A NOTE ON THE TWO-DIMENSIONAL OPERATOR WISHART AND LAGUERRE PROCESSES

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We consider the stochastic differential equations satisfied by the engenvalues of the operator Wishart and Laguerre processes. They are governed by a system of diffusions governed by Brownian motions that are not independent. It is shown that their traces are Bessel processes if and only if the corresponding operator processes are standard.

AMS 2010 Subject Classification: 60K35, 60F05.

Key words: matrix valued process, trace process, Bessel process, operator process, eigenvalue process, Dyson-Brownian motion.

1. INTRODUCTION

For $m, n \geq 1$, let $\{B_{m,n}(t)\}_{t\geq 0} = \{(B^{j,k}_{m,n}(t)); 1 \leq j \leq m, 1 \leq k \leq n\}_{t\geq 0}$ be a $m \times n$ real (resp. complex) Brownian motion; that is, the entries $(B^{j,k}_{m,n}(t))_{t\geq 0}$ (resp. $\{\mathrm{Re}(B^{j,k}_{m,n}(t))\}_{t\geq 0}$ and $\{\mathrm{Im}(B^{j,k}_{m,n}(t))\}_{t\geq 0}$) are independent one-dimensional real Brownian motions.

The continuous $n \times n$-matrix-valued process $L_{m,n}(t) = B^*_{m,n}(t)B_{m,n}(t)$, $t \geq 0$ is known as *Wishart process* (resp. *Laguerre process* or *complex Wishart process*) of size n, of dimension m and starting from $L_{m,n}(0) = B^*_{m,n}(0)B_{m,n}(0)$. For $n = 1$, $L_{m,1}(t)$ is a squared Bessel process.

Let $\{\lambda^{(m,n)}(t)\}_{t\geq 0} = \{(\lambda^{(m,n)}_1(t), \lambda^{(m,n)}_2(t), \ldots, \lambda^{(m,n)}_n(t))\}_{t\geq 0}$ be the n-dimensional stochastic process of eigenvalues of $L_{m,n}(t)$.

In the case of real Wishart processes and $m > n - 1$, Bru [1] proved that if the eigenvalues start at different positions

$$0 \leq \lambda^{(m,n)}_1(0) < \lambda^{(m,n)}_2(0) < \cdots < \lambda^{(m,n)}_n(0), \tag{1.1}$$

then they never meet at any time

$$0 \leq \lambda^{(m,n)}_1(t) < \lambda^{(m,n)}_2(t) < \cdots < \lambda^{(m,n)}_n(t) \text{ a.s. } \forall t > 0.$$

Furthermore, they are governed by a diffusion process satisfying the Itô Stochastic Differential Equation (ISDE),

$$\mathrm{d}\lambda_j^{(m,n)}(t) = 2\sqrt{\lambda_j^{(m,n)}(t)}\mathrm{d}\widetilde{B}_j^{(m,n)}(t) + \tag{1.2}$$

$$+\left(m + \sum_{k\neq j}\frac{\lambda_j^{(m,n)}(t)+\lambda_k^{(m,n)}(t)}{\lambda_j^{(m,n)}(t)-\lambda_k^{(m,n)}(t)}\right)\mathrm{d}t, \quad t\geq 0,\ 1\leq j\leq n,\ m>n-1;$$

the same holds in the case of Laguerre processes (with the same arguments as in the real case) and the ISDE's has the form

$$\mathrm{d}\lambda_j^{(m,n)}(t) = 2\sqrt{\lambda_j^{(m,n)}(t)}\mathrm{d}\widetilde{B}_j^{(m,n)}(t) + \tag{1.3}$$

$$+2\left(m + \sum_{k\neq j}\frac{\lambda_j^{(m,n)}(t)+\lambda_k^{(m,n)}(t)}{\lambda_j^{(m,n)}(t)-\lambda_k^{(m,n)}(t)}\right)\mathrm{d}t, \quad t\geq 0,\ 1\leq j\leq n,\ m>n-1,$$

where $\widetilde{B}_1^{(m,n)},\ldots,\widetilde{B}_n^{(m,n)}$ are independent one-dimensional standard Brownian motions (see for example [3], [4], [5], [9], [10], [11]).

A special feature of the systems of ISDEs (1.2) and (1.3) is that they have non smooth drift coefficients and the eigenvalues processes do not collide.

We denote by Tr is the usual unnormalized trace and $\mathrm{tr} = \frac{1}{n}\mathrm{Tr}$ is the normalized trace.

Let W be an $n\times n$-symmetric (Hermitian) non random positive definite matrix with positive eigenvalues $(\theta_j)_{1\leq j\leq n}$.

The continuous $n\times n$-matrix-valued process $L(t) = WB^*(t)B(t)W^*$, $t\geq 0$ is called *operator Wishart process* (resp. *operator Laguerre process* or *operator complex Wishart process*) of size n and dimension m and starting from $L(0) = WB^*(0)B(0)W^*$.

The purpose of this note is to study the stochastic differential equations satisfied by the eigenvalues of the operator two-dimensional Wishart and Laguerre processes. It is shown that the Brownian motions in the system of diffusions governing the eigenvalues processes are not independent. It is also shown that their traces are Bessel processes if and only if $W = I$. It is conjectured that similar results hold for higher dimensions.

2. MAIN RESULTS

Real case. We consider first the case of operator real Wishart processes.

THEOREM 1. *Let* $\{\lambda_1(t),\lambda_2(t)\}_{t\geq 0}$ *be the eigenvalues of the operator Wishart process* $L(t) = WB^*(t)B(t)W^*$, $t\geq 0$, *and assume that* $\lambda_1(0) > \lambda_2(0)$.

Then at every time $t > 0$ the eigenvalues $\lambda_1(t), \lambda_2(t)$ are distinct and satisfy the following system of stochastic differential equations

$$\begin{aligned} d\lambda_1(t) &= \sqrt{2\,(\mathrm{tr}W^2)\,\lambda_1(t)}\,\big(d\hat{B}_1(t) + d\hat{B}_2(t)\big)+ \\ &+\left\{\big(\mathrm{tr}W^2 + \theta_1^2 - \theta_2^2\big)\frac{\lambda_1(t)+\lambda_2(t)}{\lambda_1(t)-\lambda_2(t)} + 2\mathrm{tr}W^2\right\} dt, \end{aligned} \tag{2.1}$$

$$\begin{aligned} d\lambda_2(t) &= \sqrt{2\,(\mathrm{tr}W^2)\,\lambda_2(t)}\,\big(d\hat{B}_1(t) - d\hat{B}_2(t)\big)+ \\ &+\left\{\big(\mathrm{tr}W^2 + \theta_1^2 - \theta_2^2\big)\frac{\lambda_1(t)+\lambda_2(t)}{\lambda_2(t)-\lambda_1(t)} + 2\mathrm{tr}W^2\right\} dt, \end{aligned} \tag{2.2}$$

where $\hat{B}_1, \hat{B}_2$ are Brownian motions with

$$d\big[\hat{B}_1, \hat{B}_2\big]_t = \big(\theta_1^2 - \theta_2^2\big)\frac{\lambda_1(t)+\lambda_2(t)}{\lambda_1(t)-\lambda_2(t)}dt. \tag{2.3}$$

Remark 2. If $\theta_1 = \theta_2$ then $d\big[\hat{B}_1, \hat{B}_2\big]_t = 0$ and therefore $\hat{B}_1, \hat{B}_2$ are independent Brownian motions and the above result becomes the known one for real Wishart processes in the two-dimensional case.

Proof of Theorem 1. Let H be an orthogonal matrix such that $HDH^* = W$ with $D = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}$.

Then $\tilde{B}(t) = B(t)H$ is a Brownian motion and $L(t) = HD\tilde{B}^*(t)\tilde{B}(t)DH^*$. Since $L(t)$ and $\tilde{L}(t) = D\tilde{B}^*(t)\tilde{B}(t)D$ have the same eigenvalues, without loss of generality we can assume from the beginning that $W = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}$.

Define

$$\hat{B}_{11}(t) = \frac{\theta_1 b_{11}(t) + \theta_2 b_{22}(t)}{\sqrt{\theta_1^2+\theta_2^2}}, \quad \hat{B}_{12}(t) = \frac{\theta_2 b_{12}(t) - \theta_1 b_{21}(t)}{\sqrt{\theta_1^2+\theta_2^2}},$$

$$\hat{B}_{21}(t) = \frac{\theta_1 b_{11}(t) - \theta_2 b_{22}(t)}{\sqrt{\theta_1^2+\theta_2^2}}, \quad \hat{B}_{22}(t) = \frac{\theta_2 b_{12}(t) + \theta_1 b_{21}(t)}{\sqrt{\theta_1^2+\theta_2^2}},$$

$$R_1^2 = \hat{B}_{11}(t)^2 + \hat{B}_{12}(t)^2, \quad R_2^2 = \hat{B}_{21}(t)^2 + \hat{B}_{22}(t)^2.$$

We have that $\hat{B}_{jk}$ are Brownian motions such that $\hat{B}_{11}$, $\hat{B}_{12}$ (resp. $\hat{B}_{21}$, $\hat{B}_{22}$) are independent and thus R_1, R_2 are Bessel processes.

Since

$$\begin{aligned} E\big[R_1^2(t)R_2^2(t)\big] &= E\big[\hat{B}_{11}(t)^2\hat{B}_{21}(t)^2\big] + E\big[\hat{B}_{12}(t)^2\hat{B}_{22}(t)^2\big]+ \\ &+E\big[\hat{B}_{11}(t)^2\big]E\big[\hat{B}_{22}(t)^2\big] + E\big[\hat{B}_{12}(t)^2\big]E\big[\hat{B}_{21}(t)^2\big] = \end{aligned}$$

$$= \frac{1}{\left(\theta_1^2+\theta_2^2\right)^2}\left\{E\left[\left(\theta_1^2 b_{11}^2(t)-\theta_2^2 b_{22}^2(t)\right)^2\right]+E\left[\left(\theta_2^2 b_{12}^2(t)-\theta_1^2 b_{21}^2(t)\right)^2\right]\right\}+2t^2$$

$$= \frac{2\left(3\theta_1^4-2\theta_1^2\theta_2^2+3\theta_2^4\right)t^2}{\left(\theta_1^2+\theta_2^2\right)^2}+2t^2,$$

$$E\left[R_1^2(t)\right]=E\left[R_2^2(t)\right]=2t,$$

it is clear that

$$E\left[R_1^2(t)R_2^2(t)\right]\neq E\left[R_1^2(t)\right]E\left[R_2^2(t)\right],$$

and consequently the Bessel processes R_1 and R_2 are not independent if $\theta_1 \neq \theta_2$.

Next, it is well known that

$$dR_i^2(t) = 2R_i(t)d\hat{B}_i(t)+2dt, \tag{2.4}$$

$$dR_i(t) = d\hat{B}_i(t)+\frac{1}{2R_i(t)}dt, \tag{2.5}$$

with the Brownian motions $\hat{B}_1, \hat{B}_2$ given by

$$\hat{B}_i(t)=\frac{\hat{B}_{i1}(t)}{R_1(t)}d\hat{B}_{i1}(t)+\frac{\hat{B}_{i2}(t)}{R_2(t)}d\hat{B}_{i2}(t), \quad i=1,2.$$

Since the process $\left(\hat{B}_{1i}(t),\hat{B}_{2i}(t)\right)_t$ is Gaussian for $i=1,2$, it follows that its quadratic variation is

$$\left[\hat{B}_{1i},\hat{B}_{2i}\right](t)=E\left(\hat{B}_{1i}(t)\hat{B}_{2i}(t)\right)=\frac{\theta_1^2-\theta_2^2}{\theta_1^2+\theta_2^2}t. \tag{2.6}$$

Then, we have

$$d\left[R_1,R_2\right](t)=d\left[\hat{B}_1,\hat{B}_2\right](t)=$$

$$=\frac{\hat{B}_{11}(t)\hat{B}_{21}(t)}{R_1(t)R_2(t)}d\left[\hat{B}_{11},\hat{B}_{21}\right](t)+\frac{\hat{B}_{12}(t)\hat{B}_{22}(t)}{R_1(t)R_2(t)}d\left[\hat{B}_{12},\hat{B}_{22}\right](t),$$

and using (2.6) and the equality

$$\hat{B}_{11}(t)\hat{B}_{21}(t)-\hat{B}_{12}(t)\hat{B}_{22}(t)=\operatorname{Tr}L(t),$$

we obtain

$$\begin{aligned} d\left[R_1,R_2\right](t) &= d\left[\hat{B}_1,\hat{B}_2\right](t)\\ &=\frac{\theta_1^2-\theta_2^2}{\theta_1^2+\theta_2^2}\left(\frac{\hat{B}_{11}(t)\hat{B}_{21}(t)-\hat{B}_{12}(t)\hat{B}_{22}(t)}{R_1(t)R_2(t)}\right)dt\\ &=\frac{\theta_1^2-\theta_2^2}{\theta_1^2+\theta_2^2}\frac{\operatorname{Tr}L(t)}{R_1(t)R_2(t)}dt. \end{aligned} \tag{2.7}$$

In particular, from (2.7) we deduce that the Brownian motions $\hat{B}_1, \hat{B}_2$ are not independent.

We have the equalities

$$\operatorname{Tr} L(t) = \theta_1^2 \left(b_{11}^2 (t) + b_{21}^2(t)\right) + \theta_2^2 \left(b_{12}^2(t) + b_{22}^2(t)\right), \tag{2.8}$$

$$\begin{aligned} \operatorname{Det} L(t) = \theta_1^2\theta_2^2 \left[\left(b_{11}^2(t) + b_{21}^2(t)\right) \left(b_{12}^2(t) + b_{22}^2(t)\right) \right. \\ \left. - b_{11}(t)b_{12}(t) + b_{21}(t)b_{22}(t)\right]. \end{aligned} \tag{2.9}$$

It is easily seen that

$$\operatorname{Tr} L(t) = \frac{\theta_1^2 + \theta_2^2}{2} \left(R_1^2(t) + R_2^2(t)\right), \tag{2.10}$$

$$[\operatorname{Tr} L(t)]^2 - 4 \operatorname{Det} L(t) = \left(\theta_1^2 + \theta_2^2\right)^2 R_1^2(t)R_2^2(t). \tag{2.11}$$

Then

$$\begin{aligned} \lambda_{1,2}(t) &= \frac{\operatorname{Tr} L(t) \pm \sqrt{[\operatorname{Tr} L(t)]^2 - 4 \operatorname{Det} L(t)}}{2} \\ &= \frac{\frac{\theta_1^2+\theta_2^2}{2} \left(R_1^2(t) + R_2^2(t)\right) \pm \left(\theta_1^2 + \theta_2^2\right) R_1(t)R_2(t)}{2} \\ &= \frac{\theta_1^2 + \theta_2^2}{4} \left(R_1(t) \pm R_2(t)\right)^2. \end{aligned} \tag{2.12}$$

Since

$$\lambda_1(t) + \lambda_2(t) = \operatorname{Tr} L(t), \quad \lambda_1(t) - \lambda_2(t) = \left(\theta_1^2 + \theta_2^2\right) R_1(t)R_2(t), \tag{2.13}$$

and any Bessel process is positive we get that a.s.,

$$\lambda_1(t) > \lambda_2(t) \text{ for any } t > 0. \tag{2.14}$$

Using (2.13), the relation (2.7) becomes

$$d\left[R_1, R_2\right](t) = d\big[\hat{B}_1, \hat{B}_2\big](t) = \left(\theta_1^2 - \theta_2^2\right) \frac{\lambda_1(t) + \lambda_2(t)}{\lambda_1(t) - \lambda_2(t)}\, dt. \tag{2.15}$$

From (2.10) and (2.13) we also have the equality

$$\frac{\lambda_1(t) + \lambda_2(t)}{\lambda_1(t) - \lambda_2(t)} = \frac{R_1(t)}{2R_2(t)} + \frac{R_2(t)}{2R_1(t)}. \tag{2.16}$$

By using (2.4), (2.5), (2.15), (2.16) and integration by parts formula in (2.12), we obtain

$$\begin{aligned} d\lambda_{1,2}(t) = \frac{\theta_1^2 + \theta_2^2}{2} \Big[\left(R_1(t) \pm R_2(t)\right) d\hat{B}_1(t) - \left(R_1(t) \pm R_2(t)\right) d\hat{B}_2(t)\Big] \pm \\ \pm \left(\frac{\theta_1^2 + \theta_2^2}{2} + \theta_1^2 - \theta_2^2\right) \left(\frac{R_1(t)}{2R_2(t)} + \frac{R_2(t)}{2R_1(t)}\right) dt + \left(\theta_1^2 + \theta_2^2\right) dt = \end{aligned}$$

$$= \left[\sqrt{2\operatorname{tr} W^2 \lambda_{1,2}(t)}\ \left(d\hat{B}_1(t) \pm d\hat{B}_2(t)\right)\right] \pm$$

$$\pm \left(\frac{\theta_1^2+\theta_2^2}{2} + \theta_1^2 - \theta_2^2\right) \frac{\lambda_1(t)+\lambda_2(t)}{\lambda_1(t)-\lambda_2(t)} dt + 2\operatorname{tr} W^2 dt,$$

and this concludes the proof of the theorem. □

COROLLARY. *The following relations hold:*

$$d\operatorname{Tr} L(t) = \sqrt{2\operatorname{tr} W^2}\left[f_1\big(\operatorname{Tr} L(t), \det L(t)\big)\big(d\hat{B}_1(t) - d\hat{B}_2(t)\big) + \right.$$
$$\left. + f_2\left(\operatorname{Tr} L(t), \det L(t)\right)\left(d\hat{B}_1(t) + d\hat{B}_2(t)\right)\right] + 4\operatorname{tr} W^2 dt, \tag{2.17}$$

(2.18)

$$d\det L(t) = \sqrt{2\left(\operatorname{tr} W^2\right)\det L(t)}\left\{f_2\left(\operatorname{Tr} L(t), \det L(t)\right)\left(d\hat{B}_1(t) - d\hat{B}_2(t)\right) + \right.$$
$$\left. + f_1\left(\operatorname{Tr} L(t), \det L(t)\right)\left(d\hat{B}_1(t) - d\hat{B}_2(t)\right)\right\}\left(\operatorname{tr} W^2\right)\operatorname{Tr} L(t) dt,$$

where

$$f_1(x,y) = \left(\frac{x+\sqrt{x^2-4y}}{2}\right)^{\frac{1}{2}}, \quad f_2(x,y) = \left(\frac{x-\sqrt{x^2-4y}}{2}\right)^{\frac{1}{2}}.$$

Proof. From (2.13), (2.10) and (2.4), (2.5) and the integration by parts formula we have

$$[\lambda_1+\lambda_2, \lambda_1+\lambda_2] = [\lambda_1-\lambda_2, \lambda_1-\lambda_2].$$

The previous equality implies $[\lambda_1, \lambda_2] = 0$ and hence

$$d\det L(t) = d\left(\lambda_1\lambda_2\right)(t) = \lambda_1(t) d\lambda_2(t) + \lambda_2(t) d\lambda_1(t) =$$
$$= \lambda_1(t)\Big\{\sqrt{2\left(\operatorname{tr} W^2\right)\lambda_2(t)}\ \left(d\hat{B}_1(t) - d\hat{B}_2(t)\right) +$$
$$+ \left(\operatorname{tr} W^2 + \theta_1^2 - \theta_2^2\right)\frac{\lambda_1(t)+\lambda_2(t)}{\lambda_2(t)-\lambda_1(t)} dt + 2\left(\operatorname{tr} W^2\right) dt\Big\} +$$
$$+ \lambda_2(t)\Big\{\sqrt{2\left(\operatorname{tr} W^2\right)\lambda_2(t)}\left(d\hat{B}_1(t) + d\hat{B}_2(t)\right) +$$
$$+ \left(\operatorname{tr} W^2 + \theta_1^2 - \theta_2^2\right)\frac{\lambda_1(t)+\lambda_2(t)}{\lambda_1(t)-\lambda_2(t)} dt + 2\left(\operatorname{tr} W^2\right) dt\Big\} =$$
$$= \sqrt{\lambda_1(t)}\Big\{\sqrt{2\left(\operatorname{tr} W^2\right)\det L(t)}\ \left(d\hat{B}_1(t) - d\hat{B}_2(t)\right)\Big\} +$$
$$+ \lambda_2(t)\left\{\left(\operatorname{tr} W^2 + \theta_1^2 - \theta_2^2\right)\frac{\lambda_1(t)+\lambda_2(t)}{\lambda_1(t)-\lambda_2(t)} dt + 2\left(\operatorname{tr} W^2\right) dt\right\} +$$
$$+ \sqrt{\lambda_2(t)}\left\{\sqrt{2\left(\operatorname{tr} W^2\right)\det L(t)}\ \left(d\hat{B}_1(t) + d\hat{B}_2(t)\right)\right\} +$$

$$+\lambda_1(t)\left\{\left(\operatorname{tr}W^2+\theta_1^2-\theta_2^2\right)\frac{\lambda_1(t)+\lambda_2(t)}{\lambda_2(t)-\lambda_1(t)}dt+2\left(\operatorname{tr}W^2\right)dt\right\}=$$
$$=\sqrt{2\left(\operatorname{tr}W^2\right)\det L(t)}\left\{f_2\left(\operatorname{Tr}L(t),\det L(t)\right)\left(d\hat{B}_1(t)-d\hat{B}_2(t)\right)+\right.$$
$$\left.+f_1\left(\operatorname{Tr}L(t),\det L(t)\right)\left(d\hat{B}_1(t)+d\hat{B}_2(t)\right)\right\}\left(\operatorname{tr}W^2\right)\operatorname{Tr}L(t)dt.\quad\square$$

Complex case. By taking an unitary matrix H such that $HDH^*=W$ with $D=\begin{pmatrix}\theta_1 & 0\\ 0 & \theta_2\end{pmatrix}$, we can assume again that $W=\begin{pmatrix}\theta_1 & 0\\ 0 & \theta_2\end{pmatrix}$.

Similarly as in the real case we have:

THEOREM 3. *Let W be an 2×2-Hermitian positive definite matrix with positive eigenvalues $(\theta_j)_{1\le j\le n}$ and consider the 2×2-operator Laguerre process $L(t)=WB^*(t)B(t)W^*$, $t\ge 0$.*

Let $\{\lambda_1(t),\lambda_2(t)\}_{t\ge 0}$ be the eigenvalues of $L(t)$ and assume that $\lambda_1(0)>\lambda_2(0)$.

Then at every time $t>0$ the eigenvalues $\lambda_1(t),\lambda_2(t)$ are distinct and satisfy the following system of stochastic differential equations

$$d\lambda_1(t)=\sqrt{2\left(\operatorname{tr}W^2\right)\lambda_1(t)}\left(d\hat{B}_1(t)+d\hat{B}_2(t)\right)+$$
$$+2\left\{\left(\operatorname{tr}W^2+\theta_1^2-\theta_2^2\right)\frac{\lambda_1(t)+\lambda_2(t)}{\lambda_1(t)-\lambda_2(t)}+2\left(\operatorname{tr}W^2\right)\right\}dt,\tag{2.19}$$

$$d\lambda_2(t)=\sqrt{2\left(\operatorname{tr}W^2\right)\lambda_2(t)}\left(d\hat{B}_1(t)-d\hat{B}_2(t)\right)+$$
$$+2\left\{\left(\operatorname{tr}W^2+\theta_1^2-\theta_2^2\right)\frac{\lambda_1(t)+\lambda_2(t)}{\lambda_2(t)-\lambda_1(t)}+2\left(\operatorname{tr}W^2\right)\right\}dt,\tag{2.20}$$

where $\hat{B}_1,\hat{B}_2$ are Brownian motions with

$$d[\hat{B}_1,\hat{B}_2]_t=2\left(\theta_1^2-\theta_2^2\right)\frac{\lambda_1(t)+\lambda_2(t)}{\lambda_1(t)-\lambda_2(t)}dt.\tag{2.21}$$

Recall that for $\delta,x\ge 0$, the unique strong solution of the ISDE

$$Z_t=x+2\int_0^t\sqrt{Z_s}dB(s)+\delta t,\quad t\ge 0,\tag{2.22}$$

is called *squared Bessel process* starting at x and of dimension δ (we write $Z\in BESQ(x,\delta)$).

For standard Wishart and Laguerre processes it is known that their traces belong to $BESQ(\operatorname{tr}X_0,mn)$ (resp. $BESQ(\operatorname{tr}X_0,2mn)$). The next result shows that the standard case is the unique one when such a property holds.

THEOREM 4. *The 2-dimensional operator Wishart (resp. Laquerre) process L has the trace a squared Bessel process if and only if $W=I$.*

Proof. We consider only the case of Wishart processes (for Laguerre processes the proof is similar).

The if part is known (see [2], [3], [4]).

Now assume that L has its trace a squared Bessel process of the form (2.22). For simplicity we take $x = 0$.

The moments of Laguerre processes are known explicitly (see [7]). In particular (this also can be easily seen directly),

$$E(Z_t) = \delta t, \tag{2.23}$$

$$E(Z_t^2) = \delta\,(\delta + 2)\,t^2, \tag{2.24}$$

$$E(Z_t^3) = \delta\,(\delta + 2)\,(\delta + 4)\,t^3. \tag{2.25}$$

Denote

$$\begin{aligned} X_t &= \operatorname{tr} X_t, \quad \theta_1^2 = a, \quad \theta_2^2 = b, \\ S_1^2(t) &= b_{11}^2 + b_{21}^2, \quad S_2^2(t) = b_{12}^2 + b_{22}^2, \\ d\hat{b}_1(t) &= \frac{b_{11}(t)db_{11}(t) + b_{21}(t)db_{21}(t)}{S_1(t)}, \\ d\hat{b}_2(t) &= \frac{b_{12}(t)db_{12}(t) + b_{22}(t)db_{22}(t)}{S_2(t)}. \end{aligned}$$

It is clear that $\hat{b}_1, \hat{b}_2$ are independent Brownian motions and that S_1, S_2 are independent Bessel processes.

From (2.8) we have the relation

$$dX_t = 2aS_1(t)d\hat{b}_1(t) + 2bS_2(t)d\hat{b}_2(t) + 2(a + b)dt. \tag{2.26}$$

In particular, $E(X_t) = 2(a + b)t = E(Z_t) = \delta t$, and hence

$$\delta = 2(a + b). \tag{2.27}$$

Also, it is easily seen that

$$E(X_t^2) = 4\left[a^2 + b^2 + (a + b)^2\right]t^2, \tag{2.28}$$

$$E\left(X_t^3\right) = 48\left[a^3 + b^3 = ab(a + b)\right]t^3. \tag{2.29}$$

Since $E(X_t^2) = E(Z_t^2)$, (2.24), (2.28) and (2.29) yield

$$a + b = a^2 + b^2, \tag{2.30}$$

and since $E(X_t^3) = E(Z_t^3)$, the relations (2.25), (2.27) and (2.29) imply

$$6\left(a^2 + b^2 - ab\right) = 6ab = 2ab + 3(a + b) + 2. \tag{2.31}$$

Finally, from (2.31) and (2.30) we get $a = b = 1$. $\square$

Acknowledgements. This work was done while the second author was visiting CIMAT during the Summer of 2010, under the support of the Laboratorio de Estadistíca.

REFERENCES

[1] M.F. Bru, *Diffusions of perturbed principal component analysis.* J. Multivariate Anal. **29** (1989), 127–136.

[2] M.F. Bru, *Wishart processes.* J. Theoret. Probab. **9** (1991), 725–751.

[3] N. Demni, *The Laguerre process and generalized Hartman-Watson law.* Bernoulli **13** (2007), 556–580.

[4] N. Demni, *Processus Stochastiques Matriciels, Systemes de Racines et Probabilités Non Commutatives.* Thesis, Université Pierre et Marie Curie, 2007.

[5] C. Donati-Martin, Y. Doumerc and M. Yor, *Some properties of Wishart processes and a matrix extension of the Hartman-Watson laws.* Publ. RIMS, Kyoto Univ. **40** (2004), 1385–1412.

[6] U. Haagerup and S. Thorbjørsen, *Random matrices with complex Gaussian entries.* Expo. Math. **21** (2003), 293–337.

[7] P. Graczyk and L. Vostrikova, *The moments of Wishart processes via Itô calculus.* Theory Probab. Appl. **51(4)** (2007), 609–625.

[8] A. Guionnet, *Random Matrices: Lectures on Macroscopic Asymptotics.* École d'Étè des Probabilités de Saint-Flour XXXVI, 2006, Lecture Notes in Mathematics, Springer, 2008.

[9] M. Katori and K. Tanemura, *Symmetry of matrix-valued stochastic processes and non-colliding diffusion particle systems.* J. Math. Phys. **45** (2004), 3058–3085.

[10] W. Konig and N. Connell, *Eigenvalues of the Laguerre process as non-coliding squared Bessel processes.* Elec. Comm. Probab. **6** (2001), 107–114.

[11] S. Lawi, *Hermite and Laguerre polynomials and matrix-valued stochastic processes.* Bernoulli **13** (2008), 67–84.

[12] G. Letac and H. Massam, *All invariants moments of the Wishart distribution.* Scand. J. Statist. **31** (2004), 295–318.

[13] V. Pérez-Abreu and C. Tudor, *Functional limit theorems for trace processes in a Dyson Brownian motion.* Comm. Stochast. Anal. **1** (2007), 415–428.

[14] V. Pérez-Abreu and C. Tudor, *On the traces of Laguerre processes.* J. Theoret. Probab. **14** (2009), 2241–2263.

Received 1 February 2011

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