions, $\mathrm{Re}\left(b^{j,k}_{m,n}(t) \right) \sim \mathrm{Im}\left(b^{j,k}_{m,n}(t) \right) \sim N(0, (1+\delta_{jk})/(2t))$.
- Laguerre or Wishart process: $n \times n-$matrix-valued process

$$L_{m,n}(t) = B^*_{m,n}(t) B_{m,n}(t), t \geq 0.$$

- Bru (89), Graczyk (11): For eigenvalue of $L_{m,n}(t)/(2n)$

$$\mathrm{d}\lambda_j^{(m,n)}(t) = \sqrt{\frac{2\lambda_j^{(m,n)}(t)}{n}} \mathrm{d}b_j^{(m,n)}(t)$$

$$+ \frac{1}{n} \left(m + \sum_{k\neq j} \frac{\lambda_j^{(m,n)}(t) + \lambda_k^{(m,n)}(t)}{\lambda_j^{(m,n)}(t) - \lambda_k^{(m,n)}(t)} \right) \mathrm{d}t, \; 1 \leq j \leq n.$$

- PA & Tudor (09): Measure valued process & traces.

V. Brownian vs. Wishart case

- Which law plays the role of $\{\mathrm{w}_t\}_{t\geq 0}$ for measure process

$$\mu_t^{(m,n)} = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j^{(m,n)}(t)}, t \geq 0 \ ?$$

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- Not the *Free Poisson* or *Marchenko-Pastur law* m_c, $c > 0$,

$$\mathrm{m}_c(\mathrm{d}x) = \begin{cases} f_c(x)\mathrm{d}x, & c \geq 1 \\ (1-c)\delta_0(\mathrm{d}x) + f_c(x)\mathrm{d}x, & c < 1, \end{cases}$$

$$f_c(x) = \frac{c}{2\pi x}\sqrt{(x-a)(b-x)}\mathbf{1}_{[a,b]}(x)$$
$$a = (1-\sqrt{c})^2, \;\; b = (1+\sqrt{c})^2.$$

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- Rather dilations $\left\{\mu_c(t) = \mathrm{m}_c \circ h_t^{-1}\right\}_{t\geq 0},\ h_t(x) = tx$,

$$\mu_c(t)(dx) = \left\{ \begin{array}{cc} f_c^t(x)\mathrm{d}x, & c \geq 1 \\ (1-c)\,\delta_0(dx) + f_c^t(x)\mathrm{d}x, & c < 1 \end{array} \right. ,$$

$$f_c^t(x) = \frac{\sqrt{(x-at)\,(bt-x)}}{2\pi tx} 1_{(at,bt)}(x).$$

V. Marchenko-Pastur law (1967)

Universality

- $X = X_{m\times n} = (Z_{j,k} : j = 1, .., n, k = 1, ..., m)$ complex i.i.d. $\mathbb{E}(Z_{1,1}) = 0, \mathbb{E}(|Z_{1,1}|^2) = 1$.
- $W_n = X^*X$ is **Wishart matrix** if X has Gaussian entries.
- $S_n = \frac{1}{n}X^*X$, eigenvalues $0 \le \lambda_{n,1} \le ... \le \lambda_{n,n}$ & ESD

$$\widehat{F}_n(\lambda) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{\{\lambda_{n,j}\le x\}}.$$

- If $n/m \to c > 0$, $\widehat{F}_n$ converges weakly in probability to Marchenko-Pastur (MP) distribution

$$\mathrm{m}_c(\mathrm{d}x) = \begin{cases} f_c(x)\mathrm{d}x, & \textit{if} \quad c \ge 1 \\ (1-c)\delta_0(\mathrm{d}x) + f_c(x)\mathrm{d}x, & \textit{if} \quad 0 < c < 1, \end{cases}$$

$$f_c(x) = \frac{c}{2\pi x}\sqrt{(x-a)(b-x)}\mathbf{1}_{[a,b]}(x)$$
$$a = (1-\sqrt{c})^2, \quad b = (1+\sqrt{c})^2.$$

V. Marchenko-Pastur law

1. **Applications: Large Dimensional RM (LDRM):**

- Data dimension of same magnitude order than sample size. Bai & Silverstein (2010). Spectral Analysis of LDRM
- Wireless communication, MIMO channels. Couillet & Debbah (2011). RM Methods for Wireless Comm.

2. **Towards next Monday lecture:**

- $(N_t)_{t\geq 0}$ Poisson process of mean m, $(u_j)_{j\geq 1}$ i.i.d. random vectors with uniform distribution on unit sphere of $\mathbb{C}^n$.
- $n \times n$ matrix compound Poisson process

$$X_t = \sum_{j=1}^{N_t} u_j^* u_j.$$

- Distribution of X_t is invariant under unitary conjugations.
- ASD of X_t, when $n/m \to c$, is MP with parameter c.

3. **Open problem:** Measure-valued process for X_t?

V. Example: Communication Channel Capacity

Circularly symmetric complex Gaussian random matrices

A $p \times 1$ complex random vector $\mathbf{u}$ has a *Q-circularly symmetric complex Gaussian distribution* if

$$\mathbb{E}[(\mathbf{u} - \mathbb{E}[\mathbf{u}])(\mathbf{u} - \mathbb{E}[\mathbf{u}])^*] = \frac{1}{2} \begin{bmatrix} \mathrm{Re}[Q] & -\mathrm{Im}[Q] \\ \mathrm{Im}[Q] & \mathrm{Re}[Q] \end{bmatrix},$$

.for some nonnegative definite Hermitian $p \times p$ matrix Q.

$$\mathbf{u} = [\mathrm{Re}(u_1) + i\,\mathrm{Im}(u_1), ..., \mathrm{Re}(u_p) + i\,\mathrm{Im}(u_p)]^\top.$$

V. Example: Communication Channel Capacity

A Model for MIMO antenna systems

- n_T antennas at transmitter and n_R antennas at receiver
- Linear vector channel with Gaussian noise

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

- $\mathbf{x}$ is the n_T-dimensional *input vector*.
- $\mathbf{y}$ is the n_R-dimensional *output vector*.
- $\mathbf{n}$ is the received Gaussian *noise*, zero mean and $\mathbb{E}\left(\mathbf{n}\mathbf{n}^*\right) = \mathrm{I}_{n_T}$.
- The $n_R \times n_T$ random matrix $\mathbf{H}$ is the *channel matrix.*
- $\mathbf{H} = \{h_{jk}\}$ is a random matrix, it models the propagation coefficients between each pair of trasmitter-receiver antennas.
- $\mathbf{x}, \mathbf{H}$ and $\mathbf{n}$ are independent.

V. Example: Communication Channel Capacity

Raleigh fading channel

- h_{jk} are i.i.d. complex random variables with mean zero and variance one $(\mathrm{Re}(Z_{jk}) \sim N(0, \frac{1}{2})$ independent of $\mathrm{Im}(Z_{jk}) \sim N(0, \frac{1}{2}))$.

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- *Total power constraint* P is the upper bound of the variance $\mathbb{E}||\mathbf{x}||^2$ of the amplitude of the input signal.
- *Channel capacity* is the maximum data rate which can be transmitted reliably over a channel (Shannon (1948)).

V. Example: Communication Channel Capacity

- h_{jk} are i.i.d. complex random variables with mean zero and variance one ($\text{Re}(Z_{jk}) \sim N(0, \frac{1}{2})$ independent of $\text{Im}(Z_{jk}) \sim N(0, \frac{1}{2})$).
- $\mathbf{x}$ has $Q-$circularly symmetric complex Gaussian distribution.
- Signal to Noise Ratio

$$SNR = \frac{\mathbb{E}||\mathbf{x}||^2 / n_T}{\mathbb{E}||\mathbf{n}||^2 / n_R} = \frac{P}{n_T}.$$

- *Total power constraint* P is the upper bound of the variance $\mathbb{E}||\mathbf{x}||^2$ of the amplitude of the input signal.
- *Channel capacity* is the maximum data rate which can be transmitted reliably over a channel (Shannon (1948)).
- The capacity of this MIMO system channel is

$$C(n_R, n_T) = \max_{Q} \mathbb{E}_{\mathbf{H}} \left[\log_2 \det \left(\mathrm{I}_{n_R} + \mathbf{H} Q \mathbf{H}^* \right) \right]$$

V. Example: Communication Channel Capacity

Raleigh fading channel

- Maximum capacity when $Q = SNR\mathrm{I}_{n_T}$

$$C(n_R, n_T) = \mathbb{E}_{\mathbf{H}}\left[\log_2 \det\left(\mathrm{I}_{n_R} + \frac{P}{n_T}\mathbf{H}\mathbf{H}^*\right)\right]$$

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- By Marchenko-Pastur theorem, if $n_R/n_T \to c$,

$$\frac{C(n_R, n_T)}{n_R} \to \int_a^b \log_2(1 + cPx)\,\mathrm{d}\mu_c(x) = K(c, P).$$

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- *Increase capacity with more transmitter and receiver antennas without increasing the total power constraint* P.

- Arnold, L.: On the asymptotic distribution of the eigenvalues of random matrices, *J. Math. Anal. Appl.* **20** (1967), 262-268.
- Bai Z.D.: Methodology in spectral analysis of large dimensional random matrices, *Statistica Sinica* **9** (1999), 611-677.
- Bai Z.D., Yin Y.Q.: Necessary and sufficient conditions for the almost sure convergence of the largest eigenvalue of a Wigner matrix, *Ann. Probab.* **16** (1988), 1729–17401294.
- Bender M.: Global fluctuations in general β Dyson Brownian motion, *Stoch. Proc. Appl.* (2009).
- Bru M.F.: Diffusions of perturbed principal component analysis, *J. Multivariate Anal.* **29** (1989), 127-136.
- Cépa E., Lepingle D.: Diffusing particles with electrostatic repulsion, *Probab. Theory Relat. Fields* **107** (1997), 429-449.

- Chan T.: The Wigner semicircle law and eigenvalues of matrix-valued diffusions, *Probab. Theory Relat. Fields* **93** (1992), 249-272.
- Dyson F.J.: A Brownian-motion model for the eigenvalues of a random matrix, *J. Math. Phys.* **3** (1962), 1191-1198.
- Hitsuda H., Mitoma I.: Tightness problem and stochastic evolution equations arising from fluctuation phenomena for interacting diffusions, *J. Multivariate Anal.* **19** (1986), 311-328.
- Israelson S.: Asymptotic fluctuations of a particle system with singular interaction, *Stoch. Process. Appl.* **93** (2001), 25-56.
- Kallianpur, G., Perez-Abreu, V.: Stochastic evolution equations driven by nuclear space valued martingales, *Appl. Math. Optim.* **17** (1988), 237-272.
- Katori, M., Tanemura, K.: Symmetry of matrix-valued stochastic processes and noncolliding diffusion particle systems, *J. Math. Phys.* **45** (2004), 3058-3085.

McKean H.P.: Propagation of chaos for a class of nonlinear parabolic equations, in: *Lecture Series in Differential Equations* **2**, pp 177-193. Van Nostrand Math. Studies **19**. New York, 1967.

Mitoma I.: An ∞-dimensional inhomogeneous Langevin equation, *J. Funct. Anal.* **61** (1985), 342-359.

Pérez-Abreu, V. Tudor, C. (2007): Functional limit theorems for trace processes in a Dyson Brownian motion. *Comm. Stochast. Anal.* **1**, 415-428.

Pérez-Abreu, V. Tudor, C. (2009): On traces of Laguerre processes. *Elect. J. Probab.*

Rogers L.C.G., Shi Z.: Interacting Brownian particles and the Wigner law, *Probab. Theory Relat. Fields* **95** (1993), 555-570.

Sznitman A.S.: A fluctuation result for nonlinear diffusions, in: *Infinite Dimensional Analysis*. S. Albeverio (Ed.), pp 145-160. Pitman, Boston, 1985.

Sznitman A.S.: Topics in propagation of chaos, in: *Ecole d'Eté*