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Recouvrements Aléatoires et Processus de Markov Auto–Similaires

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Introduction

Cette thèse comprend quatre chapitres, les trois premiers correspondent chacun à un article qui a été publié ou soumis dans une revue scientifique à comité de lecture. Il s'agit de "On random sets connected to the partial records of Poisson point processes" paru dans *Journal of Theoretical Probability* **16** (2003), no. 1, 277–307 ; "A law of iterated logarithm for increasing self–similar Markov processes" paru dans *Stochastics and Stochastics Reports* **75** (2003), no. 6, 443–472 ; et "On recurrent extensions of self–similar Markov processes and Cramér's condition" à paraître dans *Bernoulli.* Enfin, le dernier chapitre étend certains résultats obtenus dans le troisième chapitre. Ce travail peut être divisé en deux parties : la première est consacrée à la construction et à l'étude d'une classe d'ensembles régénératifs (Chapitre I) et la seconde (Chapitres II, III & IV) est constituée de quelques contributions à la théorie des processus de Markov auto–similaires positifs. Le point commun entre ces deux parties, est les processus de Lévy à valeurs réelles, i.e. les processus en temps continu ayant des accroissements indépendants et stationnaires, et dont les trajectoires sont càdlàg (continues à droite avec limites à gauche). En effet, tout ensemble régénératif peut être identifié avec l'image d'un processus de Lévy croissant, appelé subordinateur. Quant aux processus de Markov auto–similaires, ils sont liés aux processus de Lévy par la transformation de Lamperti [19]. Cette introduction est consacrée à décrire nos principaux résultats sur chacun de ces deux sujets.

Dans une annexe à la fin de cette introduction, on rappelle quelques notations sur les processus de Lévy et la transformation de Lamperti qui relie les processus de Markov auto–similaires aux processus de Lévy.

Ensembles Régénératifs

Un ensemble régénératif M est l'analogue, pour des sous–ensembles aléatoires de $\mathbb{R}$, d'un processus de renouvellement. De façon informelle, un ensemble aléatoire fermé M a la propriété de régénération si, lorsque l'on coupe M en un point aléatoire, qui est un temps d'arrêt T à valeurs dans M, la partie de M à gauche de T est indépendante de la partie à droite et la loi de celle–ci est indépendante du choix de T. La théorie des ensembles régénératifs a été développée par plusieurs auteurs, parmi lesquels on peut citer Kingman, Krilov & Yushkevich, Hoffmann–Jørgensen et Maisonneuve. Dans l'article de Fristedt [15], on pourra trouver une présentation détaillée de cette théorie ainsi qu'une ample liste de références.

Le résultat fondamental de la théorie énonce que tout ensemble régénératif M est égal à la fermeture de l'image d'un subordinateur, i.e. $M = \overline{\{\sigma_t, t \geq 0\}}$, pour un certain subordinateur

σ. Un ensemble régénératif est donc complètement caracterisé par un triplet (a, Π, k) avec $a, k \in \mathbb{R}^+$ et Π une mesure sur $\mathbb{R}^+ \setminus \{0\}$ qui vérifie $\int_0^\infty (x \wedge 1)\Pi(dx) < \infty$. Le terme a est appelé le paramètre d'épaisseur, Π la mesure des trous et k le taux de mort. Certaines caractéristiques de l'ensemble M peuvent être déduites directement de (a, Π, k). Par exemple, M est borné si et seulement si $k > 0$; la mesure de Lebesgue de M est presque sûrement > 0 si et seulement si $a > 0$; M est d'intérieur vide si et seulement si $\Pi]0, \infty[= \infty$, ou encore, M est une union dénombrable d'intervalles non vides et fermés si $a > 0$ et $\Pi]0, \infty[< \infty$. De plus, des propriétés plus complexes pour des ensembles régénératifs peuvent être obtenues à partir de résultats connus pour les subordinateurs ; voir à ce sujet le cours de Saint-Flour de Bertoin [2] qui donne un panorama sur l'étude des ensembles régénératifs via les subordinateurs et ses diverses applications.

Un exemple d'ensemble régénératif est celui construit par Mandelbrot [20] comme une généralisation aléatoire de l'ensemble de Cantor. Plus précisement, Mandelbrot a introduit des recouvrements aléatoires de la droite réelle par des intervalles $]t, t + s[$, issus d'un processus de Poisson ponctuel $\mathcal{P}$ à valeurs sur $\mathbb{R} \times \mathbb{R}^+$. Il a étudié les caractéristiques de l'ensemble aléatoire $\mathcal{M}$ qui n'est pas recouvert par ces intervalles, i.e. $\mathcal{M} = \mathbb{R} \setminus \cup_{(t,s)\in\mathcal{P}}]t, t + s[$, et a montré que cet ensemble peut être réalisé comme la fermeture de l'image dans $\mathbb{R}$ d'un subordinateur. Fitzsimmons, Fristedt et Shepp [13] ont ensuite caracterisé la loi de ce subordinateur et en ont déduit un critère pour déterminer si la demi–droite $]0, \infty[$ est complètement couverte ou non, i.e. si $\mathcal{M} = \{0\}$ p.s. ou non, ainsi que d'autres propriétés de $\mathcal{M}$.

Ce sont ces travaux qui motivent l'étude d'un autre type d'ensemble aléatoire dont la construction a été inspiré par le travail de Marchal [21]. Tel est l'objectif du Chapitre I de cette thèse.

Chapitre I : Sur des ensembles aléatoires associés aux maxima locaux des processus de Poisson ponctuel

On considère $\mathcal{P} \subset]0, \infty[\times]0, \infty]$ un processus de Poisson ponctuel de mesure caractéristique $\lambda \otimes \nu$ sur $]0, \infty[\times]0, \infty]$ avec λ la mesure de Lebesgue sur $]0, \infty[$, ν une mesure Borelienne sur $]0, \infty[$ et $p : [0, \infty[\to [0, 1]$ une fonction mesurable. Pour tout point $(x, y) \in \mathcal{P}$ on définit x^* comme l'abscisse du premier point dans $\mathcal{P}$ à droite de x qui est dans un niveau supérieur $y^* \geq y$. C'est à dire, $x^* = \inf\{x' > x : (x', y') \in \mathcal{P}, y' \geq y\}$. De cette façon, à tout $(x, y) \in \mathcal{P}$ on associe l'intervalle $[x, x^*[$. On efface chaque intervalle $[x, x^*[$ de $\mathbb{R}^+$ avec probabilité $p(y)$, indépendamment les uns des autres, et on s'intéresse alors à l'ensemble résiduel, $\mathcal{R}$, des points qui n'ont pas été effacés :

$$\mathcal{R} := \mathbb{R}^+ \setminus \bigcup_{x\in T} [x, x^*[,$$

où T est l'ensemble des extrêmités gauche des intervalles $[x, x^*[$ qui sont effacés de $\mathbb{R}^+$.

Le but de ce Chapitre est de formaliser la construction de l'ensemble $\mathcal{R}$ et de le décrire en termes de la mesure ν et de la fonction p. Par souci de clarté on fait les hypothèses techniques suivantes sur la mesure ν. On suppose que ν est une mesure sans atomes et que sa queue $\overline{\nu}(y) := \nu(y, \infty), y > 0$, est strictement décroissante et a pour limite ∞ quand $y \to 0+$.

Le fait que les maxima locaux d'un processus de Poisson ponctuel interviennent dans la construction de l'ensemble aléatoire $\mathcal{R}$ permet d'utiliser la théorie des processus Extremes pour l'étudier. Voir Resnick & Rubinovitch [22] pour une approche des processus Extremes via les processus de Poisson ponctuels ou les sauts d'un processus de Lévy. Nous nous servons de la structure de ces processus pour établir des critères qui décrivent géométriquement l'ensemble aléatoire $\mathcal{R}$.

Proposition 1. *Soit* $S(y) := -\ln \overline{\nu}(y), y > 0$ *et pour tout* $t > 0$, $F^t(y) := \exp\{-t\overline{\nu}(y)\}, y > 0$.

(i) *Soit* $Z = \inf\{t > 0 : t \notin \mathcal{R}\}$. *Alors* $Z > 0$, $\mathbf{P}$*–p.s. si et seulement si* $\int_0^\infty p(y)\nu(dy) < \infty$. *Dans ce cas* Z *suit une loi exponentielle de paramètre* $\int_0^\infty p(y)\nu(dy)$. *En particulier,* $\mathcal{R} = [0, \infty[$, $\mathbf{P}$*–p.s. si et seulement si* $p = 0$ ν*–p.s.*

(ii) *Pour tout* $t > 0$, $\mathbf{P}(t \in \mathcal{R}) > 0$ *si et seulement si* $\int_{0+} p(y)S(dy) < \infty$. *Si la condition précédente est satisfaite alors*

$$\mathbf{P}(t \in \mathcal{R}) = \int_0^\infty F^t(dy)[1 - p(y)] \exp\left\{-\int_0^y p(w)S(dw)\right\}.$$

(iii) 0 *est un point isolé de* $\mathcal{R}$, $\mathbf{P}$*–p.s. si et seulement si* $\int_{0+}[1 - p(y)]S(dy) < \infty$.

(iv) $\mathcal{R}$ *est borné* $\mathbf{P}$*–p.s. si et seulement si* $\int^\infty[1 - p(y)]S(dy) < \infty$.

(v) $\mathcal{R} = \{0\}$ $\mathbf{P}$*–p.s. si et seulement si* $p = 1$ ν*–p.s.*

On cherche ensuite à savoir si $\mathcal{R}$, tout comme l'ensemble aléatoire $\mathcal{M}$ de Mandelbrot, est régénératif. Intuitivement, ceci devrait être vrai. Si un point t déterministe appartient à l'ensemble $\mathcal{R}$ alors la couverture des points à droite de t ne dépend que des intervalles $[x, x^*[$ pour $(x, y) \in [t, \infty[\times[0, \infty[\cap\mathcal{P}$. Ainsi la partie de $\mathcal{R}$ à gauche de t est indépendante de la partie à droite et cette dernière a la même loi que $\mathcal{R}$ par homogénéité dans le temps de $\mathcal{P}$. Cet argument est à la base de la preuve du théorème suivant.

Théorème 1. *L'ensemble aléatoire* $\mathcal{R}$ *est régénératif par rapport à la filtration naturelle engendrée par le processus de Poisson ponctuel* $\mathcal{P}$.

La propriété de régénération et la partie (iii) de la Proposition 1 montrent que l'ensemble $\mathcal{R}$ peut être discret, i.e. tous les points de $\mathcal{R}$ sont isolés $\mathbf{P}$–p.s. Au contraire des ensembles de Mandelbrot $\mathcal{M}$ qui ne peuvent être que triviaux $\mathcal{M} = \{0\}$ $\mathbf{P}$–p.s. ou sans points isolés $\mathbf{P}$–p.s.

Pour déterminer la loi de $\mathcal{R}$ on utilise le fait que $\mathcal{R}$ est l'image d'un certain subordinateur σ. En effet, la loi de σ, et donc celle de $\mathcal{R}$, est caracterisé par sa fonction de renouvellement, $U(x) = \mathbf{E}(\int_0^\infty 1_{\{\sigma_s \le x\}} ds), x > 0$, que l'on obtient dans le théorème suivant.

Théorème 2. *La fonction de renouvellement de* $\mathcal{R}$ *est donnée par*

$$U(a) = a \int_0^\infty F^a(dx) \exp\{\int_x^1 p(y)S(dy)\} \qquad \textit{pour tout } a > 0.$$

Un exemple, dont la simplicité n'enlève rien à son intérêt, est celui où la fonction p est égale à une constante $\mathrm{p} \in]0, 1[$. Dans ce cas, il est facile de voir que l'ensemble régénératif associé $\mathcal{R}_\mathrm{p}$ est auto–similaire, c'est–à–dire qu'il possède la propriété suivante : pour tout $c > 0$, $c\mathcal{R}_\mathrm{p}$ a la même loi que $\mathcal{R}_\mathrm{p}$. En conséquence, $\mathcal{R}_\mathrm{p}$ est l'image d'un subordinateur stable d'indice

$1-\mathrm{p}$, car les seuls processus de Lévy qui sont auto–similaires sont les processus stables. La détermination du paramètre se fait à l'aide du Théorème 2. La dimension de Hausdorff de l'image d'un subordinateur stable etant presque sûrement égale à son paramètre, on en déduit que l'ensemble régénératif $\mathcal{R}_{\mathrm{p}}$ a une dimension de Hausdorff égale à $1-\mathrm{p}$.

Plus généralement, on peut calculer la dimension de Hausdorff et d'autres dimensions pour $\mathcal{R}$, en se servant des résultats connus sur l'image d'un subordinateur. Tel est l'objectif du théorème suivant.

Théorème 3. *Presque surement pour tout $t>0$, les dimensions de Hausdorff et Packing de $\mathcal{R}\cap[0,t[$ sont respectivement données par*

$$dim_H(\mathcal{R}\cap[0,t]) = \liminf_{y\to 0+} \frac{\int_y^1 (1-p(w))S(dw)}{-S(y)},$$
$$Dim_P(\mathcal{R}\cap[0,t]) = \limsup_{y\to 0+} \frac{\int_y^1 (1-p(w))S(dw)}{-S(y)}.$$

Processus de Markov auto–similaires positifs

Les processus auto–similaires ont été introduits par Lamperti [18] sous le nom de processus *semi–stables*. Il s'agit de processus $(X_t, t\geq 0)$ à valeurs dans $\mathbb{R}$ qui ont la propriété dite de *scaling* : il existe $\alpha>0$ tel que pour tout $c>0$ et $x\in\mathbb{R}$ la loi du processus

$$(cX_{tc^{-1/\alpha}}, t\geq 0) \quad \text{sous} \quad \mathbb{P}_x \quad \text{est égale à} \quad \mathbb{P}_{cx},$$

où $\mathbb{P}_x$ est la loi de X issu de x. On pourra trouver une ample introduction à ce sujet dans la monographie récente d'Embrechts et Maejima [12].

Lamperti a montré que lorsque l'on fait agir une suite de changements d'echelle en temps et en espace sur un processus stochastique, la limite, si elle existe, est nécessairement un processus auto–similaire. Plus précisément, etant donnés $(Y_t, t\geq 0)$ un processus à valeurs réelles et $f:\mathbb{R}\to\mathbb{R}$ une fonction croissante telle que $f(n)\to\infty$ lorsque $n\to\infty$ on va s'intéresser au processus normalisé $(Z^n(t)=Y(nt)/f(n), t\geq 0)$. Si Z^n converge vers un processus X au sens des lois fini–dimensionnelles lorsque $n\to\infty$, alors la fonction f est nécessairement une fonction à variation régulière à l'infini d'un certain indice $\alpha>0$, et X est un processus auto–similaire d'indice α. Après ce travail fondateur, Lamperti [19] s'est intéressé au cas des processus auto–similaires qui ont la propriété de Markov homogène et dont l'espace d'états est $\mathbb{R}^+$. Des exemples de ce type de processus sont le mouvement Brownien réfléchi en 0, les processus de Bessel, les processus de Lévy stables symétriques réfléchis et les subordinateurs stables.

Le résultat principal dans [19] établit que tout processus de Markov auto–similaire positif tué en son premier temps d'atteinte de 0, est égal à l'exponentielle d'un processus de Lévy à valeurs réelles changé de temps (voir l'Annexe B pour plus de détails). Cette relation s'avère être un outil puissant dans l'étude des processus de Markov auto–similaires dans $\mathbb{R}^+$, car celle-ci permet de décrire le comportement de X, au moins avant son premier temps d'arrivée en 0, à partir de celui de ξ, voir Bertoin & Caballero [3], Bertoin & Yor [5, 4, 6], Caballero & Chaumont [9], parmi d'autres. Un autre concept intimement lié à l'étude de cette classe de processus est celui de la fonctionnelle exponentielle d'un processus de Lévy, $I := \int_0^\infty \exp\{\xi_s\} ds$. Cette variable aléatoire trouve diverses applications en mathématiques financières, en physique mathématique et dans d'autres disciplines, voir à ce sujet le compte rendu de Carmona, Petit & Yor [11] et les références qui y figurent.

Dans la suite X désigne un processus de Markov fort α–auto–similaire, avec $\alpha > 0$ et on note $\mathbb{P}_x$ sa loi issue de $x > 0$.

Chapitre II : Une loi du logarithme itéré pour des processus de Markov auto–similaires croissants

Dans ce chapitre on suppose que X est un processus de Markov auto–similaire croissant avec un temps de vie infini. Ainsi, le processus de Lévy ξ associé à X par la transformation de Lamperti est un subordinateur à durée de vie infinie. On notera ϕ son exposant de Laplace.

Récemment, Bertoin & Caballero [3] se sont intéressés au comportement d'un processus auto–similaire positif croissant lorsque le point de départ tend vers 0. Ils ont montré que si $\mathbf{E}(\xi_1) < \infty$, alors il existe une unique mesure $\mathbb{P}_{0+}$ qui est la limite dans le sens des lois fini–dimensionnelles de $\mathbb{P}_x$ lorsque $x \to 0$. Ceci est équivalent, grâce à la propriété de scaling, à l'étude du comportement à l'infini du processus X, c'est–à– dire

$$\mathbb{P}_x(X_1 \in dy) \xrightarrow[x \to 0+]{} \mathbb{P}_{0+}(X_1 \in dy),$$

si et seulement si

$$\mathbb{P}_z(t^{-\alpha} X_t \in dy) \xrightarrow[t \to \infty]{} \mathbb{P}_{0+}(X_1 \in dy) \qquad \text{pour tout } z > 0.$$

Le cas géneral a été étudié dans [4, 5, 6] et la convergence dans le sens de Skorohod a été établie dans [9].

Notre but dans le Chapitre II est de donner des estimations de la vitesse à laquelle un processus de Markov auto–similaire croissant à valeurs dans $]0, \infty[$ tend vers l'infini. Ce problème a été d'abord étudié par Fristedt [14] dans le cas où X est un processus de Markov $1/\alpha$–auto–similaire, $\alpha \in]0, 1[$, croissant avec des accroissements indépendants et stationnaires, i.e. un subordinateur stable de paramètre α. Il a montré que

$$\liminf_{t \to \infty} \frac{X_t}{t^{1/\alpha} (\log \log t)^{(\alpha-1)/\alpha}} = \alpha(1-\alpha)^{(1-\alpha)/\alpha},$$

avec probabilité 1. Ce résultat a été ensuite amélioré par Breiman [8], qui a donne un critère intégral pour déterminer si une fonction appartient ou non à l'enveloppe inférieure de X. Le résultat principal de ce chapitre donne une généralisation du résultat de Fristedt pour une classe plus large de processus auto–similaires croissants, sous une hypothèse sur les petits sauts de X.

On dit que ϕ varie régulièrement à l'infini avec indice β si pour tout $\lambda > 0$

$$\lim_{t \to \infty} \frac{\phi(t\lambda)}{\phi(t)} = \lambda^\beta.$$

On sait que la variation régulière à l'infini de l'exposant de Laplace d'un subordinateur est reliée à la taille des petits sauts du subordinateur ξ, et en conséquence à ceux de X. Voir [1] §III.1. On va aussi faire une hypothèse technique

(H) la densité ρ de $I := \int_0^\infty \exp\{-\xi_s\}ds$ est décroissante à l'infini ;

on sait que la densité ρ existe grâce aux résultats de Carmona, Petit & Yor [10].

Théorème 4. *Soit ξ un subordinateur tel que $0 < \mathbf{E}(\xi_1) < \infty$, dont l'exposant de Laplace ϕ varie régulièrement avec indice $\beta \in]0,1[$ et tel que l'hypothèse (H) soit satisfaite. Soit X le processus α–auto–similaire associé à ξ via la transformation de Lamperti. On définit,*

$$f(t) = \frac{\phi(\log\log t)}{\log\log t} \qquad t > e.$$

Alors pour tout $x > 0$,

$$\liminf_{t\to\infty} \frac{X_t}{(tf(t))^\alpha} = \alpha^{-\alpha\beta}(1-\beta)^{\alpha(1-\beta)}, \qquad \mathbb{P}_x - p.s.$$

Ce résultat reste vrai sous $\mathbb{P}_{0+}$.

Par ailleurs, l'hypothèse de variation régulière à l'infini pour ϕ avec indice $\beta \in]0,1[$, est étroitement liée au comportement de ξ près de 0. Plus précisément, soit ψ l'inverse de ϕ et g définie par

$$g(t) = \frac{\log|\log t|}{\psi(t^{-1}\log|\log t|)}, \qquad 0 < t < e^{-1}.$$

Alors

$$\liminf_{t\to 0} \frac{\xi_t}{g(t)} = c_\beta, \quad \mathbf{P}-\text{p.s.},$$

pour une certaine constante $c_\beta \in]0,\infty[$, voir [1] Théorème III.11 pour une preuve de ce résultat. On en déduit facilement le comportement de X près de 0 lorsque le point de départ est strictement positif

$$\liminf_{t\to 0} \frac{X_t - x}{g(t)} = x^{(\alpha\beta-1)/\alpha\beta} c_\beta, \qquad \mathbb{P}_x - \text{p.s.}$$

Néanmoins, le comportement de X près de 0 est assez différent sous $\mathbb{P}_{0+}$. On a le théorème suivant

Théorème 5. *Sous les mêmes hypothèses et notations du Théorème 4 on a que*

$$\liminf_{t\to 0} \frac{X_t}{(tf(1/t))^\alpha} = \alpha^{-\alpha\beta}(1-\beta)^{\alpha(1-\beta)} \qquad \mathbb{P}_{0+} - p.s.$$

La méthode utilisée pour prouver le Théorème 4 est similaire à celle utilisée par Breiman [8] et consiste à faire une transformation du processus X en un processus stationnaire U et à étudier les excursions de ce processus hors de son point de départ. Cette méthode nous permet d'établir un critère intégral en termes de la densité de I, pour déterminer si une fonction appartient ou pas à l'enveloppe inférieure de X. On donne des estimations de la densité de I pour prouver que la fonction $(1-\varepsilon)f$ (respectivement $(1+\varepsilon)f$) appartient (resp. n'appartient pas) presque sûrement à l'enveloppe inférieure de X pour tout $\varepsilon > 0$. La preuve du Théorème 5 est similaire à celle du Théorème 4 mais on utilise en plus un argument de retournement du temps.

On établit aussi une version des Théorèmes 4 & 5 lorsque le subordinateur ξ est un processus de Poisson composé, i.e. sans coefficient de dérive et avec une mesure de Lévy Π finie. On donne quelques exemples parmi lesquels se trouvent les subordinateurs stables et les processus Extrêmes avec Q–fonction $Q(x) = ax^{-b}, x > 0$, avec $a, b > 0$.

Chapitre III et IV : Extensions récurrentes des processus de Markov auto–similaires et condition de Cramér

Étant donné que la transformation de Lamperti permet de décrire les processus de Markov auto–similaires positifs qui meurent en leur premier temps d'arrivée en 0, Lamperti a posé la question suivante : quels sont les processus de Markov auto–similaires $\widetilde{X}$ positifs pour lesquels 0 est un point récurrent et régulier et qui se comportent comme X jusqu'à l'instant T_0 ? On appelle ce type de processus extensions récurrentes de X. Ce problème a été résolu par Lamperti dans le cas où X est un mouvement Brownien tué en 0, à l'aide des propriétés spécifiques du mouvement Brownien. On sait aujourd'hui que la théorie d'excursions des processus de Markov fournit un outil puissant pour obtenir une réponse plus générale. Plus précisement, Itô [17], Blumenthal [7], Rogers [24] et Salisbury [26, 27] ont montré qu'il existe une bijection entre les extensions récurrentes d'un processus de Markov Y et les mesures d'excursions qui sont compatibles avec Y. Une mesure d'excursions est une mesure sur l'espace des trajectoires absorbées en un point, sous laquelle le processus canonique est Markovien avec le même semigroupe que Y et qui intègre la fonction $1 - e^{-\zeta}$, où ζ est le temps de vie de la trajectoire. Après Lamperti, Vuolle–Apiala [28] s'est servi de ce fait pour donner, sous des hypothèses assez générales, une réponse à la question posée par Lamperti en montrant l'existence d'une unique mesure d'excursions $\mathbf{n}$ compatible avec X telle que $\mathbf{n}(X_{0+} > 0) = 0$ et d'une infinité de mesures d'excursions n^{β} tels que $n^{\beta}(X_{0+} = 0) = 0$. Ces dernières sont complètement caracterisées par les mesures de sauts $\eta_{\beta}(dx) = x^{-(1+\beta)}dx, x > 0$, $0 < \beta < 1/\alpha$ et l'extension récurrente associée part de 0 par des sauts p.s. Quant à la mesure $\mathbf{n}$, Vuolle–Apiala a montré que c'est l'unique mesure telle que son λ–potentiel $\mathbf{n}_{\lambda}$, soit donné par

$$\mathbf{n}_{\lambda}(f) = \lim_{x \to 0+} \frac{V_{\lambda} f(x)}{\mathbb{E}_x(1 - e^{-T_0})},$$

avec V_{λ} la résolvante de X tué en 0. La limite ci–dessus existe grâce aux hypothèses de [28]. L'extension récurrente associée à $\mathbf{n}$ quitte 0 de façon continue p.s.

Dans le cas où X est un mouvement Brownien tué en 0 la mesure $\mathbf{n}$ est en fait la mesure d'excursions d'Itô pour le mouvement Brownien hors de 0, et l'extension récurrente associée n'est autre que le mouvement Brownien réfléchi en 0. Toute mesure n^{β} correspond à la mesure d'excursions hors de 0 d'un mouvement Brownien changé de temps par l'inverse d'une fonctionnelle additive fluctuante (voir Rogers & Williams [25]).

Le but des Chapitres III et IV est de donner une description plus précise de la mesure $\mathbf{n}$ en nous inspirant du cas Brownien, dont on connaît plusieurs descriptions de la mesure d'Itô, voir Revuz & Yor [23] §XII. Pour cela, on va supposer que la loi $\mathbf{P}$ du processus de Lévy ξ sous-jacent (voir l'annexe B) satisfait les hypothèses suivantes :

(H1-a) ξ n'est pas arithmétique, i.e. l'espace d'états n'est pas un sous–groupe de $k\mathbb{Z}$ pour tout $k \in \mathbb{R}$;

(H1-b) il existe $\theta > 0$ tel que $\mathbf{E}(e^{\theta\xi_1}, 1 < \zeta) = 1$;

(H1-c) $\mathbf{E}(\xi_1^+ e^{\theta\xi_1}, 1 < \zeta) < \infty$, avec $a^+ = a \vee 0$.

La condition (H1-b) est la condition dite de Cramér pour le processus de Lévy ξ. L'indice de Cramér θ va jouer un rôle très important dans la détermination de certains paramètres comme on va le voir plus bas.

Lorsque le processus de Lévy ξ a une durée de vie infinie p.s., la condition de Cramér implique que le processus ξ dérive vers $-\infty$, i.e. $\lim_{t\to\infty} \xi_t = -\infty$ $\mathbf{P}$–p.s. En conséquence, le processus auto–similaire X associé à ξ atteint l'état 0 de façon continue. Par contre, si la durée de vie de ξ est finie p.s. le processus X entre dans 0 par un saut p.s. La première de ces deux familles de processus auto–similaires est étudiée dans le Chapitre III et la seconde au Chapitre IV. Les résultats principaux sur ce sujet sont analogues dans les deux cas et on va donc se contenter de les décrire sans faire de différence sauf indication contraire.

Le résultat principal de ce travail donne une description de la mesure $\mathbf{n}$ analogue à celle établie par Imhof [16], de la mesure d'excursions d'Itô pour le mouvement Brownien via la loi d'un processus de Bessel(3).

On commence par remarquer que la condition de Cramér implique que le processus $(e^{\theta\xi_t}, t \geq 0)$ est une martingale par rapport à la filtration de ξ. Par un changement de mesure à la Girsanov on peut donc construire une mesure $\mathbf{P}^\natural$ qui est absolument continue par rapport à $\mathbf{P}$ avec une derivée de Radon–Nikodym $e^{\theta\xi_t}$. On s'intéresse ensuite au processus de Markov α–auto–similaire $X^\natural$ de loi $({\mathbb{P}^\natural}_x, x > 0)$ associé au processus de Lévy de loi $\mathbf{P}^\natural$. La relation d'absolue continuité entre $\mathbf{P}$ et $\mathbf{P}^\natural$ est préservée par la transformation de Lamperti dans le sens où pour tout $x > 0$, ${\mathbb{P}^\natural}_x$ est absolument continue par rapport à $\mathbb{P}_x$ avec une dérivée de Radon–Nikodym X_t^θ. De plus, la loi ${\mathbb{P}^\natural}_x$ peut être vue comme la loi de X conditionné à ne jamais arriver en 0, car

- Pour $x > 0$ et pour toute fonctionnelle bornée F et $t > 0$ on a

$$\lim_{s\to\infty} \mathbb{E}_x(F(X_r, r \leq t)|T_0 > s) = {\mathbb{E}^\natural}_x(F(X_r, r \leq t)).$$

Donc, le processus auto–similaire $X^\natural$ est à X ce que le processus de Bessel(3) est au mouvement Brownien. Ensuite, on remarque que les hypothèses (H1) permettent d'utiliser les résultats dans [5] pour assurer qu'il existe une mesure ${\mathbb{P}^\natural}_{0+}$ qui est la limite dans le sens de lois fini dimensionnelles de ${\mathbb{P}^\natural}_x$ lorsque $x \to 0$, et de vérifier que les hypothèses de Vuolle-Apiala sont satisfaites sous nos hypothèses lorsque $0 < \alpha\theta < 1$. Dans ce cas il existe une unique mesure d'excursions $\mathbf{n}$ tel que $\mathbf{n}(X_{0^+} > 0) = 0$ et $\mathbf{n}(1 - e^{-T_0}) = 1$.

On peut maintenant énoncer notre première description de $\mathbf{n}$.

Théorème 6. *Il existe une mesure* $\mathbf{n}'$ *avec support dans l'ensemble des trajectoires positives qui sont absorbées en* 0 *tel que sous* $\mathbf{n}'$ *le processus canonique est Markovien avec le même semi–groupe que* X *tué en 0 et* $\mathbf{n}'(X_{0+} > 0) = 0$. *La loi d'entrée* $(\mathbf{n}'_s, s > 0)$ *associée a* $\mathbf{n}'$ *est donnée par*

$$\mathbf{n}'_s f = {\mathbb{E}^\natural}_{0+}(f(X_s)X_s^{-\theta}), \quad s > 0.$$

La fonction $1 - e^{-T_0}$ est intégrable par rapport à $\mathbf{n}'$ si et seulement si $0 < \alpha\theta < 1$. Dans ce cas il existe une constante $a_{\alpha,\theta}$ tel que $\mathbf{n} = (a_{\alpha,\theta})^{-1}\,\mathbf{n}'$.

Le Théorème précédent nous permet d'établir un critère pour déterminer, en fonction de θ, donc du processus de Lévy ξ, pour qu'il existe une extension récurrente de X qui quitte 0 de façon continue.

Théorème 7. *(i) On suppose $0 < \alpha\theta < 1$. Alors X admet une unique extension récurrente α–auto–similaire $\widetilde{X}$ qui quitte 0 de façon continue p.s. Le processus $\widetilde{X}$ est Fellerien.*

(ii) Si $\alpha\theta \geq 1$, alors il n'existe aucune extension récurrente de X qui quitte 0 de façon continue.

On suppose dorénavant que $0 < \alpha\theta < 1$. Motivé par la description de la mesure d'excursions d'Itô pour le mouvement Brownien via la loi d'un pont de Bessel(3) (voir [23] Théorème XII.4.2), on déduit du Théorème 6 une autre description de la mesure $\mathbf{n}$ en déterminant la loi Λ^r de l'excursion conditionnée à avoir une durée de vie donnée $r > 0$. Dans le cas où X atteint 0 de façon continue la loi Λ^r peut être interprétée comme la loi d'un pont de 0 à 0 sur $[0, r]$ pour ${\mathbb{P}^\natural}_{0+}$. En général, la loi Λ^r est une h-transformée de ${\mathbb{E}^\natural}_{0+}$. On a la proposition suivante.

Proposition 2. *La mesure $\mathbf{n}$ vérifie*

(i) $\mathbf{n}(T_0 \in dt) = (\alpha\theta/\Gamma(1-\alpha\theta))t^{-1-\alpha\theta}dt$,

(ii) Pour tout $F \in \mathcal{G}$, on a l'egalité

$$\mathbf{n}(F) = \frac{\alpha\theta}{\Gamma(1-\alpha\theta)}\int_0^\infty \Lambda^t(F \cap \{T_0 = t\})\frac{dt}{t^{1+\alpha\theta}}.$$

Par ailleurs, on sait qu'il existe deux processus $\widehat{X}$ et $\widehat{X}^\natural$ qui sont en dualité faible avec X et $X^\natural$ respectivement. Il est facile de voir que les processus X et $\widehat{X}^\natural$ sont aussi en dualité faible et que $\widehat{X}^\natural$ atteint 0 de façon continue. Dans le cadre du Chapitre III, on montre que le processus $\widehat{X}^\natural$ admet une extension récurrente Z qui quitte 0 de façon continue. On montre que les processus $\widetilde{X}$ et Z sont eux aussi en dualité faible et que la mesure d'excursions de Z, que l'on note par $\hat{\mathbf{n}}$, est l'image de $\mathbf{n}$ sous retournement du temps. En outre, on donne une généralisation de ce résultat pour des processus de Markov plus généraux que les processus α–auto–similaires. Un résultat analogue est obtenu dans le Chapitre IV à condition de prendre une extension récurrente $\widehat{Z}_\theta$ de $\widehat{X}^\natural$ qui quitte 0 par un saut. Dans ce cas l'image sous retournement de temps de $\mathbf{n}$ est, à une constante multiplicative près, la mesure $\int_0^\infty dx\ x^{-(1+\theta)}{\widehat{\mathbb{E}}^\natural}_x(\cdot)$, avec $\widehat{\mathbb{E}}^\natural$ la loi de $\widehat{X}^\natural$. En général, on a que le processus Z (resp. $\widehat{Z}_\theta$) issu de 0 est égale en loi au processus obtenu en retournant une à une les excursions hors de 0 de $\widetilde{X}$ issu de 0.

Enfin, sous des hypothèses du type Cramér avec un indice négatif, dans la Section IV.5, on construit un processus $X^\downarrow$ qui peut être interprété comme le processus X conditionné à atteindre 0 de façon continue.

Annexe A : Quelques définitions

Un processus de Lévy de loi $\mathbf{P}$ est déterminé par son exposant caractéristique, $\Psi : \mathbb{R} \to \mathbb{C}$ défini par

$$\mathbf{E}(e^{i\lambda\xi_1}) = e^{-\Psi(\lambda)}, \qquad \lambda \in \mathbb{R},$$

et la formule de Lévy–Khintchine permet d'écrire Ψ comme

$$\Psi(\lambda) = -ia\lambda + Q^2\lambda/2 + \int_{\mathbb{R}\setminus\{0\}} (1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}})\Pi(dx),$$

avec $a \in \mathbb{R}$, $Q \geq 0$ et Π une mesure sur $\mathbb{R}\setminus\{0\}$ telle que $\int_{\mathbb{R}\setminus\{0\}}(1 \wedge x^2)\Pi(dx) < \infty$. Le terme a est appelé le terme de dérive, Q le terme Gaussien et Π la mesure de Lévy ; le triplet (a, Q, Π) caractérise la loi $\mathbf{P}$. Lorsque le processus de Lévy a des trajectoires croissantes on l'appelle un subordinateur. Dans ce cas, l'exposant caractéristique peut être prolongé au semi–plan $\Im(z) \in [0, \infty[$ et la loi du subordinateur est alors caracterisée par son exposant de Laplace $\phi(\lambda) : \mathbb{R}^+ \to \mathbb{R}^+$ défini par

$$\mathbf{E}(e^{-\lambda\xi_1}) = e^{-\phi(\lambda)}, \qquad \lambda \geq 0,$$

et $\phi(\lambda) = \Psi(i\lambda)$. La croissance des trajectoires implique $Q = 0$, $\Pi]-\infty, 0[= 0$, $\int_0^\infty(1\wedge x)\Pi(dx) < \infty$, et la formule de Lévy–Khintchine pour l'exposant de Laplace du subordinateur peut se réécrire comme :

$$\phi(\lambda) = a\lambda + \int_0^\infty (1 - e^{-\lambda x})\Pi(dx), \qquad \lambda \geq 0.$$

Plus généralement on appellera également un subordinateur un processus $(\xi_t, t \geq 0)$ càdlàg croissant à valeurs dans $[0, \infty]$, ayant ∞ comme un point cimetière, tel que conditionnellement à $\xi_t < \infty$ les accroissements $\xi_{t+s} - \xi_t$ sont indépendants de $\sigma(\xi_r, 0 \leq r \leq t)$ et ont la même loi que ξ sous $\mathbf{P}$. L'exposant de Laplace de ξ défini comme avant s'écrit donc

$$\phi(\lambda) = k + a\lambda + \int_0^\infty (1 - e^{-\lambda x})\Pi(dx), \qquad \lambda \geq 0,$$

oú le terme k est appelé "taux de meurtre" puisque le temps de vie du subordinateur suit une loi exponentielle de paramètre k. La loi d'un subordinateur est donc caractérisée par un triplet (k, a, Π).

Annexe B : La transformation de Lamperti

La relation de Lamperti peut être décrite comme suit. Soit $\mathbf{P}'$ une probabilité sur l'espace $\mathbb{D}$ des trajectoires à valeurs réelles continues à droite avec limite à gauche (càdlàg), muni de la topologie de Skorohod et $\mathcal{D}'_t$ la filtration naturelle engendrée par le processus canonique. On suppose que sous $\mathbf{P}'$ le processus canonique ξ' est un processus de Lévy. Soit ξ le processus ξ' tué avec un taux $k \geq 0$, i.e. tué en un temps exponentiel de paramètre k et indépendant de ξ'. On note par Δ le point cimetière pour ξ, par ζ le temps de vie de ξ, $\zeta = \inf\{t > 0 : \xi_t = \Delta\}$ et par $\mathcal{D}_t$ la filtration du processus tué. Comme d'habitude on prolonge les fonctions $f : \mathbb{R} \to \mathbb{R}$ à $\mathbb{R} \cup \{\Delta\}$ par $f(\Delta) = 0$. Pour $\alpha > 0$ on définit une fonctionnelle exponentielle de ξ par

$$A_t = \int_0^t \exp\{(1/\alpha)\xi_s\}ds, \quad t \geq 0,$$

et par $\tau(\cdot)$ l'inverse de A,

$$\tau(t) = \inf\{s > 0 : A_s > t\},$$

en supposant que $\inf\{\emptyset\} = \infty$. Pour tout $x > 0$, on note $\mathbb{P}_x$ la loi sur $\mathbb{D}^+$, l'espace des trajectoires positives et càdlàg, du processus

$$X_t = x \exp\{\xi_{\tau(tx^{-1/\alpha})}\}, \quad t \geq 0.$$

Si $\tau(tx^{-1/\alpha}) = \infty$ alors X_t est identiquement nul. Cette supposition est assez naturelle puisque τ explose seulement dans les cas $k > 0$ ou $k = 0$ et $\lim_{s\to\infty} \xi_s = -\infty$. On note par $\mathbb{P}_0$ la loi du processus identiquement nul. Par un résultat classique sur le changement de temps dû à Volkonskii on sait que sous les lois $(\mathbb{P}_x, x \geq 0)$ le processus X est un processus de Markov fort par rapport à la filtration $(\mathcal{G}_t = \mathcal{D}_{\tau(t)}, t \geq 0)$. De plus, par construction, le processus X est un processus auto–similaire d'indice α, qui meurt en son premier temps d'arrivée en 0, $T_0 = \inf\{t > 0 : X_{t-} = 0 \text{ ou } X_t = 0\}$. Lorsque $k = 0$ et le processus de Lévy ξ ne dérive pas vers $-\infty$, le processus X ne touche jamais à 0. Or, si $k = 0$ mais ξ dérive vers $-\infty$ alors X touche 0 en un temps fini et il le fait de manière continue. Enfin, si ξ a une durée de vie finie alors X touche 0 en un temps fini et il le fait par un saut. Remarquons que ces trois cas de figure sont indépendantes du point de départ de X.

La loi de T_0 sous $\mathbb{P}_x$ est celle de $x^{1/\alpha} I$ sous $\mathbf{P}$ avec I la fonctionnelle exponentielle

$$I := \int_0^\zeta \exp\{(1/\alpha)\xi_t\}dt.$$

Le Thèoreme de Volkonskii nous permet de plus de déterminer le générateur infinitésimal pour X à partir du générateur infinitésimal pour ξ'. On va les noter par $\mathcal{L}$ et $\mathcal{A}$ respectivement. Soit $f : \mathbb{R}^+ \to \mathbb{R}$ telle que $\tilde{f}(\cdot) = f(e^{\cdot})$ appartienne au domaine de $\mathcal{A}$. On a la formule suivante

$$\begin{aligned}
\mathcal{L}f(x) &= x^{-1/\alpha}\mathcal{A}\tilde{f}(\log x) \\
&= x^{1-(1/\alpha)}(-d + \frac{1}{2}a^2)f'(x) + x^{2-(1/\alpha)}\frac{1}{2}a^2 f''(x) \\
&\quad + x^{-1/\alpha}\int_{\mathbb{R}} \left(f(xe^y) - f(x) - yxf'(x)1_{\{|y|<1\}}\right)\Pi(dy) - kx^{-1/\alpha}f(x),
\end{aligned}$$

où (d, a, Π) sont les caractéristiques de ξ'.

La réciproque de la transformation de Lamperti est vraie. On se donne un processus de Markov α–auto–similaire positif X, et $T_0 = \inf\{t > 0 : X_{t-} = 0 \text{ ou } X_t = 0\}$. On définit

$$C_t = \int_0^t X_s^{-1/\alpha} ds, \qquad t < T_0,$$

et B_t son inverse. Alors le processus $\log X_{B_t}, t \geq 0$ est un processus de Lévy à valeurs dans $\mathbb{R}$, tué en un temps exponentiel si $\mathbb{P}_x(X_{T_0-} > 0) = 1$ pour tout $x > 0$.

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Chapitre I

On random sets connected to the partial records of Poisson point processes

Abstract

Random intervals are constructed from partial records in a Poisson point process in $]0,\infty[\times]0,\infty[$. These are used to cover partially $[0,\infty[$; the purpose of this work is to study the random set $\mathcal{R}$ that is left uncovered. We show that $\mathcal{R}$ enjoys the regenerative property and identify its distribution in terms of the characteristics of the Poisson point process. As an application we show that $\mathcal{R}$ is almost surely a fractal set and we calculate its dimension.

Key Words. Poisson point process, Extremal Process, Regenerative sets, Subordinators, Fractal dimensions.
A.M.S Classification. 60 D 05

1 Introduction

Mandelbrot [14] introduced a natural and simple random generalization of Cantor's triadic set, as follows:

Let λ be the Lebesgue measure, ν an arbitrary Borel measure on $]0,\infty]$, and $\mathcal{P} \subset]-\infty,\infty[\times]0,\infty]$ a Poisson point process with characteristic measure $\lambda \otimes \nu$. This means that $\mathcal{P}$ is a countable random set with the property that for $A \subset]-\infty,\infty[\times]0,\infty]$ the cardinality of $A \cap \mathcal{P}$ is a Poisson random variable with parameter $\lambda \otimes \nu(A)$; moreover, for disjoint Borel subsets $A_i \subset]-\infty,\infty[\times]0,\infty]$ the cardinalities of $A_i \cap \mathcal{P}$ are independent random variables. For any $(x,y) \in \mathcal{P}$ he associated the open interval $]x, x+y[$. Those intervals plays the role of cut outs of $\mathbb{R}$. He then studied the structure of the so called "uncovered set"

$$\mathcal{M} = \mathbb{R} \setminus \bigcup_{(x,y)\in\mathcal{P}}]x, x+y[,$$

conditioned to contain 0. Mandelbrot has shown that the set $\mathcal{M}$ is equal in distribution to the closure of the image of a subordinator (i.e., an increasing process that has independent and homogeneous increments). He has also raised the problem of determining under which conditions $\mathbb{R}$ is completely covered by the cut outs and gave a partial solution to this problem. In a paper that was published at the same time, Shepp [21] provided a definitive answer showing that $\mathbb{R}$ is completely covered with probability one if

$$\int_0^1 dx \exp\Big\{ \int_x^\infty (y-x)\nu(dy) \Big\} = \infty,$$

and with probability zero otherwise. The fact that the closed random set $\mathcal{M}$ is equal in distribution to the closure of the image of a subordinator is equivalent to say that $\mathcal{M}$ is a regenerative random closed set in the sense of Hoffmann–Jørgensen [8], and this leads to study the random set $\mathcal{M}$ through the associated subordinator. This approach was used by Fitzsimmons, Fristedt and Shepp [6] to obtain in a simpler way the necessary and sufficient condition of Shepp and many others characteristics of $\mathcal{M}$. The problem of covering $\mathbb{R}$ or more general sets by random bodies has been studied by several authors with different approaches but we will not consider here and we refer to Kahane [9] and the references therein for an historic account.

In the present work we construct an uncovered random set $\mathcal{R}$, in a different way which is partly inspired by a paper by Marchal [15].

Let $\mathcal{P} \subset]0,\infty[\times]0,\infty[$ be a Poisson point process with characteristic measure $\lambda \otimes \nu$ and $p : [0,\infty[\rightarrow [0,1]$ be a measurable function. For every $(x,y) \in \mathcal{P}$ we define x^* as the abscissa of the first point in $\mathcal{P}$ to the right of x with a higher level, say $y^* > y$. In this way for any $(x,y) \in \mathcal{P}$ we associate the interval $[x, x^*[$. We make then a cut out $[x, x^*[$ with probability $p(y)$ and we are interested in the remainder set, $\mathcal{R}$, of points that weren't deleted from $\mathbb{R}^+$.

The class of regenerative sets that arise from our construction differs from that obtained by Mandelbrot. Example belonging to one but not both of such classes is provided (see remarks to Theorems 1 and 3). Regenerative sets that are the image of a stable subordinator can be generated with both methods.

An outline of this note now follows. Section 2 is devoted to present the setting and survey the basic elements on the theory of Extremal Process. In section 3 we obtain some integral test to decide whether $\mathcal{R}$, is bounded, has isolated points, positive Lebesgue measure and further similar properties. In section 4 we recall the definition of regenerative set, preliminaries results on subordinators and regenerative sets and establish that the uncovered random set $\mathcal{R}$ is regenerative. In Section 5 we use the knowledge about subordinators to obtain an explicit formula of the renewal function of the regenerative set $\mathcal{R}$ and an exact formula for the estimation of some fractal dimensions of $\mathcal{R}$.

2 Preliminaries

This section is subdivided in 3 subsections. Subsection 2.1 is devoted to establish mathematically the verbal construction of the uncovered random set $\mathcal{R}$. Once we have built the random set $\mathcal{R}$ we wish to know the probabilities of some related events , such like "$\mathcal{R}$ contains some interval $[0,t[$", "a given point t is in $\mathcal{R}$", "0 is isolated in $\mathcal{R}$", "$\mathcal{R}$ is bounded", etc.. The tools needed for the computation of such probabilities are essentially two well known results: one about Poisson Measures and the other on Extremal Process. These are the subjects of subsection 2.2 and 2.3, respectively.

2.1 Settings

To make precise the construction of the uncovered random set described in the preceding section, let us introduce a Marked Poisson point process, that is, we add a mark to the Poisson point process $\mathcal{P} = \big\{(t,\Delta_t), t > 0\big\}$, in the following way: suppose that to each point (t,Δ_t) we associate a random variable u_t independent of the whole Poisson point process $\mathcal{P}$ and that the u_t's are independent identically distributed (i.i.d.) with uniform law over $[0,1]$. We know by the marking Theorem (see [11]) that the process $\mathcal{P}' = \{(t, \Delta_t, u_t),\ t > 0\}$ is also a Poisson point process with characteristic measure $\mu(dt, dy, du) = dt \otimes \nu(dy) \otimes du$ on $]0,\infty[\times]0,\infty[\times[0,1]$. Let $(\mathcal{G}_t)_{t\geq 0}$ denote the completed natural filtration generated by $((t,\Delta_t,u_t); t\geq 0)$. For every $(x,y)\in\mathcal{P}$ define the associated x^* by

$$x^* = \inf\{\ x' > x | y' \geq y,\ (x',y') \in \mathcal{P}\ \}.$$

Let T be the set of left end points of the intervals $[x, x^*[$ that are deleted from $\mathbb{R}^+$, i.e.,

$$T = \{x > 0 \mid p(y) > z,\ (x,y,z) \in \mathcal{P}'\ \}.$$

Therefore the uncovered random set, $\mathcal{R}$, is given by

$$\mathcal{R} = [0,\infty) \setminus \bigcup_{x\in T} [x, x^*[. \tag{1}$$

Clearly $0 \in \mathcal{R}$.

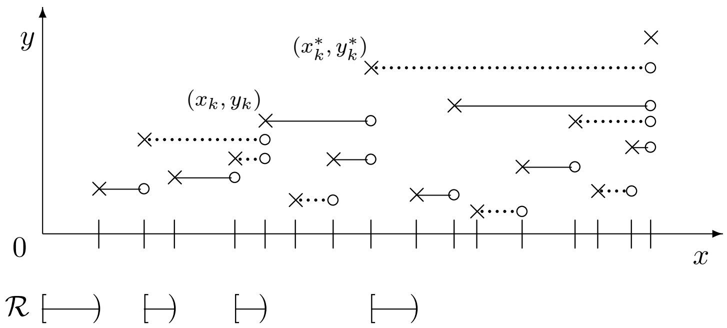

Figure 1 Uncovered set

In Figure 1, the points $\times$ are some points of a P.P.P. So the $\times$ and the $\circ$ denote respectively the left and right extremities of the possible intervals to cover $\mathbb{R}^+$. We have drawn with a doted line the intervals that are not used to cover and with a continuous line those used to cover $\mathbb{R}^+$. Last, under the graph the union of intervals shows the resulting uncovered set.

In order to get explicit and precise formulas we will make a technical assumption but the methods here used can be applied in the general case.

We assume: ν is an atom–less Borel measure such that its tail, $\overline{\nu}(y) = \nu]y,\infty[$, is finite for any $y > 0$, is strictly decreasing and its right limit at zero is infinite, i.e., $\overline{\nu}(0^+) = \infty$. This last has an immediate consequence on the points of the Poisson point process $\mathcal{P}$. If we take any right neighborhood, B_ϵ of zero in $\mathbb{R}^+$ the Poisson random variable card$\{(x,y) |\ (x,y) \in \mathcal{P} \cap \{]0,t] \times B_\epsilon\}\}$ is infinite a.s., for any $t > 0$. More precisely, the points of the Poisson point process are dense in $\mathbb{R}^+$.

It is well known that the distribution of a Poisson point process is determined by its characteristic measure. Let $\mathcal{D}$ and $\mathcal{O}$ be two Poisson point process with the same characteristic measure and a function $p : [0,\infty[\to [0,1]$. By construction we have that two uncovered random sets, say $\mathcal{R}$ and $\mathcal{R}'$, generated via p and the Poisson point process $\mathcal{D}$ and $\mathcal{O}$, respectively, are equal in distribution. To illustrate this and help the reader to become acquainted with the uncovered random sets constructed here, we present the following

Example 1. Let $\mathsf{p} \in [0,1]$ and $\nu(dx)$ an arbitrary Borel measure. Denote by $\mathcal{R}_{\mathsf{p}}$, the uncovered random set generated through the points of a Poisson point process with characteristic measure $\lambda \otimes \nu$ and a constant function p equal to p. It is plain that $\mathcal{R}_0 = \mathbb{R}^+$ a.s. and $\mathcal{R}_1 = \{0\}$ a.s. Later we shall show that the converse also holds, that is, if $\mathcal{R} = \mathbb{R}^+, (\mathcal{R} = \{0\})$ a.s. then the function p is ν–a.s. constant equal to 1 (0) (see Proposition 1 below). The structure of the uncovered random set $\mathcal{R}_{\mathsf{p}}$ with $\mathsf{p} \in]0,1[$ is not so simple; nevertheless, one can show that in this case it has the scaling property, that is, for any $c > 0$ the random sets $\mathcal{R}_{\mathsf{p}}$ and

$$c\,\mathcal{R}_{\mathsf{p}} = \{cx|\ x \in \mathcal{R}_{\mathsf{p}}\}$$

have the same distribution and we say that $\mathcal{R}_{\mathsf{p}}$ is self–similar. To show this we restrict ourselves to the case $\nu(dx) = \alpha x^{-\alpha-1}dx$, a general proof to this fact will be given as a consequence of Theorem 2 below. Indeed, let $f(x,y) = (cx, c^{1/\alpha}y)$. It is well known that

$$f(\mathcal{P}) = \{f(x,y)|\ (x,y) \in \mathcal{P}\},$$

still is a Poisson point process with characteristic measure $\lambda \otimes \nu \circ f$, i.e., for any measurable set $A \subset]0,\infty[\times]0,\infty[$

$$\lambda \otimes \nu \circ f(A) = \lambda \otimes \nu\{(x,y)|f(x,y) \in A\}.$$

Denote by $\mathcal{R}'_{\mathsf{p}}$ the uncovered random set generated via p and $f(\mathcal{P})$. It is straightforward that the measures $\lambda \otimes \nu$ and $\lambda \otimes \nu \circ f$ are equal, thus $\mathcal{R}_{\mathsf{p}}$ and $\mathcal{R}'_{\mathsf{p}}$ have the same distribution. On the other hand, as f scales the x–axis by a factor c it is immediate that $\mathcal{R}'_{\mathsf{p}}$ is equal to $c\,\mathcal{R}_{\mathsf{p}}$.

Remark 1. If a self–similar random set $\mathcal{R}_{\mathsf{p}}$ is regenerative then it must be equal in distribution to the image of a stable subordinator (see example 2 below).

2.2 Campbell's formula

Let N be the Poisson random measure on $]0,\infty[\times]0,\infty[$ defined by

$$N\big(]0,t] \times A\big) = \sum_{\{0<s\leq t;(s,\Delta_s)\in\mathcal{P}\}} 1_{\{\Delta_s \in A\}}$$

for any $t > 0$ and $A \subset]0,\infty[$ measurable. Let $f :]0,\infty[\times]0,\infty[\longrightarrow [0,\infty[$ be a positive measurable function. Define the random variable

$$< N, f >= \sum_{(s,\Delta_s)\in\mathcal{P}} f(s,\Delta_s).$$

The following Lemma provides a criteria to decide whether the random variable $< N, f >$ is finite a.s. as well as a expression of its Laplace transform. This is a classical result and can be found in any text book about Poisson random measures, we refer e.g. to [11] p. 28.

Lemma 1 (Campbell's Theorem and Exponential Formula).
The variable $< N, f >$ is finite a.s. if and only if

$$\int_{]0,\infty[\times]0,\infty[} \min\{1, f(x)\}\lambda \otimes \nu(dx) < \infty.$$

And if this last holds, the Laplace transform of $< N, f >$ is given by

$$\mathbf{E}\left[\exp\{-q < N, f >\}\right] = \exp\Big\{-\int_{]0,\infty[\times]0,\infty[} (1 - e^{-qf(x)})\lambda \otimes \nu(dx)\Big\}.$$

2.3 Some facts about Extremal Process

Let F be any distribution function on $\mathbb{R}$. We will say that a process $X(t)$ for $t \geq 0$, is a *Process Extremal–F* if its finite dimensional distribution functions are given by

$$\begin{aligned} &\mathbf{P}\left(X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_n) \leq x_n\right) \\ &= F^{t_1}(x'_1) F^{t_2-t_1}(x'_2) \cdots F^{t_n-t_{n-1}}(x'_n), \end{aligned} \tag{2}$$

for any $0 \leq t_1 < t_2 < \ldots < t_n$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}$, and $x'_k = \bigwedge_k^n x_j$. Define the process $J(t) = \sup\{\Delta_s \mid (s, \Delta_s) \in \mathcal{P}, 0 < s \leq t\}$. It is easy to verify that the process $\{J(t), t \geq 0\}$ is a process extremal–F with $F(x) = \exp\{-\overline{\nu}(x)\}$, for $x \geq 0$. Indeed, take $0 < t_1 < t_2$, and $x_1, x_2 \in \mathbb{R}^+$, the bivariate distribution of J is given by

$$\begin{aligned} &\mathbf{P}\left(J(t_1) \leq x_1, J(t_2) \leq x_2\right) \\ &= \mathbf{P}\left(J(t_1) \leq x_1 \wedge x_2, \sup_{t_1 < u \leq t_2} \{\Delta_u\} \leq x_2\right) \\ &= \exp\left\{-t_1 \overline{\nu}(x_1 \wedge x_2)\right\} \exp\left\{-(t_2 - t_1)\overline{\nu}(x_2)\right\}, \end{aligned}$$

where last equality follows from the identity

$$\left\{\mathrm{card}\left\{0 < s \leq t : \Delta_s \in (u, \infty)\right\} = 0\right\} = \left\{J(t) \leq u\right\}$$

and the independence of the counting processes. Following this pattern we verify that the n–variate distribution function of J satisfies (2). This is the constructive approach of an extremal process given by Resnick [19]. In the remainder of this subsection we recall some properties about general extremal process which can be found in [19] section(4.3), [20] and [18]. Let F be any distribution function on $\mathbb{R}$ with support $[a, b], -\infty \leq a < b \leq \infty$. Then

(i) X is stochastically continuous.

(ii) There is a version in $D(0, \infty)$, the space of right continuous functions on $(0, \infty)$, with left limits.

(iii) X has non-decreasing paths and almost surely

$$\lim_{t\to\infty} \uparrow X(t) = b, \qquad \lim_{t\to 0} \downarrow X(t) = a.$$

(iv) X is a Markov jump processes with

$$\mathbf{P}\left(X(t+s) \leq x \mid X(s) = y\right) = \begin{cases} F^t(x) & \text{if} \quad x \geq y \\ 0 & \text{if} \quad x < y \end{cases}$$

for $t > 0$ and $s > 0$. Set $Q(x) = -\log F(x)$. The parameter of the exponential holding time at x is $Q(x)$, and given that a jump is due to occur the process jumps from x to $]-\infty, y]$ with probability

$$\begin{cases} 1 - \left(Q(y)/Q(x)\right) & \text{if} \quad y > x \\ 0 & \text{if} \quad y \leq x. \end{cases}$$

The definition of extremal process is given for any distribution function but for continuous distribution functions there is essentially only one extremal process because general extremal process generated from a continuous distribution function may be obtained via a change of scale from the process extremal–Λ, where

$$\Lambda(x) = \exp\{-e^{-x}\} \quad \text{for } x \in \mathbb{R}\,.$$

The processes extremal–Λ and any process $X(t)$ extremal–F with F continuous are connected via the following measurable function. Define

$$S(x) = -\log\{-\log F(x)\} \qquad \text{for } x \in \mathbb{R}\,.$$

Note that $S(x)$ is continuous, non–decreasing and $-\infty \leq S(x) \leq \infty$. It can be verified directly from the definition that the process $\{S(X(t))\}_{t\geq 0}$ is extremal and is generated by $\Lambda(x)$. In the case of the process $\{J(t)\}_{t\geq 0}$, defined previously, the corresponding function S, is given by $S(x) = -\ln \overline{\nu}(x)$. The advantage of working with a process extremal–Λ is frequently the calculations are easier thanks to its additive structure (this is maybe the most important special property of this process). More precisely, let X be extremal–Λ. Pick t_0 arbitrary. Let $t_0 < \tau_1 < \tau_2 < \dots$ be the times of jumps of $X(t)$ in $]t_0, \infty[$ and set $Z_0 = X(t_0)$, $Z_n = X(\tau_n) - X(\tau_{n-1}), n \geq 1$. Then the random variables, $\big\{Z_n, n \geq 1\big\}$, are i.i.d. with common distribution exponential of parameter 1, independent of Z_0 which has the distribution $\Lambda^{t_0}(x)$. Remark that for $s > t_0$ this result yields the representation

$$X(s) = Z_0 + \sum_{j=1}^{\mu]t_0,s]} Z_j$$

where $\mu]t_0, s]$ is the number of jumps of X in $]t_0, s]$ and it is not independent of $\{Z_j\}$. So for a general process with continuous distribution function F, we have

$$S\big(X(s)\big) = Z_0 + \sum_{j=1}^{\mu]t_0,s]} Z_j.$$

Let S^{-1} denote the right continuous inverse of S, that is,

$$S^{-1}(x) = \inf\{z|S(z) > x\}.$$

By inversion we obtain

$$\big\{X(s), s \geq t_0\big\} =^d \big\{S^{-1}\big(Z_0 + \sum_{j=1}^{\mu]t_0,s]} Z_j\big), s \geq t_0\big\}.$$

Define the inverse process $\big\{X^{-1}(x), a \leq x \leq b\big\}$ by

$$X^{-1}(x) = \inf\big\{z \mid X(z) > x\big\}.$$

It is also directly obtained from the definition that if the process X is extremal–Λ then the process $\widetilde{X}(t) = -\log X^{-1}(-\log t)$, is also extremal–$\Lambda$. The following Lemma will be our major tool in the estimation of the probability of the event $t \in \mathcal{R}$.

Lemma 2.
Let X be extremal–F with F a continuous distribution function. Let $t > 0$ fixed. Define $T_1 = inf\{s \mid S(X(s)) = S(X(t))\}$ and for $n \geq 1$ $T_{n+1} = \inf\big\{t \mid S(X(t)) = S(X(T_n^-))\big\}$, so that $\big\{T_j, j \geq 1\big\}$ is the sequence of jump times of $S(X(\cdot))$ on $]0,t]$ ranked in decreasing order. Then

$$\big\{X(T_j^-), j \geq 1\big\} =^d \left\{S^{-1}\Big(S(X(t)) - \sum_{i=1}^{j} Z_i\Big), j \geq 1\right\}$$

Where $\big\{Z_n, n \geq 1\big\}$ are i.i.d. exponential random variables independent of $X(t)$.

The proof of this Lemma is a slight variation to that of Theorem 8 in Resnick [18] so we give a *Sketch of proof* It is clear that it is enough to consider the case $F = \Lambda$. In this case $S(x) = x,\ x \in \mathbb{R}$. As it was noted before the Process $\widetilde{X}(t) = -\log X^{-1}(-\log t)$ is a process extremal–Λ. It is well known that the jump times, $\{\tau_n\}_{n\geq 1}$, after a time $t_0 > 0$ of a extremal process have the same distribution that a function of a sum of independent identically distributed random variables (i.i.d.r.v.'s) with exponential distribution, in fact,

$$\{\tau_n, n \geq 1\} =^d \{\exp\{\log t_0 + W_n\}, n \geq 1\},$$

where $W_n = \sum_1^n Z_i$, and the random variables $Z_n, n \geq 1$ are i.i.d. with exponential distribution (this can be read from [20] p.302). This fact stills true even if t_0 is replaced by a jump time of the extremal process X. So take $\widetilde{T}_0 = \exp\{-X(T_1)\}$, which is clearly a jump time of the process $\widetilde{X}$, thus the process $\widetilde{X}(s)$ remains constant past time $\widetilde{T}_0$ except at times $\tau_1, \tau_2, \ldots$ and hence $X^{-1}(s)$ remains constant for $s < X(T_1)$ except at times $-\log \tau_1, -\log \tau_2, \ldots$ However

$$\{X(T_j^-), j \geq 1\} = \{-\log \tau_j, j \geq 1\} =^d \Big\{X(t) - \sum_1^j Z_i, j \geq 1\Big\}.$$

3 First properties of $\mathcal{R}$

By the time homogeneity of the Poisson point process we can suppose, and we shall do, that for every $t > 0$, fixed the process

$$Y_t(s) = \sup\big\{\Delta_{t-u} | (t-u, \Delta_{t-u}) \in \mathcal{P}; u \leq s\big\} \quad \text{for } 0 \leq s \leq t,$$

is a process extremal–F restricted to the time interval $[0, t]$, with F given by $F(x) = \exp\{-\overline{\nu}(x)\}$. Thus the law of $Y_t(s)$ for any $s \leq t$ is given by

$$\mathbf{P}(Y_t(s) \leq x) = F^s(x) \quad x \geq 0.$$

Throughout this note the function $S(x)$ will be defined by

$$S(x) = -\log \overline{\nu}(x)$$

and then the distribution function F and S are related by

$$F(x) = \exp\big\{-\exp\{-S(x)\}\big\}.$$

For $t > 0$ fixed, let Γ_t be the set of times between 0 and t where the process $Y_t(\cdot)$ jumps, i.e.,

$$\Gamma_t = \big\{s \mid Y_t(s) > Y_t(s^-),\ 0 \leq s \leq t\big\}$$

with $Y_t(0^-) = 0$. Note the almost sure equivalence

$$s \in \Gamma_t \iff (t-s)^* > t.$$

The proof of the direct implication is straightforward. To prove the converse suppose $0 < s \leq t$ and $s \notin \Gamma_t$. Since the points of the P.P.P. are dense in $\mathbb{R}^+$, there is at least one time r, $0 < r < s$ where the process Y_t jumps, that is, $r \in]0, s[\cap \Gamma_t$. Let

$$v_s = \inf\{r : Y_t(r) = Y_t(s)\}.$$

It is plain that $v_s \in \Gamma_t$, $v_s < s$ and $\Delta_{t-s} \leq \Delta_{t-v_s}$. Moreover, since the measure ν is atom–less the latter is a strict inequality ν–a.s. Therefore $(t-s)^* \leq t$ ν–a.s.

From the preceding equivalence we deduce that for $t > 0$, fixed the only points $x \leq t$ that can be the left extreme of an interval that covers t are those in $\Gamma_t \cap T$ (see figure 2). So we obtain the almost sure equivalence:

$$t \in \mathcal{R} \iff \forall\, s \in \Gamma_t,\; p(\Delta_{t-s}) \leq u_{t-s}, \tag{3}$$

or equivalently

$$t \in \mathcal{R} \iff \forall\, s \in \Gamma_t,\; p(Y_t(s)) \leq u_{t-s}.$$

The equivalence (3) shows two things: that the event "t belongs to $\mathcal{R}$" just depends on the Poisson point process until time t and that we can calculate the probability of the event $t \in \mathcal{R}$, in terms of the process $Y_t(\cdot)$.

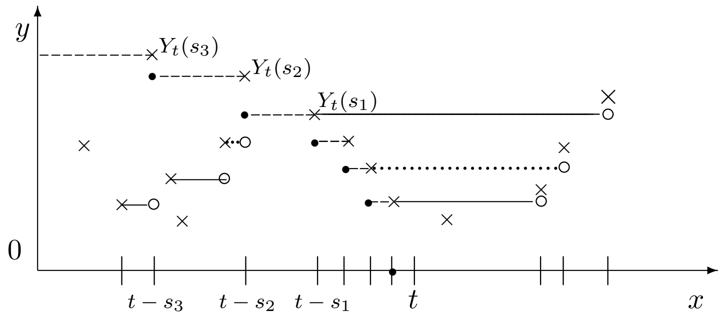

Figure 2 A t fixed that does not belong to $\mathcal{R}$.

We use the same notation as in Figure 1 and for a t fixed we draw with dashed lines the sample path of the process $Y_t(s), 0 \leq s \leq t$. So t does not belong to $\mathcal{R}$ since $t - s_1$ is the left extreme of an interval used to cover $\mathbb{R}$, i.e., $p(Y_t(s_1)) > u_{t-s_1}$.

By means of integral tests in the following results we describe the principal elementary properties of the uncovered random set $\mathcal{R}$.

Proposition 1.

i) Let $Z = \inf\{t > 0, t \notin \mathcal{R}\}$, then $Z > 0$ with probability 1 if and only if

$$\int_0^\infty p(y)\nu(dy) < \infty. \tag{4}$$

In this case Z, follows an exponential law of parameter $\int_0^\infty p(y)\nu(dy)$. In particular, $\mathcal{R} = [0, \infty[$ if and only if $p = 0$, ν–almost surely.

ii) For every $t > 0$

$$\mathbf{P}\big(t \in \mathcal{R}\big) > 0 \iff \int_{0+} p(u)S\big(du\big) < \infty. \tag{5}$$

And if the right hand side of condition (5) holds, then

$$\mathbf{P}(t \in \mathcal{R}) = \int_0^\infty F^t(dy)[1 - p(y)] \exp\Big\{ -\int_0^y p(w)S(dw)\Big\}.$$

iii) *0 is isolated in* $\mathcal{R}$ *a.s. if and only if*

$$\int_{0^+} [1 - p(y)]S(dy) < \infty. \tag{6}$$

iv) $\mathcal{R}$ *is bounded a.s. if and only if*

$$\int^{\infty +} \big[1 - p(y)\big] S(dy) < \infty. \tag{7}$$

v) $\mathcal{R} = \{0\}$ *a.s if and only if* $p = 1$, ν*–a.s.*

Proof. We begin by showing *i)*. Note the equivalence,

$$Z > t \Longleftrightarrow p(\Delta_s) \leq u_s, \forall s \leq t, (s, \Delta_s, u_s) \in \mathcal{P}' .$$

This shows in particular that Z is as $(\mathcal{G}_t)_{t\geq 0}$–stopping time. So the event $''Z > 0''$ has probability 0 or 1. From the former equivalence we also have that

$$\begin{aligned} \mathbf{P}(Z > t) &= \mathbf{E}\Big[\mathbf{E}\left(\{Z > t\} | \{(s, \Delta_s), s \leq t\}\right)\Big] \\ &= \mathbf{E}\Big[\prod_{s\leq t}[1 - p(\Delta_s)]\Big] \end{aligned} \tag{8}$$

The second equality was obtained using the fact that u's are independent identically distributed with distribution uniform on $[0, 1]$. The probability (8) is positive if and only if

$$\prod_{\{(s,\Delta_s),s\leq t\}} \big[1 - p(\Delta_s)\big] > 0 \quad \text{a.s..}$$

This is also equivalent to the convergence a.s of the series $\sum_{\{s\leq t\}} p(\Delta_s)$. We know by Campbell's Theorem that the latter converges a.s. if and only if the condition $\int_0^\infty p(y)\nu(dy) < \infty$ holds. This shows the first assertion of *i)* in Proposition 1. Suppose that $\int_0^\infty p(y)\nu(dy) < \infty$. The fact that Z follows an exponential law with parameter $\int_0^\infty p(y)\nu(dy)$ is a direct application of the exponential formula and the fact that the convergence a.s. of the sum $\sum p(\Delta_s)$ is equivalent to the convergence a.s. of $\sum \log[1 - p(\Delta_s)]$. Indeed,

$$\begin{aligned} \mathbf{P}(Z > t) &= \mathbf{E}\Big[\exp\{\sum_{s\leq t} \log[1 - p(\Delta_s)]\}\Big] \\ &= \exp\Big\{ - t\int_0^\infty p(y)\nu(dy)\Big\} \end{aligned}$$

This entails that $\mathbf{P}(Z > t) = 1$ for all $t > 0$, if and only if $p(y) = 0$ for ν–almost every y.

Next, we show *ii)*. Take $\mathcal{H}_t = \sigma\big\{Y_t(s), 0 \leq s \leq t\big\}$. By equivalence (3) and the independence of the random variables $u's$

$$\begin{aligned} \mathbf{P}\left(t \in \mathcal{R}\right) &= \mathbf{E}\left[\mathbf{E}\left(\{t \in \mathcal{R}\} \mid \mathcal{H}_t\right)\right] \\ &= \mathbf{E}\left[\mathbf{E}\left(p(Y_t(s)) \leq u_{t-s} \text{ for all } s \in \Gamma_t \mid \mathcal{H}_t\right)\right] \\ &= \mathbf{E}\Big[\prod_{y\in A_t} \left(1 - p(y)\right)\Big] \\ &= \mathbf{E}\Bigg[\mathbf{E}\Big[\prod_{y\in A_t} \big(1 - p(y)\big) \mid Y_t(t)\Big]\Bigg] \end{aligned}$$

with $A_t = \{r < \infty \mid Y_t(s) = r \text{ for some } s,\ 0 \leq s \leq t\}$. Let $(Z_k)_{k\geq 1}$ be a sequence of i.i.d.r.v.'s with common exponential distribution and independent of $Y_t(t)$. Set $W_n = \sum_{k=1}^{n} Z_k$, for $n \geq 1$. By Lemma 2

$$\mathbf{P}(t \in \mathcal{R}) = \mathbf{E}\Big(\big[1 - p\big(Y_t(t)\big)\big] H\big(Y_t(t)\big)\Big) \tag{9}$$

where $H(Y_t(t)) = \mathbf{E}\bigg(\prod_{n\geq 1}\Big[1 - p\Big(S^{-1}\big\{S(Y_t(t)) - W_n\big\}\Big)\Big]\bigg)$. Given that $Y_t(t) = y$, the term under the expectation sign is positive a.s if and only if

$$\sum_{n=0}^{\infty} p\big(S^{-1}\big[S(y) - W_n\big]\big) < \infty \text{ a.s..}$$

Since the points $\{W_n\}_{n\geq 1}$ are those of an homogeneous Poisson process (i.e. on $[0, \infty[$ with intensity given by the Lebesgue measure) by Campbell's Theorem the former holds if and only if

$$\int_0^{\infty} p\big(S^{-1}\big[S(y) - x\big]\big)dx = \int_0^{y} p(w)S(dw) < \infty.$$

As $|p(\cdot)| \leq 1$ and $\overline{\nu}(y) < \infty$ for all $y > 0$, then the integral, $\int_0^y p(w)S(dw)$, is finite for all $y > 0$ if and only if this integral is finite in some neighborhood of 0. As a consequence the convergence of the sum in question does not depend on y. This shows that $H(y)$ is strictly positive for all $y > 0$ if and only if $\int_{0+} p(w)S(dw) < \infty$. The conclusion is straightforward. To obtain the expression for the probability of the event $t \in \mathcal{R}$, suppose that the right hand side of (5) holds, by the equation (9) we just have to calculate $H(y)$ for any $y > 0$. This is a direct application of the exponential formula and the fact that the convergence a.s. of the sum

$$\sum p\big(S^{-1}\big[S(y) - S_n\big]\big)$$

is equivalent to the convergence a.s. of the sum

$$\sum \ln\big[1 - p\big(S^{-1}\big[S(y) - S_n\big]\big)\big],$$

Therefore, $H(y) = \exp\big\{-\int_0^y p(w)S(dw)\big\}$, and the result follows.

The proofs of statement in *iii)* and *iv)* are very similar to that of statement in *ii)*. So we only point out the key arguments. To deal with this task define the process $J(0) = 0$ and for $s > 0$,

$$J(s) = \sup\big\{\Delta_v \mid (v, \Delta_v) \in \mathcal{P}; 0 < v \leq s\big\},$$

and its set of jump times $\gamma_0 = \{s \mid J(s) > J(s^-)\}$. It was seen before that a such process is Extremal–F, with $F(x) = \exp -\overline{\nu}(x)$.

Sketch of proof of iii). Let T_1 be the abcissa of the first atom of $\mathcal{P}$ whose ordinate is a local maximum and whose abcissa is the left extremity of an interval that is not used to partially cover $\mathbb{R}^+$. That is,

$$T_1 = \inf\big\{t \in \gamma_0 \mid p(J(t)) \leq u_t\big\}.$$

We thus have that

$$T_1 > s \iff p(J(v)) > u_v, \quad \forall\, v \in]0, s[\cap \gamma_0; \tag{10}$$

in words, $T_1 > s$ if and only if all the jump times of J before s are the left extremities of an interval that is used to partially cover $\mathbb{R}^+$. Now we claim that if $T_1 < \infty$ then $T_1 \in \mathcal{R}$. Indeed, if $T_1 = 0$

there is nothing to prove since $0 \in \mathcal{R}$. In the case $0 < T_1 < \infty$, we have, by the way we construct $\mathcal{R}$, that the only intervals that can be used to cover T_1 are those having a left extremity $< T_1$ but, since T_1 is a local maximum, all these intervals have a right extremity $\leq T_1$. Thus no interval with a left extremity to the left of T_1 covers T_1, that is $T_1 \in \mathcal{R}$. Recall that we have assumed that the measure ν has infinite total mass which implies that 0 is an accumulation point for the jump-times of J. Thus, we have furthermore that

$$0 \text{ is isolated in} \, \mathcal{R} \iff T_1 > 0.$$

To see this we assume first that $T_1 < \infty$. If $T_1 = 0$ then there exists a random sequence of times $(t_n)_{n\in\mathbb{N}} \subset \{t \in \gamma_0 \mid p(J(t)) \leq u_t\}$ such that $t_n > 0$ and $\lim_{n\to\infty} t_n = 0$. By an argument similar to the one used to prove that $T_1 \in \mathcal{R}$ we have that $t_n \in \mathcal{R}$ for all $n \in \mathbb{N}$. Then 0 is not isolated in $\mathcal{R}$. Now, using that 0 is an accumulation point for the jump-times of J and that every jump–time of J to the left of T_1 is the left extremity of an interval used to partially cover $\mathbb{R}^+$ it is easily seen that if $T_1 > 0$ then the only uncovered point to the left of T_1 is 0, that is 0 is isolated in $\mathcal{R}$. We have proved the claim in the case $T_1 < \infty$, but the latter argument proves also that $T_1 = \infty$ implies that $\mathcal{R} = \{0\}$ and the claim follows.

So the random variable T_1 is an stopping time of the completed σ–field $(\mathcal{G}_t)_{t\geq 0}$, and by the zero-one law the event $\{T_1 > 0\}$ has probability zero or one. Therefore it is enough to show that $\mathbf{P}(T_1 > s) > 0$ for some $s > 0$. To this end we use the equivalence (10) and proceed as in the proof of *(ii)*. We omit the details.

Sketch of proof of iv). Let g_∞ be the largest element of $\mathcal{R}$. That is $g_\infty = \sup\{\, s > 0 : \; s \in \mathcal{R}\}$. It is easy to see that this random variable can be also related to the extremal process J as follows: for any $s > 0$,

$$g_\infty < s \Longrightarrow p\big(J(t)\big) > u_t \quad \text{for all} \quad t \in]s, \infty[\cap\gamma_0 \Longrightarrow g_\infty < \infty. \tag{11}$$

Indeed, let $s > 0$ and $(t_n, n \geq 1)$ be the jump times of J after s ranked in increasing order. By construction we have that $t_n^* = t_{n+1}$ for any $n \geq 1$. To see that if $g_\infty < s$ then every t_n is the left extremity of an interval that is used to partially cover $\mathcal{R}$, suppose that at least one of this times (say t_k) is not so; then by an argument similar to the one given before to prove that $T_1 \in \mathcal{R}$ we see that $t_k \in \mathcal{R}$, which is a contradiction since $g_\infty < s < t_k$. This proves the first claim. To prove the second one we use that $\cup_{n\geq 1}[t_n, t_n^*[$ forms a cover $[t_1, \infty[$ of $\mathbb{R}^+$, which implies that $\mathcal{R} \subset [0, t_1[$ and then that $g_\infty < \infty$ since $t_1 < \infty$ a.s.

Now the proof of (iv) uses the equivalence (11) and the additive structure of the extremal process J after time s stated at subsection (2.3). Indeed, proceeding as in the proof of *(ii)* we get that for any $s > 0$,

$$\begin{aligned}\mathbf{P}(g_\infty < s) &\leq \mathbf{P}(p(J(t)) > u_t, \forall t \in]s, \infty[\cap\gamma_0) \\ &= \mathbf{E}(\widetilde{H}(J(s))) \\ &\leq \mathbf{P}(g_\infty < \infty),\end{aligned}$$

where $\widetilde{H}(y) = \mathbf{E}(\prod_{n=1}^\infty p(S^{-1}(S(y) + W_n)))$ for $y > 0$ and $(W_n, n \geq 1)$ as in the proof of *(ii)*. Furthermore, using arguments similar to those given in *(ii)* we prove that $\widetilde{H}(y)$ is strictly positive of every $y > 0$ if and only if $\int^\infty (1 - p(w))S(dw) < \infty$. In this case,

$$\widetilde{H}(y) = \exp - \int_y^\infty (1 - p(w))S(dw),$$

and for any $s > 0$,

$$0 < \mathbf{E}(\widetilde{H}(J(s))) = \mathbf{E}(\exp\{-\int_{J(s)}^\infty (1 - p(w))S(dw)\}) \leq \mathbf{P}(g_\infty < \infty).$$

Thus making $s \to \infty$ we prove that $\mathbf{P}(g_\infty < \infty) = 1$. Now, if $\int^\infty (1-p(w))S(dw) = \infty$, then $H(y) = 0$ for every $y > 0$ and as a consequence $\mathbf{P}(g_\infty < \infty) = 0$.

proof of v) We know that if $p(\cdot) \equiv 1$ then $\mathcal{R} = \{0\}$ a.s.. To show the converse note that

$$\mathbf{P}\left(\mathcal{R} = \{0\}\right) = 1 - \mathbf{P}\left(p(J(s)) \le u_s \text{ for some } s \in \gamma_0\right).$$

Conditioning by $\{J(s), s > 0\}$ we see that

$$\mathbf{P}\left(p(J(s)) \le u_s \text{ for some } s > 0\right) = 0 \iff \sum_{s>0} [1 - p(J(s))] = 0 \text{ a.s.}$$

the former can only happen if $p(\cdot) = 1$ ν–a.e.. □

To continue our study of the random set $\mathcal{R}$ we adopt the approach of regenerative sets.

4 Structure of $\mathcal{R}$

In this section we show that the uncovered set $\mathcal{R}$ is regenerative and to make the paper self contained we first outline some relevant results on regenerative sets and subordinators. All the results about subordinators can be found in Bertoin [1] and those regarding regenerative sets in Kingman [10], Maisonneuve [12, 13], Fitzsimmons, Fristedt & Maisonneuve [5], Meyer[16] and Fristedt [7]. This results will be then used to characterize $\mathcal{R}$.

4.1 Regenerative Sets and Subordinators

According to Kingman [10] a random set $\mathcal{M}$ is a *Standard Regenerative Phenomena* if there exists a function $\mathbf{k} :]0, \infty[\to]0, 1]$ whose limit at zero is 1 and such that

$$\mathbf{P}\left(t_1, t_2, \cdots, t_n \in \mathcal{M}\right) = \prod_{r=1}^{n} \mathbf{k}(t_r - t_{r-1}),$$

for any $0 = t_0 < t_1 < \cdots < t_n$. The term "regenerative" comes from the following property that is obtained from the former equality. For any $l > 0$ the conditional joint distributions of $\mathcal{M} \cap [l, \infty[$ given that $l \in \mathcal{M}$ and given the past before l are the same as the unconditional joint distributions of $\mathcal{M}$. Kingman has shown that a standard regenerative phenomena is the image of a subordinator with positive drift whose law is characterized by $\mathbf{k}$ (for a proof of the latter properties see Kingman [10]). However this definition is not convenient when $\mathbf{P}(t \in \mathcal{R}) = 0$ for all $t > 0$. An adequate and easy to handle definition was given by Maisonneuve [13]. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space, $(\mathcal{Q}_t)_{t \ge 0}$ a filtration in $\mathcal{F}$ and $\mathcal{M} \subset [0, \infty[$ a closed random set in $(\Omega, \mathcal{F})$. $\mathcal{M}$ is a regenerative set relative to $(\mathcal{Q}_t)_{t \ge 0}$ if

(a) $(D_t)_{t \ge 0} = \inf\{\mathcal{M} \cap]t, \infty[\}$ is $(\mathcal{Q}_t)_{t \ge 0}$–adapted;

(b) the law of $\mathcal{M} \circ \theta_{D_t} = \{s - D_t \mid s \in \mathcal{M}, s \ge D_t\}$ given $\mathcal{Q}_t$ and $D_t < \infty$ is the same as $\mathcal{M}$.

See Fitzsimmons et al. [5] for more details. Maisonneuve [12] has shown that the closure of the image of a subordinator is a regenerative set and that any regenerative set is the closure of the image of a subordinator, determined up to *linear–equivalence*, (to be defined below). We are in position to recall some facts about subordinators. The law of a subordinator, σ, is specified by the Laplace transform of its one dimensional distribution. Its Laplace transform can be expressed in the form $\mathbf{E}(\exp\{-\lambda\sigma_t\}) = \exp\{-t\phi(\lambda)\}$ where the function $\phi : [0,\infty[\to [0,\infty[$ is called the Laplace exponent of σ. For each subordinator σ, there exist a unique pair $(k, \mathtt{d})$ of non-negative real numbers and a unique measure Π on $]0,\infty[$ such that $\int \inf\{1,x\}\Pi(dx) < \infty$, and

$$\phi(\lambda) = k + \mathtt{d}\lambda + \int_{(0,\infty)} \left(1 - e^{-\lambda x}\right)\Pi(dx).$$

Conversely, any function ϕ that can be expressed in the previous form is the Laplace exponent of a subordinator. One calls k the killing rate, $\mathtt{d}$ the drift coefficient and Π the Lévy measure of σ. Let c be a constant strictly positive. Thus σ_{tc} still is a subordinator and its Laplace exponent is characterized by $(ck, c\mathtt{d}, c\Pi)$. So the subordinator $\{\sigma_t, t \geq 0\}$ and $\{\sigma_{tc}, t \geq 0\}$ have the same range. Two such subordinators are called linearly equivalent. The measure potential $U(dx)$ of the subordinator σ is often called the *Renewal Measure* and it is given by

$$\int_{[0,\infty[} f(x)U(dx) = \mathbf{E}\left(\int_0^\infty f(\sigma_t)dt\right).$$

The distribution function of the renewal measure $U(x) = \mathbf{E}\left(\int_0^\infty 1_{\{\sigma_t \leq x\}}dt\right)$ for $x \geq 0$, is called *renewal function*. The Laplace transform of the renewal measure is related to the Laplace exponent of the subordinator by

$$\int_{[0,\infty)} e^{-\lambda x}U(dx) = \frac{1}{\phi(\lambda)}.$$

Denote by $\mathcal{M}$ the closure of the image of a subordinator σ so the renewal measure characterizes the law of the regenerative set $\mathcal{M}$ since ϕ characterizes the law of σ and from the previous identity ϕ is characterized by the renewal measure U. By using Fubini's Theorem we obtain that $\mathcal{M}$ has zero Lebesgue measure a.s. if and only if $\mathtt{d} = 0$, and we then say that $\mathcal{M}$ is light. Otherwise we say that $\mathcal{M}$ is heavy. We will also need the following Lemma that relies the renewal measure with the probability that $x \in \mathcal{M}$ for any $x > 0$, fixed.

Lemma 3.

- *(Kesten) If the drift* $\mathtt{d} = 0$, *then* $\mathbf{P}(x \in \mathcal{M}) = 0$ *for every* $x > 0$.
- *(Neveu) If* $\mathtt{d} > 0$, *then the function* $\mathtt{d}^{-1}\,\mathbf{P}(x \in \mathcal{M})$ *is a version of the renewal density* $dU(x)/dx$ *that is continuous and everywhere positive on* $[0,\infty[$.

Concerning regenerative sets:

$0 \in \mathcal{M}$. If 0 is isolated at $\mathcal{M}$, then $\mathcal{M}$ has only isolated points and we say that $\mathcal{M}$ is discrete. If 0 is not isolated then $\mathcal{M}$ does not have any isolated points, we then say that $\mathcal{M}$ is perfect. A right closed random set is regenerative if and only if its closure is regenerative, this can be read from Fitzsimmons et al. [5], page 158. Let $\underline{\mathcal{M}}$ be the set of isolated points and right accumulation points of $\mathcal{M}$ then for every $t > 0$, $\mathbf{P}\left(t \in \mathcal{M} \setminus \underline{\mathcal{M}}\right) = 0$, then $\mathbf{P}(t \in \mathcal{M}) = \mathbf{P}(t \in \underline{\mathcal{M}})$. Let Z be the first time after 0 when t does not belong to $\mathcal{M}$, then there exist a constant $q \in [0,\infty]$ such that $\mathbf{P}(Z > t) = e^{-qt}$ for all t. If $q = 0$, then $\mathcal{M} = \mathbb{R}^+$ a.s. If $0 < q < \infty$, then $\mathcal{M}$ is a.s. the union of a sequence of closed disjoints intervals. If $q = \infty$, then $\mathcal{M}$ has a.s. empty interior. For us one of the most useful results on regenerative sets will be

Lemma 4. *(Fitzsimmons et al. [5])*
Let $(\mathcal{R}_n, n \geq 1)$ be a decreasing sequence of regenerative sets with corresponding renewal functions U_n. Then $\cap_{n=1}^{\infty} \mathcal{R}_n$ is a regenerative set with a corresponding renewal function equal to the vague limit of $c_n U_n$ as $n \to \infty$, where the $(c_n, n \geq 1)$ is an appropriate sequence of constants, in fact we may choose $c_n = c/U_n(1)$, with $c > 0$, a constant.

4.2 $\mathcal{R}$ as a Regenerative Set

Theorem 1.
The uncovered random set $\mathcal{R}$ is a regenerative set relative to $(\mathcal{G}_t)_{t \geq 0}$.

To prove Theorem 1 we will show that if $\mathbf{P}(t \in \mathcal{R})$ is strictly positive for all $t > 0$, then $\mathcal{R}$ is a *Standard Regenerative Phenomena* with function $\mathbf{k}(t) = \mathbf{P}(t \in \mathcal{R})$ and then we proceed by approximation using Lemma 4.

Proof. Let $p : [0, \infty[\to [0, 1]$ be a measurable function continuous at 0 such that $p(0) = 0$ and $\int_{0+} p(y) S(dy) < \infty$. So by Proposition 1, $\mathbf{P}(t \in \mathcal{R}) > 0$ for all $t > 0$. We begin by showing that

$$\lim_{t \to 0} \mathbf{P}(t \in \mathcal{R}) = 1.$$

We know by *ii)* in Proposition 1 that

$$\mathbf{P}(t \in \mathcal{R}) = \int_0^{\infty} F^t(dy) h(y)$$

with $h(y) = [1 - p(y)] \exp\left\{ -\int_0^y p(w) S(dw) \right\}$. Since the measure $F^t(dy)$ converges weakly to the Dirac mass at zero as t goes to zero, $h(y) \leq 1$ for all $y \geq 0$ and p is continuous at 0, then

$$\lim_{t \to 0} \mathbf{P}(t \in \mathcal{R}) = h(0) = 1.$$

Let $0 < t_1 < t_2$. We next show that

$$\mathbf{P}(t_1, t_2 \in \mathcal{R}) = \mathbf{P}(t_1 \in \mathcal{R})\, \mathbf{P}(t_2 - t_1 \in \mathcal{R}).$$

As $\mathbf{P}(t_1 \in \mathcal{R}) > 0$ then

$$\mathbf{P}(t_1, t_2 \in \mathcal{R}) = \mathbf{P}(t_1 \in \mathcal{R})\, \mathbf{P}(t_2 \in \mathcal{R} \,|t_1 \in \mathcal{R}).$$

Given that $t_1 \in \mathcal{R}$, every interval having left end point in $T \cap]0, t_1[$ can not cover any point $s > t_1$, since it does not do for t_1. So the coverage of any point $s > t_1$ just depends on the points of the Poisson point process $\mathcal{P}$ that fall in $]t_1, \infty[\times]0, \infty[$. Moreover, the shifted point process

$$\mathcal{P}^{t_1} = \left\{ (t_1 + s, \Delta_{t_1+s}, u_{t_1+s}) \in \mathcal{P}'; s > 0 \right\},$$

is independent of $\mathcal{G}_{t_1}$ and still is a Poisson point process with characteristic measure $dt \otimes \nu(dy) \otimes du$ (see Meyer [17]). Let T^{t_1}, be the set of points that are the left end points of the intervals that are deleted from $\mathbb{R}^+$, corresponding to $\mathcal{P}^{t_1}$, that is

$$T^{t_1} = \left\{ r > 0 |\ p(\Delta_{t_1+r}) > u_{t_1+r} \right\}.$$

So

$$\mathcal{R}^{t_1} = [0,\infty[\setminus \bigcup_{x\in T^{t_1}} [x, x^*[,$$

enjoys the property

$$\mathcal{R} =^d \mathcal{R}^{t_1} = \mathcal{R} \circ\theta_{t_1} \mid t_1 \in \mathcal{R},$$

with $\mathcal{R} \circ\theta_t(\omega) = (\mathcal{R} - t)^+(\omega) = \{s - t | \ s \in \mathcal{R}, s \geq t\}$. In particular,

$$\begin{aligned}\mathbf{P}(t_2 \in \mathcal{R} | t_1 \in \mathcal{R}) &= \mathbf{P}(t_2 - t_1 \in \mathcal{R} \circ\theta_{t_1} | t_1 \in \mathcal{R}) \\ &= \mathbf{P}(t_2 - t_1 \in \mathcal{R}).\end{aligned}$$

The argument for any $0 < t_1 < t_2 < \cdots < t_n$, is exactly the same if we note the obvious fact $\mathbf{P}(t_n \in \mathcal{R} | t_1, t_2, \cdots t_{n-1} \in \mathcal{R}) = \mathbf{P}(t_n \in \mathcal{R} | t_{n-1} \in \mathcal{R})$. So we have showed that $\mathcal{R}$ is the image of a subordinator with positive drift. To conclude the proof, let $p : [0,\infty[\to [0,1]$, be any measurable function. Set

$$p_n(y) = \begin{cases} p(y) & \text{if } y > 1/n \\ 0 & \text{if } 0 \leq y \leq 1/n \end{cases},$$

and $\mathcal{R}_n$ its associated uncovered set. The function p_n satisfies condition (5) for any Borel measure ν and $n \geq 1$, is continuous at zero and $p_n(0) = 0$. Denote by $\overline{\mathcal{R}}_n$ the closure of $\mathcal{R}_n$. So $\left(\overline{\mathcal{R}}_n : n \in \mathbb{N}\right)$ is a decreasing sequence of regenerative closed random sets and $\overline{\mathcal{R}} = \bigcap_{n\in\mathbb{N}} \overline{\mathcal{R}}_n$. Therefore, by Lemma 4 it follows that $\overline{\mathcal{R}}$ is regenerative and by consequence $\mathcal{R}$ is regenerative. □

Remark 2. Let $p : [0,\infty[\to [0,1]$ and ν a Borel measure such that condition (6) holds. Then the associated uncovered random set $\mathcal{R}$ is a discrete regenerative set. This provides an example that does not belong to Mandelbrot's class of regenerative sets, since the latter are always perfect or trivial (equal to $\{0\}$ a.s.), see e.g. Theorem 1 and corollary 1 in Fitzsimmons et al. [6] or Theorem 7.2 in Bertoin [1].

The following statements rephrases Proposition 1 in terms of subordinators.

Let Z be the first time after 0 when t does not belong to $\mathcal{R}$, it was shown that Z follows an exponential law with parameter q, given by $q = \int_0^\infty p(x)\nu(dx)$. As the only Regenerative sets that are union of disjoint closed intervals are those that are the image of a compound Poisson process with drift. Then, $\mathcal{R}$, is the image of a compound Poisson process with drift if and only if $\int_0^\infty p(x)\nu(dx) < \infty$.

Given that the only Regenerative sets that have isolated points are the image of compound Poisson process without drift, $\mathcal{R}$ is the image of a compound Poisson process without drift if and only if $\int_{0+}[1 - p(x)]S(dx) < \infty$.

If now we are interested in the Lebesgue measure of the regenerative set $\mathcal{R}$, by applying Fubini's Theorem we obtain that $\mathcal{R}$ is heavy if and only if $\int_{0+} p(y)S(dy) < \infty$. Which is equivalent to $\mathcal{R}$ is light if and only if $\int_{0+} p(y)S(dy) = \infty$.

Last, $\mathcal{R}$ is perfect, equivalently, is the image of a subordinator with Lévy measure Π such that $\Pi]0,\infty[= \infty$ if and only if $\int_{0+}[1 - p(y)]S(dy) = \infty$.

5 Further properties of $\mathcal{R}$

Firstly, we will calculate the renewal function of the set $\mathcal{R}$. In the case $\mathcal{R}$ has positive Lebesgue measure a.s., i.e.,

$$\int_{0+} p(y)S(dy) < \infty,$$

from Lemma 3 the function

$$c\int_0^\infty F^t(dy)[1-p(y)]\exp\Big\{\int_0^y p(w)S(dw)\Big\}$$

is a version of the density of the renewal measure of $\mathcal{R}$, for c a positive constant. This means that for $a>0$, the renewal function is given by

$$U[0,a]=c\int_0^a dt\int_0^\infty F^t(dy)[1-p(y)]\exp\big\{-\int_0^y p(w)S(dw)\big\}.$$

We will generalize this result for any measurable function p. Our argument is similar to the analogue of Fitzsimmons et al. [6], Theorem 1. To tackle this problem we will use the following

Lemma 5.
Let v_0, be the first time when $S(x)=0$, that is, $v_0=S^{-1}(0)$. The integral

$$\int_0^a dt\int_0^{v_0} F^t(dy)[1-p(y)]\exp\Big\{\int_y^{v_0} p(w)S(dw)\Big\},$$

is finite for all $a>0$.

Proof. Let $h(y)=[1-p(y)]\exp\Big\{\int_y^{v_0} p(w)S(dw)\Big\}$ and note that $F^t(dy)=te^{-S(y)}F^t(y)S(dy)$. By Fubini's Theorem

$$\begin{aligned}
&\int_0^a dt\int_0^{v_0} F^t(dy)h(y)\\
&=\int_0^{v_0} S(dy)h(y)e^{-S(y)}\Big(\int_0^a dt\, t\exp\big\{-te^{-S(y)}\big\}\Big)\\
&\le\int_0^{v_0} S(dy)h(y)e^{S(y)}\\
&=e^{S(v_0)}\int_0^{v_0} S(dy)[1-p(y)]\exp\Big\{-\int_y^{v_0}[1-p(w)]S(dw)\Big\}\\
&=\int_0^{v_0} d\Big(\exp\Big\{-\int_y^{v_0}[1-p(w)]S(dw)\Big\}\Big)\\
&=\Big(1-\exp\big\{-\int_0^{v_0}[1-p(w)]S(dw)\big\}\Big)
\end{aligned}$$

the second inequality was obtained from an integration by parts in

$$\int_0^a dt\, t\exp\{-tc_y\}=-\frac{a}{c_y}e^{-ac_y}+\frac{1}{c_y^2}\big(1-e^{-ac_y}\big)\le\frac{1}{c_y^2},$$

where $c_y=e^{-S(y)}$. $\square$

Just to ease the notation, in the sequel we will suppose that $v_0=S^{-1}(0)=1$. Now we have all the elements to show the

Theorem 2.
Let $p:[0,\infty[\to[0,1]$ be a measurable function, ν be a atom-less measure such that $\overline{\nu}(x)=\nu]x,\infty[$

is finite, strictly decreasing and $\overline{\nu}(0^+) = \infty$. *Set* $F(x) = \exp\{-\overline{\nu}(x)\}$ *and* $S(x) = -\log\{\overline{\nu}(x)\}$ *for all* $x \geq 0$. *Then the renewal function of* $\mathcal{R}$ *is given by*

$$U[0, a] = a \int_0^\infty F^a(dx) \exp\Big\{ \int_x^1 p(y) S(dy) \Big\}$$

for all $a > 0$.

Proof. When $p(\cdot) = 1$ ν–a.s., it is the subject of *v)* in Proposition (1), that $\mathcal{R} = \{0\}$ a.s., which implies in particular that $U[0, a] \equiv 1$, for all $a > 0$. On the other hand, for any $a > 0$,

$$a \int_0^\infty F^a(dy) \exp\Big\{ \int_y^1 p(x) S(dx) \Big\} = \int_0^\infty dx\ x e^{-x} = 1 = U[0, a]$$

where the first equality was obtained by the change of variables $x = ae^{-S(y)}$. So it remains to study the case $p(\cdot) \not\equiv 1$ in a set of positive ν–measure. For this we build a decreasing sequence of regenerative right closed random sets $\mathcal{R}_n$ as the uncovered random sets generated via

$$p_n(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq \frac{1}{n} \\ p(y) & \text{if } y > \frac{1}{n} \end{cases}$$

and note that for this family of functions the condition (5) holds. By *ii)* in Proposition 1 and Lemma 3 the renewal function is given by

$$U_n[0, a] = \frac{1}{\gamma_n} \int_0^a dt \int_0^\infty F^t(dy) h_n(y)$$

with

$$h_n(y) = \big[1 - p_n(y)\big] \exp\Big\{ - \int_0^y p_n(w) S(dw) \Big\}.$$

By construction

$$\begin{aligned} U_n[0, a] &= \frac{1}{\gamma_n} \int_0^a dt F^t(1/n) + \frac{1}{\gamma_n} \int_0^a dt \int_{1/n}^\infty F^t(dy) h_n(y) \\ &= I_n + II_n. \end{aligned}$$

Take $\gamma_n = \exp\Big\{ - \int_0^1 p_n(y) S(dy) \Big\}$ and note that by monotone convergence

$$II_n \xrightarrow[\text{n}\to\infty]{} \int_0^a dt \int_0^\infty F^t(dy) [1 - p(y)] \exp\Big\{ \int_y^1 p(w) S(dw) \Big\}.$$

Now if 0 is isolated, i.e., if $\int_{0+} [1 - p(w)] S(dw) < \infty$ then

$$\begin{aligned} I_n &= \frac{1}{\gamma_n} \frac{1}{\overline{\nu}(1/n)} \big[1 - e^{-a\overline{\nu}(1/n)}\big] \\ &= \big[1 - e^{-a\overline{\nu}(1/n)}\big] \exp\Big\{ - \int_{1/n}^1 [1 - p_n(y)] S(dy) \Big\} \\ &\xrightarrow[\text{n}\to\infty]{} \exp\Big\{ - \int_0^1 [1 - p(y)] S(dy) \Big\}. \end{aligned}$$

Otherwise, $I_n \to 0$ as $n \to \infty$. From the previous calculations we obtain the expression

$$U[0,a] = U\{0\} + \int_0^a dt \int_0^\infty F^t(dy)[1-p(y)] \exp\big\{\int_y^1 p(w)S(dw)\big\}, \tag{12}$$

for any $a > 0$, with $U\{0\} = \exp\Big\{-\int_0^1 [1-p(y)]S(dy)\Big\}$. Next we deduce the result from the identity (12) by means of some relatively elementary calculations. Let dA_y denotes the measure induced by the increasing function $A_y = \exp\Big\{-\int_y^1 [1-p(w)]S(dw)\Big\}$. From equation (12) and Fubini's Theorem

$$\begin{aligned} U]0,a] &= \int_0^a dt \int_0^\infty F^t(dy)e^{-S(y)}[1-p(y)]A_y \\ &= \int_0^\infty dA_y e^{-S(y)} \int_0^a dt\ te^{-S(y)} \exp\big\{-te^{-S(y)}\big\} \\ &= \int_0^\infty dA_y \Big(\big[1-F^a(y)\big] - ae^{-S(y)}F^a(y)\Big) \\ &= -A_0 + \int_0^\infty S(dy)a^2 e^{-2S(y)}F^a(y)A_y, \end{aligned}$$

the fourth equality was obtained via an integration by parts using that

$$\begin{aligned} &d\Big(\big[1-F^a(y)\big] - ae^{-S(y)}F^a(y)\Big) \\ &= -ae^{-S(y)}F^a(y)S(dy) + ae^{-S(y)}F^a(y)S(dy) - a^2e^{-2S(y)}F^a(y)S(dy), \end{aligned}$$

and since $F^a(y) \sim 1 - ae^{-S(y)}$, as y goes to ∞,

$$\begin{aligned} &\Big(\big[1-F^a(y)\big] - ae^{-S(y)}F^a(y)\Big)A_y\Big|_0^\infty \\ &= -A_0 + \lim_{y\to\infty} A_y\big[1-F^a(y) - ae^{-S(y)}F(y)\big] \\ &= -A_0 + a\lim_{y\to\infty} A_y e^{-S(y)}[1-F(y)] \\ &= -A_0 + a\lim_{y\to\infty} A_y e^{-2S(y)} \\ &= -A_0 + a\lim_{y\to\infty} \exp\Big\{-\int_1^y p(w)S(dw) - S(y)\Big\} \\ &= -A_0. \end{aligned}$$

Therefore,

$$U[0,a] = U\{0\} - A_0 + a\int_0^\infty F^a(dy)\exp\Big\{\int_y^1 p(w)S(dw)\Big\},$$

which ends the proof since $U\{0\} = A_0$. □

Remark 3. Results iii)–v) in Proposition 1 could be obtained as a corollary to Theorem 2. To see iii), recall that 0 is isolated in $\mathcal{R}$ if and only if the renewal function has an atom at 0. It has been showed at the first stage of the proof of Theorem 2 that U has an atom at 0 if an only if $\int_0^1 [1-p(y)]S(dy) < \infty$. To get iv), recall that $U[0,\infty[< \infty$ if and only if $\mathcal{R}$ is bounded a.s. Use (12) and proceed as in the proof of Lemma 5 to show that

$$U[0,\infty[= \exp\Big\{\int_1^\infty [1-p(y)]S(dy)\Big\}.$$

Last, if $\mathcal{R} = \mathbb{R}^+$ by Lemma 3 $U(dx) = cdx$, we can suppose without loss of generality that $c = 1$. Use (12) to conclude that $p = 0$ ν-a.s.

Example 2 (continuation example 1). Let the function $p(y) = \mathsf{p}$ for all $y \geq 0$ with $\mathsf{p} \in (0,1)$. Then the associated set $\mathcal{R}_{\mathsf{p}}$ is indistinguishable of the image of a subordinator stable$(1-\mathsf{p})$.

To show this we just have to calculate the renewal function. By Theorem 2

$$\begin{aligned} U[0,a] &= a\int_0^\infty F^a(dy)e^{-\mathsf{p}S(y)} \\ &= a\int_0^\infty dS(y)ae^{-(1+\mathsf{p})S(y)}\exp\{-a\exp\{-S(y)\}\} \\ &= a^{1-\mathsf{p}}\int_0^\infty x^{\mathsf{p}}e^{-x}dx \\ &= a^{1-\mathsf{p}}\Gamma(1+\mathsf{p}) \end{aligned}$$

where $\Gamma(x)$ denotes the function gamma calculated in x. So the Laplace exponent is given by

$$\phi(\lambda) = c_{\mathsf{p}}\lambda^{1-\mathsf{p}},$$

with $c_{\mathsf{p}} = \big(\Gamma(1+\mathsf{p})\Gamma(2-\mathsf{p})\big)^{-1}$.

5.1 Fractal Dimensions of $\mathcal{R}$

In this subsection we study some fractal dimensions of the regenerative set $\mathcal{R}$. To this end we next introduce two of the most important notions of fractal indices used in probability *Hausdorff and Packing dimensions*. We refer to Falconer [3] for a detailed account on these and other definitions of dimension.

Hausdorff measures and dimension. Let h be a strictly increasing continuous function on $\mathbb{R}^+$ such that $h(0) = 0$ and $h(\infty) = \infty$ and F be a Borel subset of $\mathbb{R}$. A δ–cover of a subset F is a collection $\{U_i\}$ countable (or finite) of subsets of diameter, $|U_i|$, at most $\delta > 0$ that covers F, i.e., $F \subset \bigcup_i U_i$. For any δ we define

$$\mathcal{H}^h_\delta(F) = \inf\left\{\sum_{i=1}^\infty h\big(|U_i|\big) : \{U_i\} \text{ is } \delta\text{–cover of } F\right\}.$$

As δ decreases the class of permissible δ–covers of F is reduced. Therefore the number $\mathcal{H}^h_\delta$ increases and so approaches a limit as $\delta \to \infty$. The Hausdorff h–measure of F is the number

$$\mathcal{H}^h(F) = \lim_{\delta\to 0}\mathcal{H}^h_\delta(F) \in [0,\infty].$$

It can be shown that the mapping $F \longrightarrow \mathcal{H}^h(F)$ defines a measure on a σ–field that includes the Borel sets (see Falconer [4]). Of special interest is the case where $h(x) = x^s$, $s > 0$ in which we write $\mathcal{H}^s$ and speak of s–measure. For any F it is clear that $\mathcal{H}^s(F)$ is non–decreasing as s increases. Furthermore, if $t < s$ then

$$\mathcal{H}^s_\delta(F) \leq \delta^{s-t}\mathcal{H}^t_\delta(F),$$

which implies that if $\mathcal{H}^t(F)$ is positive then $\mathcal{H}^s(F)$ is infinite. Thus there exist a critical value, $\dim_H F$, called the Hausdorff dimension of F such that

$$\begin{aligned} \mathcal{H}^s(F) &= \infty \quad \text{if } 0 \leq s < \dim_H(F) \\ \mathcal{H}^s(F) &= 0 \quad \text{if } \dim_H(F) < s < \infty. \end{aligned}$$

Packing measures and dimension. Let F be a Borel subset of $\mathbb{R}$, $s, \delta > 0$ and $B_r(x)$ a ball of radii r with center in x. Consider

$$\mathcal{P}^s_\delta(F) = \sup\left\{\sum_i |B_{r_i}|^s : \{B_{r_i}(x_i)\} \text{ disjoints such that } x_i \in F, r_i < \delta\right\}$$

Since $\mathcal{P}^s_\delta(F)$ decreases with δ, the limit

$$\mathcal{P}^s_0(F) = \lim_{\delta\to 0} \mathcal{P}^s_\delta(F)$$

exist. It may be shown that the mapping

$$F \longrightarrow \mathcal{P}^s(F) = \inf\left\{\sum_i \mathcal{P}^s_0(F_i) : F \subset \bigcup_i F_i\right\}$$

defines a measure on $\mathbb{R}$, known as the s–dimensional packing measure. Analogous to the case of the Hausdorff dimension we define the fractal index

$$\mathrm{Dim}_P(F) = \inf\big\{s > 0 : \mathcal{P}^s(F) = 0\big\},$$

which is known as the packing dimension. The definition of packing measure and dimension where introduced by Taylor and Tricot [22]. Its well known that for any Borel subset of $\mathbb{R}$

$$0 \le \dim_H(F) \le \mathrm{Dim}_P(F) \le 1,$$

Suitable examples shows that none of the inequalities can be replaced by equality. These fractal indices have the advantage of being defined for any set through measures which are relatively easy to manipulate. A major disadvantage is that in many cases it is hard to calculate or to estimate by computational methods. Although, for regenerative sets there exists some refined results that allow us to obtain its exact Hausdorff and Packing dimension. Let $\phi(\lambda)$ be the Laplace exponent of the regenerative set $\mathcal{R}$, i.e., for any $\lambda > 0$

$$\phi(\lambda) = \left(\int_0^\infty e^{-\lambda t} U(dt)\right)^{-1}$$

with U the renewal function of $\mathcal{R}$ given by Theorem 2. Define the so called lower and upper indices, respectively, of the Laplace exponent ϕ by

$$\underline{\mathrm{Ind}}\ \phi = \sup\big\{\alpha > 0 : \lim_{\lambda\to\infty} \phi(\lambda)\lambda^{-\alpha} = \infty\big\},$$
$$\overline{\mathrm{Ind}}\ \phi = \inf\big\{\alpha > 0 : \lim_{\lambda\to\infty} \phi(\lambda)\lambda^{-\alpha} = 0\big\}.$$

with the usual convention $sup\ \emptyset = 0$. We recall the following results

Lemma 6.
We have a.s for every $t > 0$

$$\underline{Ind}\ \phi = dim_H(\mathcal{R} \cap [0, t])$$
$$\overline{Ind}\ \phi = Dim_P(\mathcal{R} \cap [0, t])$$

For a proof of these facts see chapter 5 section 1 in [1]. In the following Theorem we give formulas to calculate the Hausdorff and Packing dimensions of $\mathcal{R}$ in terms of p and S.

Theorem 3.
Almost surely for every $t > 0$, the Hausdorff and Packing dimensions of $\mathcal{R} \cap [0, t[$ are given by

$$dim_H(\mathcal{R} \cap [0,t]) = \liminf_{y \to 0+} \frac{\int_y^1 (1 - p(w))S(dw)}{-S(y)},$$
$$Dim_P(\mathcal{R} \cap [0,t]) = \limsup_{y \to 0+} \frac{\int_y^1 (1 - p(w))S(dw)}{-S(y)}.$$

Proof. It is well known that $\phi(\lambda) \asymp (U[0, 1/\lambda])^{-1}$, i.e., there exist two positive constants c, c', such that $c(U[0, 1/\lambda])^{-1} \leq \phi(\lambda) \leq c'(U[0, 1/\lambda])^{-1}$. (see [1] Proposition 1.8, page 12). So it is immediate that

$$\underline{\text{Ind}}\ \phi = \sup\big\{\alpha > 0 : \lim_{\lambda \to \infty} U[0, 1/\lambda]\lambda^\alpha = 0\big\},$$
$$\overline{\text{Ind}}\ \phi = \inf\big\{\alpha > 0 : \lim_{\lambda \to \infty} U[0, 1/\lambda]\lambda^\alpha = \infty\big\}.$$

This result will be our major tool in the estimation of the lower and upper indices of ϕ. The conclusion is then obtained by Lemma 6.

When 0 is isolated the affirmation is obvious. Indeed, in one hand, by Proposition 1 we know that $\int_{0+}[1 - p(x)]S(dx) < \infty$ so

$$\frac{\int_y^1 [1 - p(w)]S(dw)}{-S(y)} \xrightarrow[y \to 0]{} 0 = \rho.$$

On the other hand, in the proof of Theorem 2 we have shown that

$$U[0, a] = U\{0\} + \int_0^a dt \int_0^\infty F^t(dy)[1 - p(y)] \exp\Big\{\int_y^1 p(w)S(dw)\Big\}$$

with $U\{0\} = \exp\Big\{-\int_0^1 [1 - p(y)]S(dy)\Big\}$, then $\lambda^\alpha U[0, \frac{1}{\lambda}] \to \infty$ as $\lambda \to \infty$ for any $\alpha > 0$. Therefore $\underline{\text{Ind}}\ \phi = \overline{\text{Ind}}\ \phi = 0 = \rho$.

It remains to show the statement of Theorem 3 when 0 is not isolated. To reach our goal, we will first show that the function $\lambda U[0, 1/\lambda]$ is related to the Laplace-Stieltjes transform of an increasing extended regularly varying function (say h). Then we will use a Tauberian theorem to determine the behavior at infinity of $\lambda U[0, 1/\lambda]$ through that of h. (See e.g. Bingham et al. [2] Chapter 2 for background on extended regularly varying functions.) We first introduce some notation. Let $f_\lambda(x) = (1/\lambda)\exp\{-S(x)\}$ and $f_\lambda^{-1}(\cdot)$ the inverse in the variable x of $f_\lambda(x)$. Observe that $S\big(f_\lambda^{-1}(x)\big) = -\log \lambda x$, for all $x > 0$, thus

$$\lim_{\lambda \to \infty} S\big(f_\lambda^{-1}(x)\big) = -\infty, \quad \text{for all } x > 0,$$

and since S is continuous

$$\lim_{\lambda \to \infty} f_\lambda^{-1}(x) = 0, \quad \text{for all } x > 0.$$

Because of Theorem 2 and making the change of variables $y = f_\lambda(x)$ we get that

$$\begin{aligned}
\lambda U[0, 1/\lambda] &= \int_0^\infty F^{1/\lambda}(dx) \exp\left\{\int_x^1 p(w)S(dw)\right\} \\
&= \int_0^\infty S(dx)\frac{1}{\lambda}e^{-S(x)}F^{1/\lambda}(x) \exp\{\int_x^1 p(w)S(dw)\} \\
&= \int_0^\infty dy e^{-y} \exp\{\int_{f_\lambda^{-1}(y)}^1 p(w)S(dw)\}.
\end{aligned}$$

Now, let S^{-1} be the right-continuous inverse of S, that is $S^{-1}(t) = \inf\{x > 0 : S(x) > t\}$. By a change of variables for Stieltjes integrals and a change of variables $u = e^{-w}$ we get that for any $\lambda, y > 0$

$$\begin{aligned}\exp\left\{\int_{f_\lambda^{-1}(y)}^{1} p(w)S(dw)\right\} &= \exp\left\{\int_{S(f_\lambda^{-1}(y))}^{0} p\left(S^{-1}(w)\right) dw\right\} \\ &= \exp\left\{\int_{1}^{\lambda y} p\left(S^{-1}\left(-\ln(u)\right)\right) \frac{du}{u}\right\} := h(\lambda y).\end{aligned} \tag{13}$$

In short, for every $\lambda > 0$,

$$\lambda U[0,1/\lambda] = \frac{1}{\lambda}\int_0^\infty dy\, e^{-y/\lambda} h(y) = \widehat{h}(1/\lambda),$$

where $\widehat{h}$ denotes the Laplace–Stieltjes transform of h. By the representation theorem for extended regularly varying functions (Theorem 2.2.6 in [2]) we have that the function h is indeed an increasing extended regularly varying function. Furthermore, by a Tauberian theorem (Theorem 2.10.2 in [2]) we have that $h(\lambda) = O(\widehat{h}(1/\lambda))$ and $\widehat{h}(1/\lambda) = O(h(\lambda))$ as $\lambda \to \infty$. We deduce therefrom that

$$\begin{aligned}\underline{\mathrm{Ind}}\ \phi &= \sup\left\{\alpha > 0 : \lim_{\lambda\to\infty} U[0,1/\lambda]\lambda^\alpha = 0\right\}, \\ &= \sup\left\{\alpha > 0 : \lim_{\lambda\to\infty} \lambda^{\alpha-1}\widehat{h}(1/\lambda) = 0\right\} \\ &= \sup\left\{\alpha > 0 : \lim_{\lambda\to\infty} \lambda^{\alpha-1}h(\lambda) = 0\right\} \\ &= \sup\left\{\alpha > 0 : \lim_{\lambda\to\infty} \frac{\lambda/h(\lambda)}{\lambda^\alpha} = \infty\right\} \\ &= \liminf_{\lambda\to\infty} \frac{\log(\lambda/h(\lambda))}{\log(\lambda)}.\end{aligned}$$

Analogously, we get that

$$\overline{\mathrm{Ind}}\ \phi = \limsup_{\lambda\to\infty} \frac{\log(\lambda/h(\lambda))}{\log(\lambda)}.$$

Last, by the fact that

$$\lambda/h(\lambda) = \exp\left\{\int_1^\lambda \left(1 - p\left(S^{-1}\left(-\ln(u)\right)\right)\right) \frac{du}{u}\right\}, \qquad \lambda > 0,$$

and reversing the change of variables done in equation (13) we deduce that

$$\begin{aligned}\liminf_{\lambda\to\infty} \frac{\log(\lambda/h(\lambda))}{\log(\lambda)} &= \liminf_{\lambda\to\infty} \frac{\int_{f_1^{-1}(\lambda)}^1 (1-p(w))S(dw)}{-S(f_1^{-1}(\lambda))} \\ &= \liminf_{y\to 0^+} \frac{\int_y^1 (1-p(w))S(dw)}{-S(y)}.\end{aligned}$$

Analogously, we prove the claim for the $\limsup$. □

Example 3. Let $p(x) = \beta e^{-x}$ for $x > 0, \beta \in]0,1]$ and $\overline{\nu}(x) = x^{-\alpha}$ for $\alpha > 0$. So $S(x) = \alpha \ln x$ and the associated uncovered random set $\mathcal{R}$ has zero Lebesgue measure, is perfect if $\beta \in]0,1[$ and discrete if $\beta = 1$, unbounded and with fractal dimension $1 - \beta$.

Example 4. Let $S(x)$ be as in the previous example and

$$p(x) = \cos^2(1/x).$$

Then the associated uncovered set $\mathcal{R}$ has zero Lebesgue measure, is perfect, bounded and with fractal dimension 1/2.

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Chapitre II

A law of iterated logarithm for increasing self–similar Markov processes

Abstract

We consider increasing self–similar Markov processes $(X_t, t \geq 0)$ on $]0,\infty[$. By using the Lamperti's bijection between self–similar Markov processes and Lévy processes, we determine the functions f for which there exists a constant $c \in \mathbb{R}_+ \setminus \{0\}$ such that $\liminf_{t\to\infty} X_t/f(t) = c$ with probability 1. The determination of such functions depends on the subordinator ξ associated to X through the distribution of the Lévy exponential functional and the Laplace exponent of ξ. We provide an analogous result for the self–similar Markov process associated to the opposite of a subordinator.

Key Words. Self–similar Markov processes, Subordinators, Exponential functional of Lévy process, weak duality of Markov processes.

A.M.S Classification. 60G18, 60G17, 60F15.

1 Introduction

Let $X = (X_s, s \geq 0)$ be a strong Markov process with values in $]0,\infty[$ and denote by $\mathbb{P}_x$ its law starting from $X_0 = x > 0$. For $\alpha > 0$, we say that X is α–self–similar (α–ss), whenever it fulfills the scaling property: for any $c > 0$ and $x > 0$

$$\text{the distribution of } \Big(cX_{(tc^{-1/\alpha})}, t \geq 0\Big) \text{under } \mathbb{P}_x \text{ is } \mathbb{P}_{cx}. \tag{1}$$

Such processes have been introduced by Lamperti [20, 21] under the name of semi–stable processes. We refer to Embrechts and Maejima [12] for some account of their properties and applications.

Recently, Bertoin and Caballero [3] studied the weak behavior of $t^{-\alpha}X_t$ as $t \to \infty$, in the case when X has increasing sample paths (see also Bertoin and Yor [5] for the general case). For any $y > 0$ fixed, they established the weak convergence

$$\mathbb{P}_y\big(t^{-\alpha}X_t \in \cdot\big) \xrightarrow[\text{t}\to\infty]{} \mathbb{P}_{0+}\big(X_1 \in \cdot\big),$$

where $\mathbb{P}_{0^+}\big(X_1 \in \cdot\big)$ is the so-called entrance law from 0^+. The problem that we consider here concerns the rate at which an increasing α–ss process goes to infinity. More precisely, we should like to determine the functions $f : [0,\infty[\to [0,\infty[$, for which, for any $x > 0$

$$\liminf_{t\to\infty} \frac{X_t}{f(t)} \in \,]0,\infty[\qquad \mathbb{P}_x\text{–a.s.} \tag{2}$$

Fristedt [15] (see also Breiman [8]) provided an answer to (2) when X has moreover independent and stationary increments, that is X is a stable subordinator. Later, the problem was solved by Watanabe [32] for increasing ss–process with independent increments. In this paper we treat the case that does not assume neither stationarity nor independence of the increments. Namely, under a rather natural hypothesis on the entrance laws, we provide an explicit characterization of the functions that satisfies (2). Our approach is based, essentially on the main result of Lamperti [21] about the existence of a bijection between self–similar and Lévy processes. Specifically, let $\xi = (\xi_t, t \geq 0)$ be a Lévy process and $(\mathcal{F}_t, t \geq 0)$ its natural filtration. Denote by $\mathbf{P}$ and $\mathbf{E}$ the probability and expectation with respect to ξ. Suppose that ξ does not drift to $-\infty$. For $\alpha > 0$, define

$$A_t = \int_0^t e^{\xi_s/\alpha} ds, \qquad t \geq 0,$$

and the time change associated to A by

$$\tau(t) = \inf\{s : A_s > t\}.$$

For an arbitrary $x > 0$, write by $\mathbb{P}_x$ the law of the process

$$X_t = x \exp \xi_{\tau(tx^{-1/\alpha})}, \qquad t \geq 0.$$

It is straightforward that under $\mathbb{P}_x$, X has the scaling property defined in (1). A classical result on time changes shows that the process X inherits the strong Markov property from ξ. So X is an α–ss Markov process. Conversely any α-ss Markov process can be obtained in this way.

In our setting X is an increasing process so ξ is a subordinator (see Bertoin [1] § 3, for background). The law of a subordinator is characterized by its Laplace transform,

$$\mathbf{E}(e^{-\lambda\xi_t}) = \exp -t\phi(\lambda) \qquad \lambda \geq 0, t \geq 0,$$

where ϕ is the so called Laplace exponent of ξ and can be expressed thanks to the Lévy–Khintchine's formula as

$$\phi(\lambda) = \mathrm{d}\lambda + \int_{]0,\infty[} \big(1 - e^{-\lambda x}\big)\Pi(dx),$$

The term d is called the drift coefficient and Π is the Lévy measure associated to the subordinator ξ, that is, a positive measure such that $\int_{]0,\infty[}(1 \wedge x)\Pi(dx) < \infty$. We suppose henceforth that the drift coefficient is $\mathrm{d} = 0$, and we shall exclude the case ξ is arithmetic, that is when Π is supported by $k\,\mathbb{N}$, for some $k > 0$.

Bertoin and Caballero [3] showed that if

$$0 < \mu = \mathbf{E}\big(\xi_1\big) = \phi'(0+) < \infty,$$

then the α–ss Markov process X started at $x > 0$ converges in the sense of finite dimensional distributions when $x \longrightarrow 0^+$ (cf. Bertoin and Yor [5] for the general case). We then denote by $\mathbb{P}_{0^+}$ the limiting law. Moreover, the law of X_1 under $\mathbb{P}_{0^+}$ is related to the law under $\mathbf{P}$ of the Lévy exponential functional associated to the subordinator ξ, i.e.

$$I = \int_0^\infty e^{-\xi_s/\alpha} ds, \tag{3}$$

by the formula

$$\mathbb{E}_{0^+}\left(f(X_1^{1/\alpha})\right) = \frac{\alpha}{\mu}\mathbf{E}\left(I^{-1}f(1/I)\right), \tag{4}$$

where $f : [0,\infty[\to [0,\infty[$ is a measurable and bounded function. Besides, provided that $\phi'(0+) < \infty$ Carmona, Petit and Yor [10], showed (c.f. Proposition 2.1 in [10]) that the law of I admits a density ρ which is infinitely differentiable on $]0,\infty[$. Furthermore, Proposition 3.3 op. cit. establishes that the law of I is determined by its integral moments, which in turn are given by the formulae

$$\mathbf{E}\left(I^n\right) = \prod_{k=1}^{n} \frac{k}{\phi(k/\alpha)} \quad n \in \mathbb{N},$$

and that

$$\mathbf{E}(e^{rI}) < \infty,$$

for every $0 < r < \phi(\infty)$. Let us introduce the following technical hypothesis

(H) The density ρ is decreasing in a neighborhood of ∞, and bounded.

Examples which satisfy hypothesis **(H)** are given in Section 6.

Recall that a Borel function $f : \mathbb{R}_+ \to \mathbb{R}_+$, is regularly varying at infinity (resp. at 0) with index β if

$$\frac{f(xt)}{f(t)} \longrightarrow x^\beta, \qquad \text{as} \qquad t \to \infty \quad (resp.\ as\ t \to 0)$$

for every $x > 0$. We refer to Bingham et al. [7] for a complete account of the theory of regular variation.

It is well known in the theory of subordinators that the regular variation at infinity (resp. at 0) of the Laplace exponent ϕ, is related to the behavior at 0 (resp. at ∞) of the subordinator ξ associated to it. So it is natural to expect that the regular variation at ∞ of the Laplace exponent should also be related to the local behavior of any α–ss process associated to ξ. This is indeed the case, but we first need to recall a result on subordinators in order to give a precise statement: let ϕ be regularly varying at infinity with index $\beta \in]0,1[$, let ψ be the inverse of ϕ and

$$g(t) = \frac{\log|\log t|}{\psi(t^{-1}\log|\log t|)}, \qquad 0 < t < e^{-1},$$

then

$$\liminf_{t\to 0} \frac{\xi_t}{g(t)} = (1-\beta)^{(1-\beta)/\beta} \qquad \mathbf{P}\text{–a.s.},$$

see e.g. Bertoin [1], section III.4. It follows easily that g is regularly varying at 0 with index $1/\beta$ and

$$\lim_{t\to 0} \frac{\tau(t)}{t} = 1, \qquad \mathbf{P}\text{–a.s.}$$

This being said, it is straightforward that for any $x > 0$ and X an α–ss process associated to ξ we have that

$$\liminf_{t\to 0} \frac{X_t - X_0}{g(t)} = X_0^{(\alpha\beta-1)/\alpha\beta}(1-\beta)^{(1-\beta)/\beta} \qquad \mathbb{P}_x\text{–a.s.} \tag{5}$$

On the other hand, contrary to what we might expect, it is also the regular variation at infinity of the Laplace exponent that gives us the means to determine the behavior at infinity of an increasing self–similar Markov process. Indeed, we have the following

Theorem 1. *Let ξ be a subordinator such that $0 < \mu = \mathbf{E}(\xi_1) < \infty$ and whose Laplace exponent ϕ is regularly varying at infinity with index $\beta \in]0,1[$. Suppose that the density ρ, of the Lévy exponential functional I of ξ satisfies hypothesis (**H**). For $\alpha > 0$, let X be the α–ss process associated to the subordinator ξ. Define*

$$\mathrm{f}(t) = \frac{\phi(\log\log t)}{\log\log t}, \qquad t > e.$$

Then for any $x > 0$

$$\liminf_{t\to\infty} \frac{X_t}{(t\mathrm{f}(t))^{\alpha}} = \alpha^{-\alpha\beta}(1-\beta)^{\alpha(1-\beta)} \qquad \mathbb{P}_x\text{–}a.s.$$

This result also holds true under $\mathbb{P}_{0^+}$.

From the equation (5) only the local behavior of X under $\mathbb{P}_{0^+}$ remains to be determined. In the next result we fill this gap.

Theorem 2. *Under the hypotheses and notations of Theorem 1, we have that*

$$\liminf_{t\to 0} \frac{X_t}{(t\mathrm{f}(1/t))^{\alpha}} = \alpha^{-\alpha\beta}(1-\beta)^{\alpha(1-\beta)} \qquad \mathbb{P}_{0^+}\text{–}a.s.$$

The rest of this note is organized as follows. In section 2 we state two propositions that enable us to prove Theorem 1. Section 3 is devoted to the proof of these propositions. The proof of Theorem 2 is given in section 4 where we obtain some results on time reversal for a self–similar Markov process. There we also obtain a result analog to Theorem 1 for the self–similar Markov process associated to the opposite of a subordinator near the first time that it hits 0. Finally in section 6 we give some examples.

2 Preliminaries

Let X be an α-ss Markov process with $\alpha > 0$. It is plain that the process $Y = X^{1/\alpha}$, is a 1-ss Markov process, in fact it is the 1–ss process associated to $(1/\alpha)\xi$. Conversely if Y is a 1-ss Markov process then, for any $\alpha > 0$, the process $X = Y^{\alpha}$ is an α–ss Markov process. So we can assume henceforth, without loss of generality, that $\alpha = 1$.

We can deduce from equation (4) that the entrance law $\mathbb{P}_{0^+}(X_1 \in dx)$ has a density

$$p_1(x) = \begin{cases} (\mu x)^{-1}\rho(x^{-1}) & \text{if } 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

with ρ the density of the law of I.

Denote by $U = (U_s, s \geq 0)$ the Ornstein–Uhlenbeck (OU) process associated to the 1–ss Markov process X, (or to the underlying subordinator ξ through Lamperti's transformation if $X_0 = x$ for some $x > 0$) that is

$$U_t = e^{-t} X_{e^t - 1} \qquad t \geq 0.$$

This process inherits the homogeneity and strong Markov property from X, has transition probabilities

$$\widetilde{P}_s f(x) = \mathbb{E}_x\left(f(e^{-s}X_{e^s-1})\right) \quad s \geq 0,$$

for every Borel function f. Moreover, it has a unique invariant probability measure given by the entrance law $p_1(x)dx$. See e.g. Carmona, Petit and Yor [10] for a proof of these facts.

The asymptotic behavior of the OU process U, defined above is described in the next proposition.

Proposition 1. *Let ξ be a subordinator such that $0 < \mu = \mathbf{E}(\xi_1) < \infty$ and whose Laplace exponent is regularly varying at infinity with index $\beta \in]0,1[$. Suppose that the density, $\rho(\cdot)$, of the exponential functional I satisfies **(H)**. Let U be the Ornstein–Uhlenbeck process associated to ξ. If $h :]0,\infty[\to]0,\infty[$, is a decreasing function then for every $x > 0$*

$$\mathbb{P}_x\big(U_s < h(s) \quad i.o. \quad s \to \infty\big) = 0 \quad or \quad = 1,$$

according whether

$$\int^{\infty} \rho(1/h(s))ds < \infty \quad or \quad = \infty.$$

This result also holds true if we suppose that the Lévy measure is finite, $\Pi]0,\infty[< \infty$, instead of the regular variation at infinity of ϕ.

Remark 1. Of course one can derive an integral test from Proposition 1 for the 1–ss Markov process associated to ξ. Indeed, if h is a decreasing function then

$$\mathbb{P}_x\big(X_s < sh(s) \quad \text{i.o. } s \to \infty\big) = 0 \quad \text{or} \quad 1$$

according whether

$$\int^{\infty} \rho(1/h(s))\frac{ds}{s} < \infty \quad \text{or} \quad = \infty.$$

However this result is not really satisfactory unless one has good estimates of ρ.

Despite the characterization of the law of the exponential functional I it is not always possible get an explicit representation of its density. But to obtain the result stated at Theorem 1 we will only need estimations of the behavior of $\log \rho(\cdot)$ near infinity. That is the purpose of the following

Proposition 2. *Let I be the exponential functional associated to a subordinator $(\xi_s, s \geq 0)$ whose Laplace exponent ϕ, varies regularly at infinity with index $\beta \in]0,1[$. Then*

$$-\log \mathbf{P}(I > t) \sim (1-\beta)\varphi^{\leftarrow}(t), \quad t \to \infty, \tag{6}$$

where

$$\varphi^{\leftarrow}(t) = \inf\left\{s > 0\ ,\ \frac{s}{\phi(s)} > t\right\}.$$

If moreover, the density $\rho(\cdot)$ of the law of I, is decreasing on some neighborhood of ∞, then

$$-\log \rho(t) \sim (1-\beta)\varphi^{\leftarrow}(t), \quad t \to \infty. \tag{7}$$

Remark 2. The fact that the tail distribution of I has this asymptotic form implies that the law of I cannot be infinitely divisible (see e.g. Steutel [30] or Bingham et al. [7] section 8.2.8).

If we take for granted Propositions 1 and 2 the proof of Theorem 1 follows by standard arguments.

Proof of Theorem 1. By Proposition 2 and the fact that $\varphi^{\leftarrow}$ is regularly varying with index $\frac{1}{1-\beta}$ we have that for any constant $c > 0$

$$-\log \rho\Big(\frac{1}{c\mathrm{f}(t)}\Big) \sim (1-\beta)c^{-\frac{1}{1-\beta}}\varphi^{\leftarrow}\Big(\frac{1}{\mathrm{f}(t)}\Big) \quad \text{as} \quad t \to \infty.$$

Since $\varphi^{\leftarrow}$ is the inverse of $s/\phi(s)$ we then have that

$$-\log \rho\Big(\frac{1}{c\mathrm{f}(t)}\Big) \sim (1-\beta)c^{-\frac{1}{1-\beta}}\log\log t \quad \text{as} \quad t \to \infty. \tag{8}$$

The statement in Theorem 1 is equivalent to the property (to be proven) that for any $\epsilon > 0$,

$$\mathbb{P}_x\big(X_s < (1-\epsilon)c_\beta s\mathrm{f}(s) \quad \text{i.o. } s \to \infty\big) = 0,$$

and

$$\mathbb{P}_x\big(X_s < (1+\epsilon)c_\beta s\mathrm{f}(s) \quad \text{i.o. } s \to \infty\big) = 1,$$

where $c_\beta = (1-\beta)^{(1-\beta)}$. From the remark after Proposition 1 the former and later equations hold if for any $\epsilon > 0$,

$$\int^\infty \rho(1/\mathrm{f}_{1,\epsilon}(s))\frac{ds}{s} < \infty,$$
$$\int^\infty \rho(1/\mathrm{f}_{2,\epsilon}(s))\frac{ds}{s} = \infty,$$

where

$$\mathrm{f}_{1,\epsilon}(s) = (1-\epsilon)c_\beta\mathrm{f}(s), \qquad \mathrm{f}_{2,\epsilon}(s) = (1+\epsilon)c_\beta\mathrm{f}(s),$$

respectively. Indeed, let $\epsilon > 0$, by equation (8) there exists an s_ϵ such that for every $s > s_\epsilon$,

$$\begin{aligned}-\log \rho\Big(\frac{1}{\mathrm{f}_{1,\epsilon}(s)}\Big) &\geq (1-\beta)(1-\epsilon)(1-\epsilon)^{-\frac{1}{1-\beta}}c_\beta^{-\frac{1}{1-\beta}}\log\log s\\ &= (1-\epsilon)^{-\frac{\beta}{1-\beta}}\log\log s.\end{aligned}$$

Therefore, taking $k_\epsilon = (1-\epsilon)^{-\frac{\beta}{1-\beta}}$, we have

$$\int_{s_\epsilon}^\infty \rho(1/\mathrm{f}_{1,\epsilon}(s))\frac{ds}{s} \leq \int_{s_\epsilon}^\infty (\log s)^{-k_\epsilon}\frac{ds}{s} < \infty,$$

since $k_\epsilon > 1$. Similarly, one shows the divergence of

$$\int^\infty \rho(1/\mathrm{f}_{2,\epsilon}(s))\frac{ds}{s}.$$

We have showed the statement of Theorem 1 for $\alpha = 1$, to show that the result holds for any α, consider the 1–ss process Y associated to the subordinator $\alpha^{-1}\xi$. This subordinator has Laplace exponent $\phi_\alpha(\lambda)$, such that

$$\phi_\alpha(\lambda) = \phi(\alpha^{-1}\lambda) \sim \alpha^{-\beta}\phi(\lambda) \qquad \lambda \to \infty,$$

owed to the regular variation of ϕ. Then one obtain the result readily by means of the α–ss process $X = Y^\alpha$.

□

3 Proofs

This section contains two parts. In the first one, we give the proof of the Proposition 1, which is rather technical so that we decompose it in to several Lemmas. The second part contains the proof of the Proposition 2.

3.1 Proof of Proposition 1

Let $\widetilde{U}$ be process

$$\left\{\widetilde{U}_s = e^{-s}X_{e^s}, s \in \mathbb{R}\right\}.$$

Under $\mathbb{P}_{0+}$ the process $\widetilde{U}$ is a stationary strong Markov process, whose transition probabilities are those of the OU process U defined in the preceding section. In fact, the law of the process $(\widetilde{U}_s, s \geq 0)$ under $\mathbb{P}_{0+}$ is the same as that of the OU process $(U_s, s \geq 0)$ with initial measure the entrance law $\mathbb{P}_{0+}(X_1 \in dx) = p_1(x)dx$. This process will enable us to describe the local behavior of the OU process U and in section 4 prove the Theorem 2.

The first ingredient in the proof of Proposition 1 is the following

Lemma 1. *For any $x > 0$*

$$\mathbb{P}_{0+}\left(\lim_{h\to 0}\frac{\widetilde{U}_h - \widetilde{U}_0}{h} = -\widetilde{U}_0 \quad \middle| \widetilde{U}_0 = x\right) = 1.$$

Remark 3. In Lemma 1 we do not impose any constraint in the way we make h tend to 0. That is why we postpone its proof until section 4.

We suppose in the sequel that the starting point of the OU process U is fixed, $U_0 = x > 0$, unless otherwise stated. The main argument in the proof of Proposition 1 is that of Breiman's [8] proof of a law of iterated logarithm for stable subordinators, which in turn is an adaptation of Motoo's [24] proof of Kolmogorov's test for diffusions. Here is an outline of such a method, see e.g. Ito and McKean [18] for Motoo's proof of Kolmogorov's test. Let $\{R_n, n \geq 0\}$ be the successive return times of the OU process U to its starting point, i.e.,

$$R_{n+1} = \inf\{t > R_n : U_t = U_0\},$$

with $R_0 = 0$. Denote by $R = R_1$ and T_y the first hitting time of a level $y > 0$ by the OU process U, i.e.,

$$T_y = \inf\{t > 0 : U_t = y\}.$$

Define the function $g(x, y)$ by

$$g(x, y) = \mathbb{P}_x(T_y < R) = \mathbb{P}_x\Big(\inf_{t\in[0,R]} U_t < y\Big), \qquad y < x.$$

By the homogeneity and strong Markov property of U the random variables

$$\{R_{n+1} - R_n, n \geq 0\}$$

are independent and identically distributed with the same law as R. The fact that the OU process U has a unique invariant probability implies that

$$\mathbb{E}_x(R) < \infty.$$

Then, by the strong law of large numbers

$$\frac{R_n}{n} \xrightarrow[\text{n}\to\infty]{} \mathbb{E}_x(R), \qquad \mathbb{P}_x\text{–a.s.}$$

Besides, we can deduce from the Lemma 1, using the homogeneity and the strong Markov properties of the OU process $\widetilde{U}$ that the OU process U hits points from above and it leave it from below and more importantly the range of the excursion process $\{U_s, s \in [0, R]\}$ is a compact interval with U_0 in its interior. Thanks to these facts Motoo's arguments apply to show that for any decreasing function $h :]0, \infty[\to]0, \infty[$ we have the $\mathbb{P}_x$–a.s. inclusion of sets

$$\big\{U_s < h(s) \quad \text{i.o.} \quad s \to \infty\big\} \subseteq \big\{\inf_{s\in[R_n,R_{n+1}]} U_s < h(c_1 n) \quad \text{i.o.} \quad n \to \infty\big\}$$

$$\big\{\inf_{s\in[R_n,R_{n+1}]} U_s < h(c_2 n) \quad \text{i.o.} \quad n \to \infty\big\} \subseteq \big\{U_s < h(s) \quad \text{i.o.} \quad s \to \infty\big\}$$

with $c_1, c_2 > 0$ constants that depend only on $\mathbb{E}_x(R)$. Therefore, by a standard application of the Borel–Cantelli Lemma we get that if the integral

$$\int^{\infty} g(x, h(s))ds \tag{9}$$

converges then

$$\mathbb{P}_x\left(\inf_{t\in[R_n,R_{n+1}]} U_t < h(c_1 n) \quad \text{i.o.} \quad n \to \infty\right) = 0,$$

whereas if (9) diverges then

$$\mathbb{P}_x\left(\inf_{t\in[R_n,R_{n+1}]} U_t < h(c_2 n) \quad \text{i.o.} \quad n \to \infty\right) = 1.$$

The proof reduces then to estimate the function $g(x, y)$, that is, estimate the distribution of the depth of the excursion and to show that the criterion does not depend of x. Namely that

$$g(x, y) \asymp \rho(1/y) \qquad \text{as} \qquad y \to 0, \tag{10}$$

that is, there exists two positive constants b_1, b_2 such that

$$b_1\rho(1/y) \leq g(x, y) \leq b_2\rho(1/y) \qquad \text{as} \qquad y \to 0.$$

In [2] Bertoin gets an estimate for the function g when the underlying self–similar process is a stable subordinator. His proof provides the key steps for our estimation of the function g.

Lemma 1 enable us to follow the arguments of section 3 in [2] and this yields

Lemma 2. *Assume ρ is bounded. For every $x, y > 0$ and $q > 0$ we have*

(i)

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^R 1_{\{U_s\in[y-\epsilon,y]\}} ds = \frac{1}{y} Card\big\{t \in [0, R[: U_t = y\big\}$$

both $\mathbb{P}_x$–a.s. and in $L^1(\mathbb{P}_x)$.

(ii)

$$\lim_{\epsilon\to 0} \mathbb{E}_y\left(\frac{1}{\epsilon}\int_0^R e^{-qs} 1_{\{U_s\in[y-\epsilon,y]\}} ds\right) = \frac{1}{y}.$$

(iii)

$$\mathbb{E}_x(R) = \frac{\mu}{\rho(1/x)}$$

(iv)

$$\mathbb{E}_x\Big(\sum_{t\in[0,R[} 1_{\{U_t=y\}}\Big) = \frac{\rho(1/y)}{\rho(1/x)}.$$

Proof. First note that an application of Dynkin's formula shows that the measure

$$\nu(f) = \frac{\mathbb{E}_x(\int_0^R f(U_s)ds)}{\mathbb{E}_x(R)},$$

is an invariant law for the OU process. Moreover, by the uniqueness of the invariant law we have that

$$\nu(f) = \int_0^\infty f(z)(\mu z)^{-1}\rho(1/z)dz, \tag{11}$$

for every function f non–negative and measurable. Next, if we take for granted Lemma 1, then we may simply repeat the arguments of [2] to prove (i–iii). The statement in (iv) follows from (i), (iii) and the identity in equation (11). □

A standard application of the strong Markov property at time R shows that for every $y > 0$

$$\begin{aligned}\mathbb{E}_x\Big(\sum_{t\in[0,R[} 1_{\{U_t=y\}}\Big) &= \mathbb{P}_x(T_y < R)\left(1 + \mathbb{P}_y(R < T_x) + (\mathbb{P}_y(R < T_x))^2 + \cdots\right)\\ &= \frac{\mathbb{P}_x(T_y < R)}{\mathbb{P}_y(T_x \le R)}.\end{aligned} \tag{12}$$

Therefore, by comparing (iv) in Lemma 2 and equation (12) we get that

$$\mathbb{P}_x(T_y < R) = \mathbb{P}_y(T_x \le R)\frac{\rho(1/y)}{\rho(1/x)}.$$

Since by hypothesis **(H)** we have that

$$\lim_{y\to 0}\rho(1/y) = 0,$$

then we may conclude that the statement in (10) is equivalent to

$$\liminf_{y\to 0}\mathbb{P}_y(T_x \le R) > 0. \tag{13}$$

We next focus in the proof of (13). To that end, we will obtain more precise information on the duration R of the excursion as the starting point tends to 0 using the well known fact that the distribution of R can be characterized in terms of the resolvent density. We introduce some notation.

Define by $\{L_t^y, t > 0\}$ the "local time" at $y > 0$ of the OU process U, that is

$$L_t^y = \frac{1}{y}\sum_{0<s\le t} 1_{\{U_s=y\}}.$$

Let $x \ge 0, y > 0$, and $u_1(x, y)$ the 1-potential of L_t^y under $\mathbb{P}_x$, for $x > 0$ and $\mathbb{P}_{0+}$ for $x = 0$ i.e.,

$$u_1(x, y) = \mathbb{E}_x\Big(\int_{]0,\infty[} e^{-s}dL_s^y\Big).$$

We have by the strong Markov property that

$$u_1(x,y) = y^{-1}\frac{\mathbb{E}_x(e^{-T_y})}{1-\mathbb{E}_y(e^{-R})}. \tag{14}$$

Lemma 3. *For every $y>0$, we have*

$$u_1(0,y) = \frac{1}{\mu}\int_0^{1/y} dz\rho(z).$$

In particular $u_1(0,y)$ is a bounded and continuous function.

Proof. Let $\mathcal{R}_1$ denote the 1-resolvent operator of the OU process, that is,

$$\mathcal{R}_1 f(x) = \mathbb{E}_x\Big(\int_0^\infty e^{-s} f(U_s)ds\Big) = \int \mathcal{R}_1(x,dy)f(y),$$

for any Borel positive function f, and $x\geq 0$. Our first aim is to show that the measure $\mathcal{R}_1(0,dy)$ has a density that coincides with $u_1(0,y)$. Indeed, by a change of variables, an application of Fubini's Theorem and the self–similarity of X, we get

$$\begin{aligned}
\mathcal{R}_1 f(0) &= \mathbb{E}_{0^+}\Big(\int_0^\infty e^{-s} f(U_s)ds\Big)\\
&= \mathbb{E}_{0^+}\Big(\int_0^1 f(uX_{(1-u)/u})du\Big)\\
&= \int_0^1 \mathbb{E}_{0^+}\big(f(uX_1)\big)du.\\
&= \int_0^1 du\int_0^\infty dx(\mu x)^{-1}\rho(1/x)f(xu).
\end{aligned}$$

Straightforward calculations shows that

$$\mathcal{R}_1 f(0) = \int_0^\infty dy f(y)v(y),$$

with $v(y) = \mu^{-1}\int_0^{1/y} dx\rho(x)$. This shows that $\mathcal{R}_1(0,dy)$ has a density $v(y)$, that is continuous and bounded. In particular

$$v(y) = \lim_{\epsilon\to 0}\epsilon^{-1}\int_{y-\epsilon}^{y} v(x)dx.$$

On the other hand,

$$\begin{aligned}
\int_{y-\epsilon}^{y} v(x)dx &= \mathbb{E}_{0^+}\Big(\int_0^{T_y} e^{-s}1_{\{U_s\in[y-\epsilon,y]\}}ds\Big) + \mathbb{E}_{0^+}\Big(\int_{T_y}^\infty e^{-s}1_{\{U_s\in[y-\epsilon,y]\}}ds\Big)\\
&= I_\epsilon + II_\epsilon.
\end{aligned}$$

By the strong Markov property

$$II_\epsilon = \frac{\mathbb{E}_{0^+}(e^{-T_y})}{1-\mathbb{E}_y(e^{-R})}\,\mathbb{E}_y\Big(\int_0^R e^{-s}1_{\{U_s\in[y-\epsilon,y]\}}ds\Big).$$

Using (ii) in Lemma 2 and equation (14) we get that

$$\lim_{\epsilon\to 0}\epsilon^{-1}II_\epsilon = y^{-1}\frac{\mathbb{E}_{0^+}(e^{-T_y})}{1-\mathbb{E}_y(e^{-R})} = u_1(0,y).$$

Thus the proof will be completed if we show that, $\epsilon^{-1}I_\epsilon \to 0$ as $\epsilon \to 0$. Let H_y be the first time that the OU process jumps above the level $y>0$, $H_y = \inf\{s>0 : U_s > y\}$. Indeed, by the Markov property applied at the first passage time of the OU process above the level $y-\epsilon$ we get

$$I_\epsilon \le \mathbb{E}_{0^+}\left(1_{\{H_{y-\epsilon}<H_y\}}e^{-H_{y-\epsilon}}\right)\sup_{z\in[y-\epsilon,y]}\mathbb{E}_z\left(\int_0^{H_y} e^{-s}1_{\{U_s\in[y-\epsilon,y]\}}ds\right).$$

Applying repeatedly the Markov property at the stopping time R we get for every $z>0$,

$$\mathbb{E}_z\left(\int_0^{H_y} e^{-s}1_{\{U_s\in[y-\epsilon,y]\}}ds\right) \le \frac{\mathbb{E}_z\left(\int_0^R e^{-s}1_{\{U_s\in[y-\epsilon,y]\}}ds\right)}{1-\mathbb{E}_z\left(e^{-R}1_{\{R<H_y\}}\right)}.$$

The claimed result now follows from an application of (ii) in Lemma 2 and the fact that $H_{y-\epsilon} \longrightarrow H_y$ as $\epsilon \to 0$ a.s. □

We assume throughout the rest of this section that either ϕ is regularly varying at ∞ with index $\beta\in]0,1[$ or the Lévy measure is finite, $\Pi]0,\infty[<\infty$.

Lemma 4. *One has*

$$\liminf_{y\to 0^+} y^{-1}\,\mathbb{E}_{0^+}\left(e^{-T_y}\right) > 0.$$

Before proving this Lemma let us define a function that will be used in the sequel. Since the function $\phi(\lambda)/\lambda$ is decreasing, there exists a function β_y such that

$$\phi(\beta_y)/\beta_y = y,$$

we denote $\delta_y = e^{1/\beta_y}-1$.

Proof. The statement in Lemma 4 means that

$$\liminf_{y\to 0^+} y^{-1}\mathbb{P}_{0^+}(T_y<\mathbf{e})>0,$$

with $\mathbf{e}$ an exponential random variable independent of the OU process. To show this fact we will need to introduce some notation and recall some results. Let H_y^X be the first passage time above the level y by the 1-ss process X, that is

$$H_y^X = \inf\left\{s>0 : X_s > y\right\}.$$

Bertoin and Caballero [3] showed that under the entrance law $\mathbb{P}_{0^+}$, the law of the pair

$$(H_y^X, X_{H_y^X})$$

is the same as that of

$$(yI\exp\{-VZ\}, y\exp\{(1-V)Z\}),$$

where V, Z and I are independent and V is uniformly distributed on $[0,1]$ and the law of Z is given by

$$\mathbf{P}(Z\in dz) = \mu^{-1}z\Pi(dz) \qquad z>0.$$

So by taking $S_y = \log(1 + H_y^X)$ we get that

$$\big(U_{S_y}/y, S_y\big) \xrightarrow[y\to 0]{\mathfrak{D}} (e^K, 0),$$

where K is a random variable with law

$$\mathbf{P}(K \in dk) = \mu^{-1}\overline{\Pi}(k)dk,$$

and $\overline{\Pi}(k) = \Pi(k, \infty)$. Recall that H_y is the first time that the OU process U jumps above the level y. It is plain that the OU process hits a level $[y, \infty[$ only if the ss process X is already at this level, i.e.

$$\log(1 + H_y^X) \le H_y,$$

for every $y > 0$. Moreover, the weak convergence of U_{S_y}/y implies that

$$\mathbb{P}_{0^+}\big(\log(1 + H_y^X) < H_y\big) \le \mathbb{P}_{0^+}(U_{S_y} \in [0, y[) \longrightarrow 0 \quad \text{ as } y \to 0.$$

So we can suppose henceforth that $\log(1 + H_y^X) = H_y$, for all y small enough.

Let $t > 0$ fixed and ϵ_y an arbitrary function vanishing at 0. For every $y > 0$ such that $t > \epsilon_y$ we have by the strong Markov property applied at time S_y, that

$$\begin{aligned}
&\mathbb{P}_{0^+}(T_y < t)\\
&= \int_y^\infty \int_0^t \mathbb{P}_{0^+}(U_{S_y} \in dz, S_y \in dr)\mathbb{P}_z(\exists s \in [0, t-r], U_s = y)\\
&\ge \int_y^{y(1+\delta_y)} \int_0^{t-\epsilon_y} \mathbb{P}_{0^+}(U_{S_y} \in dz, S_y \in dr)\mathbb{P}_z(\exists s \in [0, \epsilon_y], U_s \le y),
\end{aligned} \tag{15}$$

the inequality in the former equation is owed to the fact that the OU process does not have negative jumps and hits the points from above. Using the Lamperti's transformation, it is straightforward that for every $z \in]y, y(1 + \delta_y)[$

$$\begin{aligned}
\mathbb{P}_z\left(\exists s \in [0, \epsilon_y], U_s \le y\right) &= \mathbf{P}\left(\exists s \in [0, \epsilon_y],\ ze^{-s}\exp\{\xi_{\tau((e^s-1)/z)}\} \le y\right)\\
&\ge \mathbf{P}\left(\exists s \in [0, \epsilon_y],\ y(1+\delta_y)e^{-s}\exp\{\xi_{\tau((e^s-1)/y)}\} \le y\right)\\
&= \mathbb{P}_y\left(\exists s \in [0, \epsilon_y], (1+\delta_y)U_s \le y\right).
\end{aligned} \tag{16}$$

The weak convergence of $(U_{S_y}/y, S_y)$ as $y \to 0$, implies that

$$\mathbb{P}_{0^+}(U_{S_y} \in [y, y(1+\delta_y)], S_y \in [0, t-\epsilon_y]) \sim \mathbf{P}\left(e^K \in]1, 1+\delta_y[\right),$$

as $y \to 0$. Putting together equations (15) & (16) and the later fact we get the estimation

$$\begin{aligned}
&\mathbb{P}_{0^+}(T_y < t)\\
&\ge \mathbb{P}_{0^+}(U_{S_y} \in [y, y(1+\delta_y)], S_y \in [0, t-\epsilon_y])\mathbb{P}_y\left(\exists s \in [0, \epsilon_y], (1+\delta_y)U_s \le y\right)\\
&\sim \mathbf{P}\left(e^K \in]1, 1+\delta_y[\right)\mathbb{P}_y\left(\exists s \in [0, \epsilon_y], (1+\delta_y)U_s \le y\right),
\end{aligned} \tag{17}$$

as $y \to 0$. Furthermore, by the definition of δ_y we have that

$$\begin{aligned}
\mathbf{P}(e^K \in [1, 1+\delta_y]) &= \mathbf{P}\left(K \in [0, \beta_y^{-1}]\right)\\
&= \frac{1}{\mu}\int_0^{\beta_y^{-1}} \overline{\Pi}(r)dr,
\end{aligned} \tag{18}$$

and the last term in the former equation can be estimated in terms of the Laplace exponent. Specifically, there exist two constant c_1, c_2 depending only on ϕ such that

$$c_1 \frac{\phi(\beta_y)}{\beta_y} \leq \frac{1}{\mu} \int_0^{\beta_y^{-1}} \overline{\Pi}(r)dr \leq c_2 \frac{\phi(\beta_y)}{\beta_y},$$

see e.g. [1] Proposition III.1. Since β_y is the inverse of $\phi(z)/z$ we have by equations (17) & (18) that

$$\mathbb{P}_{0^+}(T_y < t) \geq yc_1 \mathbb{P}_y \left(\exists s \in [0, \epsilon_y], (1+\delta_y)U_s \leq y\right),$$

for every y small enough. Now, we shall show in Lemma 5 below that the function ϵ_y can be chosen such that

$$\liminf_{y \to 0} \mathbb{P}_y \left(\exists s \in [0, \epsilon_y], (1+\delta_y)U_s \leq y\right) = \vartheta > 0. \tag{19}$$

Taking for granted this statement we end the proof since we have showed that for all $t > 0$,

$$\begin{aligned} \mathbb{P}_{0^+}(T_y < \mathbf{e}) &\geq e^{-t}\mathbb{P}_{0^+}(T_y < t) \\ &\geq e^{-t}Cy \qquad \text{as} \qquad y \to 0, \end{aligned}$$

where $\mathbf{e}$ is an exponential random variable independent of U and $C = c_1\vartheta$. □

Lemma 5. *We may choose ϵ_y such that (19) holds true.*

Proof. Recall that β_y is determined by $\phi(\beta_y)/\beta_y = y$ and that $\delta_y = e^{1/\beta_y} - 1$. The regular variation at infinity of ϕ will enable us to show that the functions

$$\epsilon_y = \frac{e^{d/\beta_y} - 1}{y} \qquad \text{and} \qquad a_y = (d-1)/\beta_y,$$

with $d > 1$ arbitrary, are such that

(i)
$$\epsilon_y,\ a_y, \longrightarrow 0,\ \text{as } y \to 0,$$

(ii)
$$\lim_{y \to 0} \mathbf{P}(\xi_{\epsilon_y} \leq a_y) = \vartheta_1 > 0.$$

The reason why we require the functions ϵ_y and a_y to have this behavior is the following. Let

$$s_y = \log(1 + y\epsilon_y),\ y > 0$$

and note that $\tau(s) \leq s$, for every $s \geq 0$, since $A_s \geq s$ for every $s \geq 0$. Then on the event

$$\xi_{\epsilon_y} \leq a_y,$$

we have the inequalities

$$\exp\{\xi_{\tau((e^{s_y}-1)/y)}\} \leq \exp\{\xi_{\epsilon_y}\} \leq e^{a_y},$$

due to the fact that ξ is an increasing process and

$$\tau\left((e^{s_y} - 1)/y\right) \leq (e^{s_y} - 1)/y = \epsilon_y.$$

So, on this event, we have also the inequalities

$$y(1+\delta_y)e^{-s_y}\exp\{\xi_{\tau\left((e^{s_y}-1)/y\right)}\} \leq y(1+\delta_y)e^{-s_y+a_y} \leq y,$$

from the definition of the functions s_y and a_y. Since $s_y \leq \epsilon_y$ for every y small enough and the OU process does not have negative jumps, we can conclude by (ii) that

$$\liminf_{y\to 0} \mathbf{P}\left(\exists\ s \in [0,\epsilon_y],\ y(1+\delta_y)e^{-s}\exp\{\xi_{\tau((e^s-1)/y)}\} \leq y\right) \geq \lim_{y\to 0} \mathbf{P}(\xi_{\epsilon_y} \leq a_y) > 0.$$

Then the proof reduces to show that the functions ϵ_y and a_y so defined satisfies (i,ii).

Let $\phi'(\cdot)$ be the derivative of ϕ,

$$\Lambda(u) = \phi(u) - u\phi'(u), \qquad u > 0,$$

and λ_y the function determined by the relation

$$\phi'(\lambda_y) = \frac{a_y}{\epsilon_y}.$$

Since $\phi(\lambda)$ is concave and regularly varying with index $\beta \in]0,1[$ then $\phi(\lambda)/\lambda$ is regularly varying with index $\beta - 1$ and $\phi'(\lambda) \sim \beta\phi(\lambda)/\lambda$. This implies in turn that $\beta_y \longrightarrow \infty$ and $y\beta_y \longrightarrow \infty$ as $y \to 0$. Thus it is straightforward that ϵ_y, and a_y satisfy (i), and moreover $a_y = O(y\epsilon_y)$. This and the regular variation of ϕ imply that $\lambda_y = O(\beta_y)$.

According to Jain and Pruitt [19] Theorem 5.1 the statement in (ii) is equivalent to

$$\lim_{y\to 0} \epsilon_y \Lambda(\lambda_y) < \infty.$$

The former is indeed true in our construction,

$$\begin{aligned}\Lambda(\lambda_y) &= \lambda_y\Big(\frac{\phi(\lambda_y)}{\lambda_y} - \phi'(\lambda_y)\Big)\\ &\sim \lambda_y\phi'(\lambda_y)\Big(\frac{1-\beta}{\beta}\Big)\\ &= \lambda_y\frac{a_y}{\epsilon_y}\Big(\frac{1-\beta}{\beta}\Big).\end{aligned}$$

Therefore

$$\epsilon_y\Lambda(\lambda_y) \sim \frac{\lambda_y}{\beta_y}(d-1)\Big(\frac{1-\beta}{\beta}\Big) = O(1).$$

This ends the proof in the case ϕ is regularly varying at infinity. When the Lévy measure is a finite measure, that is ξ is a compound Poisson process, we can take $a_y \equiv 0$ and

$$\epsilon_y = (e^{1/\beta_y} - 1)/y \qquad y > 0.$$

This choice of the functions a_y, ϵ_y is due to the fact that a compound Poisson process remains at zero during an exponential time and a fortiori

$$\lim_{y\to 0^+} \mathbf{P}\left(\xi_{\epsilon_y} \leq a_y\right) > 0.$$

The rest of the proof follows as in the case ϕ is regularly varying at ∞. □

The last ingredient in the proof of Proposition 1 is the following result.

Lemma 6. *One has*

(i)
$$\limsup_{y\to 0^+} \mathbb{E}_y(e^{-R}) < 1,$$

(ii)
$$\liminf_{y\to 0} \mathbb{P}_y(T_x \le R) > 0.$$

Proof. (i) We know from equation (14) that

$$u_1(0,y) = y^{-1}\frac{\mathbb{E}_{0^+}\left(e^{-T_y}\right)}{1-\mathbb{E}_y(e^{-R})} \ge \mathbb{E}_{0^+}\left(e^{-T_y}\right) u_1(y,y).$$

Moreover, by Lemma 3 one has

$$\lim_{y\to 0} u_1(0,y) = \lim_{y\to 0}\frac{1}{\mu}\int_0^{1/y} dz\rho(z) = \frac{1}{\mu}.$$

Thus Lemma 4 implies

$$\limsup_{y\to 0^+} y u_1(y,y) = \theta < \infty.$$

In particular, using equation (14) one gets

$$\limsup_{y\to 0} \mathbb{E}_y(e^{-R}) = \frac{\theta}{1+\theta}.$$

(ii) The statement in (i) shows that for every $t > 0$

$$\limsup_{y\to 0} \mathbb{P}_y(R \le t) \le \frac{\theta}{1+\theta}.$$

Since the OU process U hits the points continuously from above, it is plain that for every $y < x$

$$\mathbb{P}_y(T_x < R) = \mathbb{P}_y(H_x < R).$$

Thus, for every $t > 0$

$$\mathbb{P}_y(H_x < R) \ge \mathbb{P}_y(H_x < t) - \mathbb{P}_y(R \le t),$$

and as a consequence

$$\begin{aligned}\liminf_{y\to 0^+} \mathbb{P}_y(H_x < R) &\ge \mathbb{P}_{0^+}(H_x < t) - \limsup_{y\to 0} \mathbb{P}_y(R \le t)\\ &\ge \mathbb{P}_{0^+}(H_x < t) - \frac{\theta}{1+\theta}.\end{aligned}$$

Since the OU process U is recurrent and without negative jumps we can ensure that

$$\mathbb{P}_{0^+}(H_x < \infty) = 1.$$

Then there exists a $t > 0$ such that the right hand term in the former inequality is strictly positive. □

Lemma 6 ends the proof of Proposition 1 since we have noted that (10) is equivalent to (13).

3.2 Proof of Proposition 2

This proof is based on the fact that one can relate the behavior of $\mathbf{P}(I > t)$ to that of the Laplace exponent ϕ of ξ by using connections between the behavior of $\mathbf{E}(e^{\lambda I})$ as $\lambda \to \infty$ and that of $\mathbf{P}(I > t)$ as $t \to \infty$. This result can be proved using the results in Geluk [16]. However, for ease of reference we provide a complete proof based on a result due to Kasahara. We note that a similar result has been obtained in Haas [17] Proposition 11.

Proof of Proposition 2. Since the moment generating function of I, is well defined, that is,

$$\widehat{\rho}(s) = \mathbf{E}(e^{sI}) < \infty \quad \forall s > 0,$$

we have the conditions to use Kasahara's Tauberian Theorem (Bingham et al. [7] Theorem 4.12.3), it links the regular variation of $\log \widehat{\rho}(s)$ as $s \to \infty$ with that of $-\log \mathbf{P}(I > t)$ as $t \to \infty$. On the other hand the characteristic function of I, say f, is an entire function, admits a Taylor series

$$f(z) = \sum_n a_n z^n, \quad \text{with} \quad a_n = i^n \frac{\mathbf{E}(I^n)}{n!} = \frac{i^n}{\prod_{k=1}^n \phi(k)} \qquad \forall n \in \mathbb{N},$$

and its maximum modulus,

$$M(s, f) = \sup \{|f(z)| : |z| \leq s\},$$

coincides with $\widehat{\rho}(s)$, that is

$$M(s, f) = \widehat{\rho}(s), \qquad \forall s > 0,$$

e.g. Lukacs [23] Theorem 7.1.2.

In order to apply Kasahara's Theorem we must check that $\log \widehat{\rho}(s)$, i.e. $\log M(s, f)$, is asymptotically regularly varying. To this end, we recall that we can estimate the behavior of $\log M(s, f)$ in terms of the coefficients of the Taylor expansion of f. More precisely, suppose that

$$\limsup_{n\to\infty} \frac{n \log n}{\log(1/|a_n|)} = \frac{1}{\beta}. \tag{20}$$

By Levin [22] (section 1.13), if there exists a regularly varying function with index β, say ψ, such that

$$\lim_{n\to\infty} |a_n|^{1/n} \psi(n) = e^{\beta}, \tag{21}$$

then

$$\lim_{s\to\infty} \frac{\log M(s, f)}{\psi^{\leftarrow}(s)} = \beta, \tag{22}$$

with $\psi^{\leftarrow}$ the asymptotic inverse of ψ. A version of $\psi^{\leftarrow}$ is

$$\psi^{\leftarrow}(s) = \inf\{r > 0 | \psi(r) > s\}.$$

With the aim of obtaining the asymptotic behavior of $-\log \mathbf{P}(I > t)$, let $\varphi(s) = s/\psi(s)$. Then φ is a regularly varying function with index $1 - \beta$ and its asymptotic inverse, $\varphi^{\leftarrow}$, varies regularly with index $(1 - \beta)^{-1}$. Using equation (22), a straightforward application to Theorem 4.12.7 in Bingham et al. [7] leads to

$$-\log \mathbf{P}(I > t) \sim (1 - \beta)\varphi^{\leftarrow}(t), \qquad t \to \infty,$$

and, provided that ρ decreases in some neighborhood of ∞, we can apply Theorem 4.12.10 op. cit. to get

$$-\log \rho(t) \sim (1 - \beta)\varphi^{\leftarrow}(t), \qquad \text{as} \qquad t \to \infty.$$

The rest of the proof is devoted to the proof of (20) and the fact that ϕ satisfies the equation (21). With this aim, recall that

$$|a_n| = \mathbf{E}(I^n)/n! = \left(\prod_{k=1}^{n} \phi(k)\right)^{-1}.$$

As ϕ is regularly varying with index β, it can be expressed as $\phi(s) = s^\beta l(s)$, with l a slowly varying function. Moreover, there exist two functions ε and c and a positive constant a, such that

$$l(t) = \exp\left\{c(t) + \int_a^t \varepsilon(s)\frac{ds}{s}\right\}$$

and $\varepsilon(t) \to 0$ and $c(t) \to c$ with $c \in \mathbb{R}$, as $t \to \infty$. Therefore

$$\log 1/|a_n| = \sum_{k=1}^{n} \log \phi(k) = \beta \sum_{k=1}^{n} \log k + \sum_{k=1}^{n} \log l(k).$$

Since

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \log k}{n \log n} = 1,$$

and for every slowly varying function l we have

$$\lim_{t\to\infty} \frac{\log l(t)}{\log t} = 0,$$

it is straightforward that the lim sup in (20) is in fact a limit and equals $1/\beta$. Next, we show that

$$\lim_{n\to\infty} \phi(n)|a_n|^{1/n} = e^\beta.$$

To do this, observe that due to the fact that $(n!)^{1/n} \sim ne^{-1}$ we get

$$|a_n|^{1/n} \sim (ne^{-1})^{-\beta} \exp\{-\frac{1}{n}\sum_{k=1}^{n} \log l(k)\}.$$

Moreover,

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n} \log l(k)\\
=&\frac{1}{n}\left(n\int_a^1 \varepsilon(s)\frac{ds}{s} + \sum_{k=1}^{n-1}(n-k)\int_k^{k+1} \varepsilon(s)\frac{ds}{s}\right) + \frac{1}{n}\sum_{k=1}^{n} c(k)\\
=&\int_a^n \varepsilon(s)\frac{ds}{s} - \frac{1}{n}\sum_{k=1}^{n-1} k\int_k^{k+1} \varepsilon(s)\frac{ds}{s} + \frac{1}{n}\sum_{k=1}^{n} c(k)\\
\sim&\int_a^n \varepsilon(s)\frac{ds}{s} + c,
\end{aligned}$$

the last line is a consequence of Cesaro's theorem since $c(k) \to c$, and

$$k\int_k^{k+1} \varepsilon(s)\frac{ds}{s} \longrightarrow 0,$$

as $k \to \infty$. Therefore

$$|a_n|^{1/n} \sim e^\beta(\phi(n))^{-1}.$$

□

4 On time reversal of X.

The aim of this section is to obtain a result on time reversal for a self–similar process and then use it to prove Lemma 1 and Theorem 2.

Let $z > 0$ and $\widehat{\mathbb{P}}_z$ the law of the process $\widehat{X}$ defined by

$$\widehat{X}_t = z \exp -\xi_{\widehat{\tau}(t/z)}, \quad t \geq 0$$

with the time change

$$\widehat{\tau}(t) = \inf\{s > 0, \int_0^s e^{-\xi_r} dr > t\},$$

and the convention that $\widehat{X}_t = 0$ if $\widehat{\tau}(t/z) = \infty$. Define $\widehat{\mathbb{P}}_0$ the law of the process identical to 0. Then, under the family $(\widehat{\mathbb{P}}_z, z \geq 0)$ the process $\widehat{X}$ is Markovian and has the scaling property defined in equation (1) with $\alpha = 1$. We will say that $\widehat{X}$ is the dual 1–self–similar Markov process, cf. Bertoin and Yor [5] and the reference therein. Observe that 0 is an absorbing state for $\widehat{X}$ and let $\mathfrak{J}$ be its lifetime, i.e.,

$$\mathfrak{J} = \inf\{t \geq 0 : \widehat{X}_t = 0\}.$$

It should be clear that the distribution of $\mathfrak{J}$ under $\widehat{\mathbb{P}}_z$ is that of zI, where I is the Lévy exponential functional defined in (3). Last, denote $(\widehat{\mathcal{F}}_t, t \geq 0)$ the natural filtration and $\widehat{P}_t(z, dy)$ the semigroup of the dual 1–ss Markov process.

Lemma 2 in [5], states that the q–resolvents R_q and $\widehat{R}_q$ of the processes X and $\widehat{X}$, respectively, are in weak duality with respect to the Lebesgue measure (cf. Vuolle-Apiala and Graversen [31] for a related discussion). Thus duality also holds for the respective semigroups. We will refer to this result as the *"duality Lemma"* and we will use it to show, roughly speaking, that the law of the process

$$\left(X_{(r-t)^-}, 0 \leq t < r \mid X_{r^-} = x\right),$$

under $\mathbb{P}_{0+}$ is the same as that

$$\left(\widehat{X}_t, 0 \leq t < r \mid \mathfrak{J} = r\right),$$

under $\widehat{\mathbb{P}}_x$, with $r, x > 0$ fixed. A rigorous statement will be done by using the method of h–transform of Doob, see e.g. Sharpe [29] section 62, Fitzsimmons et al. [13],... To this end, note that

- by the self similarity of X, for any $s > 0$ the law of the random variable X_s under $\mathbb{E}_{0+}$ has a density
$$p_s(z) = (\mu z)^{-1} \rho(s/z), \qquad z > 0,$$
- for every $s, t > 0$ and $z > 0$
$$\widehat{P}_t p_s(z) = p_{t+s}(z). \tag{23}$$

The identity (23) follows from the duality Lemma and the fact that $(\mathbb{P}_{0^+}(X_s \in dz), s > 0)$ is a family of entrance laws for the semigroup P_t, of the 1–ss Markov process X.

Equation (23) and the Markov property of $\widehat{X}$ implies that for any $r, x > 0$ the process

$$h^r_t = \frac{p_{r-t}(\widehat{X}_t)}{p_r(\widehat{X}_0)} 1_{\{t<r\}}, \qquad t > 0,$$

is a $\widehat{\mathbb{P}}_x$ martingale. Let Ω be the space of càdlàg maps from $[0,\infty[$ to $[0,\infty[$ killed at the first hitting time of 0. After Sharpe [29] Theorem 62.19, there exists a unique probability measure $\mathbb{Q}^r_x$ on Ω equipped with its natural filtration, rendering the process $\widehat{X}$ an inhomogeneous Markov process with semigroup

$$Q^r_{t,t+s}(z,dy) = \frac{\widehat{P}_s(z,dy)p_{r-t-s}(y)}{p_{r-t}(z)}, \tag{24}$$

and such that $\mathbb{Q}^r_x(\widehat{X}_0 = x) = 1$. The measure $\mathbb{Q}^r_x$ has the property that for any $s > 0$

$$\mathbb{Q}^r_x(F1_{\{s<\mathfrak{J}\}}) = \widehat{\mathbb{P}}_x\left(Fh^r_s\right), \tag{25}$$

for every F in $\widehat{\mathcal{F}}_s$.

Lemma 7. *(i) If F is $\widehat{\mathcal{F}}_{\mathfrak{J}^-}$–measurable and $g \geq 0$ is a Borel function, then*

$$\widehat{\mathbb{E}}_x\left(Fg(\mathfrak{J})\right) = \mu\int dr p_r(x)g(r)\mathbb{Q}^r_x\left(F\right). \tag{26}$$

Thus,

$$(\mathbb{Q}^r_x)_{r>0}$$

is a regular version of the family of conditional probabilities

$$\widehat{\mathbb{P}}_x\left(\,\cdot\,|\mathfrak{J} = r\right), \qquad r > 0.$$

(ii) Let $r > 0$ fixed and $G \geq 0$ a bounded functional then

$$\mathbb{E}_{0^+}\left(G\left(X_{(r-t)-}, 0 \leq t < r\right)\right) = \mathbb{Q}^r_{p_r}\left(G(\widehat{X}_t, 0 \leq t < r)\right), \tag{27}$$

where $\mathbb{Q}^r_{p_r}$ denotes the law of the process $\widehat{X}$ under $\mathbb{Q}^r_x$ with initial measure $\mathbb{P}_{0^+}(X_r \in dx)$.

It is implicit in the statement in (ii) of Lemma 7 that $\mathbb{Q}^r_x$ is the image under time reversal of a measure $\widetilde{\mathbb{Q}}^r_x$ on Ω corresponding to $(X_t, 0 \leq t < r)$ under the conditional law

$$\mathbb{P}_{0^+}(\,\cdot\,|X_{r-} = x).$$

So the support of $\mathbb{Q}^r_x$ is the set Ω_r of càdlàg paths that start at x and are absorbed at 0 at time r.

Proof. (i) By the Monotone class Theorem, to prove (26), it suffices to check that for any $s \geq 0$ the formula holds for every element of the form $F = F' \cap \{\mathfrak{J} > s\}$ with F' in $\widehat{\mathcal{F}}_s$. Indeed, note that on the set $\{s < xI\}$ we have that

$$xI = x\int_0^{\widehat{\tau}(s/x)} e^{-\xi_t}dt + xe^{-\xi_{\widehat{\tau}(s/x)}}I' = s + xe^{-\xi_{\widehat{\tau}(s/x)}}I',$$

with I' independent of $(\xi_{\widehat{\tau}(u/x)}, u \leq s)$ and equal in law to I, owed to the strong Markov property of ξ. Using the fact that under $\widehat{\mathbb{P}}_x$ the law of $\mathfrak{J}$ is that of xI, the former equality and the strong Markov property of $\widehat{X}$ we get that

$$\widehat{\mathbb{E}}_x\left(g(\mathfrak{J})1_{\{s<\mathfrak{J}\}}|\widehat{\mathcal{F}}_s\right) = W(s, \widehat{X}_s),$$

where

$$\begin{aligned} W(s,z) &= \widehat{\mathbb{E}}_z\left(1_{\{0<\mathfrak{J}\}}g(s+\mathfrak{J})\right) \\ &= \mathbf{E}\left(g(s+zI)\right) \\ &= \int_s^\infty dr z^{-1}\rho\left((r-s)/z\right)g(r) \\ &= \mu\int_s^\infty dr p_{r-s}(z)g(r). \end{aligned}$$

Note that

$$\frac{\widehat{\mathbb{P}}_z(\mathfrak{J}\in dr)}{dr} = \mu p_r(z).$$

Thereby an application of formula (25) gives

$$\begin{aligned} \widehat{\mathbb{E}}_x\left(Fg(\mathfrak{J})\right) &= \widehat{\mathbb{E}}_x\left(F'\ W(s,\widehat{X}_s)\right) \\ &= \mu\widehat{\mathbb{E}}_x\left(F'\int_s^\infty dr\ p_{r-s}(\widehat{X}_s)g(r)\right) \\ &= \mu\int_0^\infty dr p_r(x)g(r)\widehat{\mathbb{E}}_x\left(F'h_s^r\right) \\ &= \int_0^\infty dr\mu p_r(x)g(r)\mathbb{Q}_x^r(F). \end{aligned}$$

(ii) We first verify that under $\mathbb{P}_{0^+}$ the process $Y_t = X_{r-t}, 0<t<r$, admits the semigroup defined in equation (24). Let $a,b:[0,\infty[\to]0,\infty[$ be Borel functions and $t, t+s\in[0,r[$. Indeed, by the duality lemma for ss Markov processes, we have that

$$\begin{aligned} \mathbb{E}_{0^+}\left(a(Y_t)b(Y_{t+s})\right) &= \int dz\ p_{r-t-s}(z)b(z)\,\mathbb{E}_z(a(X_s)) \\ &= \int dz a(z)\widehat{\mathbb{E}}_z(p_{r-t-s}(\widehat{X}_s)b(\widehat{X}_s)) \\ &= \int dz a(z)p_{r-t}(z)\frac{\widehat{\mathbb{E}}_z(p_{r-t-s}(\widehat{X}_t)b(\widehat{X}_t))}{p_{r-t}(z)} \\ &= \mathbb{E}_{0^+}(a(Y_t)Q_{t,t+s}^r b(Y_t)), \end{aligned}$$

with $Q_{t,t+s}^r$ the semigroup defined in equation (24).

By the Monotone class theorem, to prove (ii) it suffices to check that equation (27) holds for every G of the form $f_1(X_{(r-t_1)-})\cdots f_n(X_{(r-t_n)-})$ with $f_1,\ldots,f_n$ positive bounded Borel functions and $0\le t_1<\cdots<t_n<r$. Using the fact that the ss process X does not have fixed jumps we get that for $n=2$

$$\begin{aligned} \mathbb{E}_{0^+}\left(f_1(X_{(r-t_1)-})f_2(X_{(r-t_2)-})\right) &= \mathbb{E}_{0^+}\left(f_1(X_{(r-t_1)})f_2(X_{(r-t_2)})\right) \\ &= \mathbb{E}_{0^+}\left(Q_{0,t_1}^r f Q_{t_1,t_2}^r f_2(X_r)\right) \\ &= \int dx p_r(x)\mathbb{Q}_x^r\left(f_1(\widehat{X}_{t_1})f_2(\widehat{X}_{t_2})\right), \end{aligned}$$

the general case follows by iteration. □

Now we have all the elements to provide a

Proof of Lemma 1. When $h \to 0^+$, thanks to the Markov property of X, applied at time 1, our problem reduces to show that for every $x > 0$

$$\mathbb{P}_x\left(\lim_{h\to 0^+}\frac{U_h - U_0}{h} = -U_0\right) = 1.$$

To this end, we recall that since ξ is a subordinator we have

(i)
$$\lim_{s\to 0}\frac{\xi_s}{s} = 0,$$

(ii) ξ at time $\tau(1/x)$ is right continuous and

(iii)
$$\lim_{s\to 0}\frac{\tau(s)}{s} = 1 \qquad \mathbf{P}\text{–a.s.}$$

Using these facts and Lamperti's transformation it is straightforward that

$$\lim_{\epsilon\to 0^+}\frac{X_\epsilon - X_0}{\epsilon} = 0, \qquad \mathbb{P}_x\text{–a.s.}$$

The rest of the proof, in the case $h \to 0^+$, follows by standard arguments.

Next we use Lemma 7 to study the case $h \to 0^-$. By equation (27) we know that

$$\mathbb{P}_{0^+}\left(\lim_{h\to 0^-}\frac{\widetilde{U}_h - \widetilde{U}_0}{h} = -\widetilde{U}_0 \left|\widetilde{U}_0 = x\right.\right) = \mathbb{Q}^1_x\left(\lim_{h\to 0^+}\frac{e^h\widehat{X}_{(1-e^{-h})} - \widehat{X}_0}{-h} = -\widehat{X}_0\right).$$

Since for any $x > 0$ and $\epsilon > 0$ the measure $\mathbb{Q}^1_x$ is absolutely continuous with respect to $\widehat{\mathbb{P}}_x$ on the trace of $\{\epsilon < \mathfrak{J}\}$ in $\widehat{\mathcal{F}}_\epsilon$, the result follows as in the case $h \to 0^+$ but this time for the dual self–similar process $\widehat{X}$. □

Other interesting results on time reversal can be deduced from the duality Lemma by using the classical Theorem on time reversal of Nagasawa or its generalized version in Theorem 47 chapter XVIII Dellacherie et al. [11]. We will content ourselves with the following result and refer to Bertoin and Yor [5] and the reference therein for a related discussion.

Proposition 3. *Let $x > 0$ fixed. Under $\mathbb{Q}^1_x$ the dual Ornstein–Uhlenbeck process*

$$\widehat{U} = \{e^t\widehat{X}_{1-e^{-t}}, t > 0\},$$

is an homogeneous strong Markov process with semigroup

$$Q^1_{0,1-e^{-s}}H_{e^s}f(\cdot),$$

where H_t is the dilatation $H_tf(z) = f(tz)$.

Proof. The homogeneity is obtained from the expression of the semigroup in (24) using the self–similarity enjoyed by $\widehat{X}$ under $\widehat{\mathbb{P}}_x$. Indeed, let f, g positive Borel functions then

$$\mathbb{Q}^1_x\left(f(e^t\widehat{X}_{1-e^{-t}})g(e^{t+s}\widehat{X}_{1-e^{-(t+s)}})\right) = \mathbb{Q}^1_x\left(f(e^t\widehat{X}_{1-e^{-t}})Q^1_{1-e^{-t},1-e^{-(t+s)}}H_{e^{t+s}}g(\widehat{X}_{1-e^{-t}})\right).$$

The expression of the semigroup can be reduced to

$$\begin{aligned} Q^1_{1-e^{-t},1-e^{-(t+s)}}H_{e^{t+s}}g(z) &= (p_{e^{-t}}(z))^{-1}\,\widehat{\mathbb{E}}_z\left(g\left(e^{t+s}\widehat{X}_{e^{-t}(1-e^{-s})}\right)p_{e^{-(t+s)}}\left(\widehat{X}_{e^{-t}(1-e^{-s})}\right)\right) \\ &= \left(p_1(e^t z)\right)^{-1}\widehat{\mathbb{E}}_{e^t z}\left(g(\widehat{U}_s)p_{e^{-s}}(\widehat{X}_{1-e^{-s}})\right), \\ &= Q^1_{0,1-e^{-s}}H_{e^s}g(e^t z) \end{aligned}$$

where the second equality is owed to the self–similarity and the obvious identity

$$cp_{rc}(u) = p_r(c^{-1}u).$$

The strong Markov property follows from (25) by the optional stopping theorem using standard arguments. □

We have now the elements to prove the Theorem 2.

Proof of Theorem 2. The statement in (ii) in Lemma 7 shows that for every positive and bounded functional F,

$$\mathbb{Q}^1_x\left(F(\widehat{U}_t, 0 \le t \le \widehat{R})\right) = \mathbb{E}_{0+}\left(F(e^t X_{e^{-t}}, 0 \le t \le R') \mid X_1 = x\right),$$

with R' (resp. $\widehat{R}$) the first return time of the process $\{e^t X_{e^{-t}}, t \ge 0\}$ (resp. of $\widehat{U}$), to its starting point. Moreover, by the stationarity of the OU process $\widetilde{U}$ defined at the beginning of the subsection 3.1, one gets that

$$\mathbb{E}_{0+}(R'|X_1 = x) = \mathbb{E}_x(R),$$

and

$$\mathbb{P}_{0+}\left(\inf_{0<t<R'} e^t X_{e^{-t}} > y \mid X_1 = x\right) = \mathbb{P}_x\left(\inf_{0<t<R} e^{-t}X_{e^t-1} > y\right).$$

Recall that our proof of Proposition 1 is based on the fact that the OU process U is homogeneous and strong Markov and the probabilities that we considered there depend only on the excursion away its starting point. It should be then clear that thanks to Proposition 3 one can repeat the arguments in the proof of Proposition 1 to show that for any decreasing Borel function h we have

$$\mathbb{Q}^1_x\left(\widehat{U}_t < h(t) \quad \text{i.o.} \quad t \to \infty\right) = 0 \quad \text{or} \quad 1,$$

according whether

$$\int^{\infty} \rho(1/h(s))ds < \infty \quad \text{or} \quad = \infty.$$

We deduce from this criterion, the equation (27) and a time change that for any increasing Borel function ℓ such that $\ell(0) = 0$ we have

$$\mathbb{P}_{0+}\left(X_t < t\ell(t) \quad \text{i.o.} \quad t \to 0\right) = 0 \quad \text{or} \quad 1,$$

according whether

$$\int_{0+} \rho(1/\ell(s))\frac{ds}{s} < \infty \quad \text{or} \quad = \infty.$$

Rewriting the arguments in the proof of Theorem 1 we obtain the result. □

The former proof provides further information on the behavior of the dual 1–ss Markov process near its lifetime.

Corollary 1. *Let ξ be a subordinator such that its Laplace exponent ϕ is regularly varying at infinity with index $\beta \in]0,1[$ and $0 < \phi'(0^+) < \infty$. Suppose that the density of the Lévy exponential functional associated to ξ satisfies hypothesis **(H)**. If $\widehat{X}$ is the dual 1–ss process associated to ξ with lifetime $\mathfrak{I}$ and f is the function defined in Theorem 1 then for any $x > 0$*

$$\liminf_{s\to 0} \frac{\widehat{X}_{r(1-s)}}{s\mathrm{f}(1/s)} = r(1-\beta)^{(1-\beta)} \qquad \mathbb{Q}^r_x\text{–}a.s.$$

Proof. In the previous proof we showed that for any $x > 0$ and ℓ an increasing Borel function we have

$$\mathbb{Q}^1_x\left(\widehat{X}_{(1-s)} < s\ell(s) \quad \text{i.o.} \quad s \to 0\right) = 0 \qquad \text{or} \qquad = 1$$

according whether

$$\int_{0^+} \rho(1/\ell(s))\frac{ds}{s} < \infty \quad \text{or} \quad = \infty.$$

Moreover, a straightforward verification of the finite dimensional distributions shows that the scaling property of $\widehat{X}$ under $\widehat{\mathbb{P}}$ is translated for the dual OU process in the form: under $\mathbb{Q}^r_x$ the law of the process

$$\frac{1}{r}e^t\widehat{X}_{r(1-e^{-t})}, t > 0$$

is that of the dual OU under $\mathbb{Q}^1_{x/r}$. The result follows as in the proof of Theorem 1. □

5 Examples

Example (Watanabe process) Let ξ be a subordinator with zero drift and Lévy measure $\nu(dx) = abe^{-bx}dx$, with $a, b > 0$. That is, ξ a compound Poisson process with jumps having an exponential distribution. Carmona et al. [10] §2 showed that in this case the density of the law of $I = \int_0^\infty e^{b\xi_s}ds$ is given by

$$\rho(x) = a^2xe^{-ax}, \qquad x > 0.$$

So $\rho(x)$ satisfies the hypothesis **(H)**. The $(1/b)$–ss Markov process associated to ξ by Lamperti transformation is a process that arises in the study of extremes. More precisely, the $(1/b)$–ss Markov process associated to ξ is a Q–Extremal process with

$$Q(x) = \begin{cases} \infty & x \le 0, \\ ax^{-b} & x > 0 \end{cases}.$$

See Resnick [25]. This family of process is usually called generalized Watanabe process in honor to Watanabe S. who studied them, when $b = 1$, using the theory of Brownian excursions, see e.g. Revuz et Yor [26] pp. 504. We refer also to Carmona et al. [9] and the reference therein for the study of this process as a ss Markov process and its generalizations. Hence, thanks to Proposition 1 we obtain

Corollary 2. *Let X be a generalized Watanabe process and h an increasing function such that $(h(s))^b/s$ is a decreasing function. Then*

$$\mathbb{P}_x\big(X_s < h(s) \quad i.o.\ s \to \infty\big) = 0 \quad or \quad 1$$

according whether

$$\int^\infty (1/h(s))^b e^{-as(h(s))^{-b}} ds < \infty \quad or \quad = \infty.$$

This result appears in Yimin Xiao [33] Corollary 4.1 in the case $b = a = 1$.

With the aim of providing a larger class of examples, in the following construction we make some assumptions on the subordinators that ensure that the density of I satisfies hypothesis **(H)**. It uses the recent results of Bertoin and Yor [6, 4].

Let $U(dx)$, be the renewal measure of ξ, i.e.

$$\mathbf{E}\Big(\int_0^\infty f(\xi_s)ds\Big) = \int_{[0,\infty)} f(x)U(dx).$$

If the renewal measure is absolutely continuous with respect to Lebesgue measure, the function $u(x) = U(dx)/dx$, is usually called the renewal density.

Proposition 4. *Let ξ be a subordinator. Suppose that its renewal measure is absolutely continuous with respect to Lebesgue measure and that its renewal density $u(x)$, is a decreasing and convex function such that*

$$\lim_{t\to\infty} u(t) = \frac{1}{\mu} \in]0,\infty[,$$

*i.e., $\mathbf{E}(\xi_1) = \mu$. Then the density ρ, of the exponential functional associated to ξ satisfies the hypothesis **(H)**.*

Examples of such subordinators are those arising in Mandelbrot's construction of regenerative sets (see e.g. Fitzsimmons et al. [14]).

Proof. It is well known, that the renewal measure and the Laplace exponent of ξ are related by the formula

$$\frac{1}{\phi(\lambda)} = \int_0^\infty e^{-\lambda x}u(x)dx. \tag{28}$$

An integration by parts in the former equation leads

$$\kappa(\lambda) = \frac{\lambda}{\phi(\lambda)} = \frac{1}{\mu} + \int_0^\infty (1 - e^{-\lambda x})g(x)dx,$$

where $-g(x)$ is the left hand derivative of $u(x)$. That is, κ is the Laplace exponent of a subordinator with killing term $\frac{1}{\mu}$, zero drift and Lévy measure with density $g(x)$. Integrating by parts, once more, we obtain that

$$\psi(\lambda) = \lambda\kappa(\lambda) = \frac{\lambda}{\mu} + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x)\nu(-dx),$$

with $\nu(dx) = -dg(x)$ a Stieltjes measure. Specifically, $\psi(\lambda)$ is the Laplace exponent of a Lévy process, say $(\zeta_s, s \geq 0)$, with no-positive jumps, drift term $1/\mu$ and no Gaussian component. We have furthermore, that

$$\mathbf{E}(\zeta_1) = \psi'(0^+) = \frac{1}{\mu} \in]0,\infty[,$$

then ζ drifts to ∞. This implies that the law of the exponential functional, I_ψ, associated to ζ, is self–decomposable, i.e., for every $0 < a < 1$, there exists an independent random variable J_a such that $J_a + aI_\psi$ has the same law as I_ψ, we refer to Sato [27] for background on self–decomposable laws. To see this, consider the first passage time above the level $-\log a$, that is $\varrho_a = \inf\{s > 0 : \zeta_s > -\log a\}$. By the strong Markov property of ζ we have that

$$\zeta'_s = \zeta_{\varrho_a+s} - \zeta_{\varrho_a}$$

is a Lévy process independent of $\{\zeta_r, r < \varrho_a\}$ and the same law as ζ. Moreover, by the absence of positive jumps and the fact that $\mathbf{E}(\zeta_1) \in]0, \infty[$, we have that $\zeta_{\varrho_a} = -\log a$ a.s. Therefore,

$$\begin{aligned}\int_0^\infty e^{-\zeta_s} ds &= \int_0^{\varrho_a} e^{-\zeta_s} ds + e^{-\zeta_{\varrho_a}} \int_0^\infty e^{-\zeta'_s} ds \\ &= J_a + aI'_\psi.\end{aligned}$$

As a consequence the density ρ_ψ, of the law of I_ψ, is unimodal, i.e., there exists a $b > 0$ such that $\rho_\psi(x)$ is increasing on $]0, b[$ and decreasing on $]b, \infty[$, see e.g. Sato [27] Theorem 53.1. Besides, Bertoin and Yor [4] section 3, showed that

$$\frac{1}{\mu} \mathbf{E}\left(f(I_\phi)\right) = \mathbf{E}\left(I_\psi^{-1} f(I_\psi^{-1})\right),$$

for every positive measurable function f, in the obvious notation. In particular, the densities of I_ψ and I_ϕ are related by

$$\frac{1}{\mu}\rho_\phi(x) = \frac{1}{x}\rho_\psi\Big(\frac{1}{x}\Big), \qquad \text{for every} \qquad x > 0. \tag{29}$$

We derive from this that ρ_ϕ is a bounded and decreasing function on some neighborhood of ∞. □

Remark 4. Equation (29) and the uniqueness of the invariant law for the OU process show that the law of I_ψ is the invariant law of the OU process associated to the subordinator with Laplace exponent ϕ.

Remark 5. Since every self–decomposable law is infinitely divisible, then the law of I_ψ is infinitely divisible. According to Steutel [30] its tail distribution is of the form

$$-\log \mathbf{P}(I_\psi > x) = O(x \log x),$$

and since its density is decreasing on a set $]b, \infty[$ it follows by Theorem 4.12.10 in Bingham et al. [7] that its density has the same behavior at infinity, i.e.

$$-\log \rho_\psi(x) = O(x \log x) \qquad x \to \infty.$$

This provides a complementary result to Proposition 2,

$$-\log x\rho_\phi(x) = O(x^{-1} \log(1/x)), \qquad x \to 0.$$

We take the following examples from Fitzsimmons et al. [14] and Bertoin and Yor [4], respectively.

Example Let ξ be a subordinator without killing term, with zero drift and Lévy measure

$$\Pi(dx) = \frac{\beta e^x}{\Gamma(1-\beta)(e^x - 1)^{1+\beta}} dx,$$

with $\beta \in]0, 1[$. An integration by parts in the Lévy–Khintchine formula and a use of the beta integral show that the Laplace exponent of ξ is given by

$$\phi(\lambda) = \frac{\Gamma(\lambda + \beta)}{\Gamma(\lambda)}.$$

Using equation (28) we get that the potential measure of ξ is absolutely continuous with respect to Lebesgue measure and that the renewal density is given by

$$u(x) = \frac{1}{\Gamma(\beta)}\Big(\frac{e^x}{e^x - 1}\Big)^{1-\beta} \qquad x > 0.$$

Therefore u is a convex decreasing function. Moreover,

$$\phi(\lambda) \sim \lambda^{\beta} \quad \text{as} \quad \lambda \to \infty.$$

According to Lamperti [21], the increasing $1/\beta$–ss Markov process X, associated to ξ is a β–stable subordinator. Then by Theorem 1 one gets

$$\liminf_{t\to\infty} \frac{X_t}{t^{1/\beta}\big(\log\log t\big)^{(\beta-1)/\beta}} = \beta(1-\beta)^{\frac{(1-\beta)}{\beta}}.$$

That is we recover the law of iterated logarithm for stable subordinators of Fristedt [15]. Furthermore, since under $\mathbb{P}_{0+}$ the law of $X(1)$ is that of an β–stable random variable one can use Proposition 1 and the estimations of the stable density, see e.g. Zolotarev [34], to recover the Breiman's [8] test for stable subordinators.

Example Let $\beta \in]0,1[$ and ξ be a subordinator with zero drift and Lévy measure

$$\Pi(dx) = \frac{e^{-x/\beta}}{\Gamma(1-\beta)(1-e^{-x/\beta})^{1+\beta}}dx.$$

By straightforward calculations we get that its Laplace exponent, say ϕ, can be expressed as

$$\phi(\lambda) = \frac{\Gamma(\beta\lambda+1)}{\Gamma(\beta(\lambda-1)+1)},$$

and by the Stirling formula

$$\phi(\lambda) \sim \beta^{\beta}\lambda^{\beta} \quad \text{as} \quad \lambda \to \infty.$$

Proceeding as in the former example we get that the renewal density of ξ is given by

$$u(x) = \frac{1}{\Gamma(1+\beta)}(e^{x/\beta}-1)^{-(1-\beta)},$$

and is a convex decreasing function. Besides, since the law of the exponential functional I associated to this subordinator is characterized by its entire moments it is immediate that its Laplace transform is given by

$$\mathbf{E}(e^{-sI}) = E_{\beta}(-s) = \sum_{n=0}^{\infty} \frac{(-s)^n}{\Gamma(n\beta+1)}.$$

The function $E_{\beta}(x)$ is the so called Mittag-Leffler function. Hence, I follows the Mittag–Leffler distribution, that is, I follows the same distribution as $\gamma_{\beta}^{-\beta}$ with γ_{β} a β–stable random variable. Furthermore, it can be showed without the use of Proposition 2 that

$$-\log \mathbf{P}(I > x) \sim (1-\beta)\beta^{\frac{\beta}{(1-\beta)}} x^{\frac{1}{(1-\beta)}}, \qquad x \to \infty,$$

see e.g. Bingham et al. [7] Theorem 8.1.12 or Sato [27] solution to exercise 29.19. This fact can be considered as a motivation for our proof of Proposition 2.

More generally, one can consider the subordinator with Laplace exponent

$$\phi_{\theta}(\lambda) = \frac{\Gamma(\beta\lambda+\theta)}{\Gamma(\beta(\lambda-1)+\theta)}, \tag{30}$$

for $\beta \in]0,1[$ and $\theta \geq \beta$. See Bertoin and Yor [4] for a description of the Lévy measure corresponding to this Laplace exponent. The renewal density associated to this Laplace exponent admits the expression

$$u_\theta(x) = \frac{1}{\Gamma(\theta+1)} e^{-x(\theta-1)/\beta}\big(e^{x/\beta}-1\big)^{-(1-\beta)}, \quad x \geq 0.$$

Which is easily seen to be a decreasing and convex function. The entire moments of the exponential functional I_θ associated to this subordinator are given by

$$\mathbf{E}(I_\theta^n) = \frac{n!\Gamma(\theta)}{\Gamma(\beta n+\theta)}, \qquad n \geq 1.$$

We recognize in this formula the entire moments of a generalized Mittag–Leffler distribution see e.g. Schneider [28]. Schneider showed that this distribution admits a density $\rho_{\beta,\theta}(x)$, whose behavior at infinity is

$$\rho_{\beta,\theta}(x) \sim Bx^\delta \exp\{c_\beta x^\sigma\}, \qquad x \to \infty,$$

with

$$\sigma = 1/(1-\beta), \qquad \delta = \frac{(\beta-\theta+1/2)}{1-\beta}, \qquad c_\beta = (1-\beta)\beta^{\frac{\beta}{1-\beta}}, \tag{31}$$

and $B = (2\pi)^{-1/2}\Gamma(\theta)\sigma^{1/2}\beta^\delta$. This fact enables us to state the sharper result

Corollary 3. *Let X be the 1–ss process associated to a subordinator ξ with Laplace exponent defined by (30). If $h : [0,\infty[\to [0,\infty[$ is a decreasing function then*

$$\mathbb{P}_x(X_s < sh(s) \quad i.o\ s \to \infty) = 0 \quad or \quad 1,$$

according whether

$$\int^\infty (h(s))^{-\delta} \exp\big\{-c_\beta (h(s))^{-\sigma}\big\}\frac{ds}{s} < \infty \quad or \quad = \infty$$

with σ, c_β and δ as in (31).

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Chapitre III

Recurrent extensions of self–similar Markov processes and Cramér's condition

Abstract

Let ξ be a real valued Lévy process that drifts to $-\infty$ and satisfies Cramér's condition, and X a self–similar Markov process associated to ξ via Lamperti's [22] transformation. In this case, X has 0 as a trap and fulfills the assumptions of Vuolle-Apiala [34]. We deduce from [34] that there exists a unique excursion measure $\mathbf{n}$, compatible with the semigroup of X and such that $\mathbf{n}(X_{0+} > 0) = 0$. Here, we give a precise description of $\mathbf{n}$ via its associated entrance law. To this end, we construct a self–similar process $X^{\natural}$, which can be viewed as X conditioned to never hit 0, and then we construct $\mathbf{n}$ in a similar way to the way in which the Brownian excursion measure is constructed via the law of a Bessel(3) process. An alternative description of $\mathbf{n}$ is given by specifying the law of the excursion process conditioned to have a given length. We establish some duality relations from which we determine the image under time reversal of $\mathbf{n}$.

Key words. Self–similar Markov processes, description of excursion measures, weak duality, Lévy processes.
A.M.S. Classification. 60 J 25 (60 G 18).

1 Introduction

Let $X = (X_t, t \geq 0)$ be a strong Markov process with values in $[0, \infty[$ and for $x \geq 0$, denote by $\mathbb{P}_x$ its law starting from x. Assume that X fulfills the scaling property: there exists some $\alpha > 0$ such that

$$\text{the law of } \ (cX_{tc^{-1/\alpha}}, t \geq 0) \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{cx}, \tag{1}$$

for any $x \geq 0$ and $c > 0$. Such processes were introduced by Lamperti [22] under the name of semi–stable processes, nowadays they are called α–self–similar Markov processes. We refer to Embrechts and Maejima [14] for a recent account on self–similar processes.

Lamperti established that for each fixed $\alpha > 0$, there exists a one to one correspondence between α–self–similar Markov processes on $]0, \infty[$ and Lévy processes which we now sketch. Let $(\mathbb{D}, \mathcal{D})$ be the space of càdlàg paths $\omega : [0, \infty[\to] - \infty, \infty[$ endowed with the σ–algebra generated by the coordinate maps and the natural filtration $(\mathcal{D}_t, t \geq 0)$ satisfying the usual conditions of right continuity and completeness. Let $\mathbf{P}$ be a probability measure on $\mathcal{D}$ such that under $\mathbf{P}$ the coordinate process ξ is a Lévy process that drifts to $-\infty$, i.e. $\lim_{s\to\infty} \xi_s = -\infty$. Set for $t \geq 0$

$$\tau(t) = \inf\{s > 0, \int_0^s e^{\xi_r/\alpha} dr > t\},$$

with the usual convention that $\inf\{\emptyset\} = \infty$. For an arbitrary $x > 0$, let $\mathbb{P}_x$ be the distribution on $\mathbb{D}^+ = \{\omega : [0,\infty[\to [0,\infty[\text{ càdlàg}\}$, of the time–changed process

$$X_t = x \exp\left(\xi_{\tau(tx^{-1/\alpha})}\right), \qquad t \geq 0,$$

where the above quantity is assumed to be 0 when $\tau(tx^{-1/\alpha}) = \infty$. We agree that $\mathbb{P}_0$ is the law of the process identical to 0. Classical results on time change yield that under $(\mathbb{P}_x, x \geq 0)$ the process X is Markovian with respect to the filtration $(\mathcal{G}_t = \mathcal{D}_{\tau(t)}, t \geq 0)$. Furthermore, X has the scaling property (1). Thus, X is a self–similar Markov process on $[0,\infty[$ having 0 as trap or absorbing point. It should be clear that the distribution of the first hitting time of 0 for X,

$$T_0 = \inf\{t > 0 : X_t = 0\}$$

under $\mathbb{P}_x$ is the same as that of $x^{1/\alpha} I$ under $\mathbf{P}$, with I the so–called Lévy exponential functional associated to ξ and α, that is

$$I = \int_0^\infty \exp\{\xi_s/\alpha\} ds. \tag{2}$$

Since ξ drifts to $-\infty$ we have that $I < \infty$, $\mathbf{P}$–a.s. and

$$\mathbb{P}_x(X_{T_0-} = 0, T_0 < \infty) = 1 \quad \text{for all} \quad x > 0.$$

We will say that X hits 0 continuously. Besides, if in the former construction we use a Lévy process killed at an independent exponential time the resulting process is a self–similar Markov process X that hits 0 by a jump

$$\mathbb{P}_x(X_{T_0-} > 0, T_0 < \infty) = 1 \quad \text{for all} \quad x > 0.$$

Conversely, any self–similar Markov process that has 0 as a trap and hits 0 continuously (resp. by a jump) is the exponential of Lévy process (resp. killed at an independent exponential time) time changed, cf. [22]. In this chapter we will restrict ourselves to the case where X hits 0 continuously and we will devote the Chapter IV to study the case where X hits 0 by a jump.

Denote P_t and V_q the semigroup and resolvent for the process X killed at time T_0, say (X, T_0),

$$P_t f(x) = \mathbb{E}_x(f(X_t), t < T_0), \qquad x > 0,$$

$$V_q f(x) = \int_0^\infty e^{-qt} P_t f(x) dt, \qquad x > 0,$$

for non–negative or bounded measurable functions f. It is customary to refer to (X, T_0) as the minimal process.

Given that the preceding construction enables us to describe the behavior of the self–similar Markov process X until its first hitting time of 0, Lamperti [22] raised the following question: *What are the self–similar Markov processes $\widetilde{X}$ on $[0,\infty[$ which behave like (X, T_0) up to the time $\widetilde{T}_0$?* Lamperti solved this problem in the case where the minimal process is a Brownian motion killed at 0. Then Vuolle-Apiala [34] tackled this problem using excursion theory for Markov processes and assuming that the following hypotheses hold. There exists $\kappa > 0$ such that

(H1-a) the limit

$$\lim_{x\to 0}\frac{\mathbb{E}_x(1-e^{-T_0})}{x^{\kappa}},$$

exists and is strictly positive;

(H1-b) the limit

$$\lim_{x\to 0}\frac{V_q f(x)}{x^{\kappa}},$$

exists for all $f\in C_K]0,\infty[$ and is strictly positive for some such functions,

with $C_K]0,\infty[=\{f:\mathbb{R}\to\mathbb{R},\text{ continuous and with compact support on }]0,\infty[\}$. The main result in [34] is the existence of an unique entrance law $(\mathbf{n}_s, s>0)$ such that

$$\lim_{s\to 0}\mathbf{n}_s\, B^c=0,$$

for every neighborhood B of 0 and

$$\int_0^{\infty} e^{-s}\,\mathbf{n}_s\, 1 ds=1.$$

This entrance law is determined by its q–potential via the formula

$$\int_0^{\infty} e^{-qs}\,\mathbf{n}_s\, f ds=\lim_{x\to 0}\frac{V_q f(x)}{\mathbb{E}_x(1-e^{-T_0})},\qquad q>0, \tag{3}$$

for $f\in C_K]0,\infty[$. Then, using the results of Blumenthal [7], Vuolle-Apiala proved that associated to the entrance law $(\mathbf{n}_s, s>0)$ there exists a unique recurrent Markov process $\widetilde{X}$ having the scaling property (1) which is an extension of the minimal process (X,T_0), that is $\widetilde{X}$ killed at time $\widetilde{T}_0$ is equivalent to (X,T_0) and 0 is a recurrent regular state for $\widetilde{X}$, i.e.

$$\widetilde{\mathbb{P}}_x(T_0<\infty)=1,\quad \forall x>0,\quad \widetilde{\mathbb{P}}_0(T_0=0)=1,$$

with $\widetilde{\mathbb{P}}$ the law on $\mathbb{D}^+$ of $\widetilde{X}$. Furthermore, we know from [7] that there exists a unique excursion measure say $\mathbf{n}$, on $(\mathbb{D}^+,\mathcal{G}_\infty)$ compatible with the semigroup P_t such that its associated entrance law is $(\mathbf{n}_s, s>0)$; the property $\lim_{s\to 0}\mathbf{n}_s\, B^c=0$, for any B neighborhood of 0, is equivalent to $\mathbf{n}(X_{0+}>0)=0$, that is the process leaves 0 continuously under $\mathbf{n}$. Then the excursion measure $\mathbf{n}$ is the unique excursion measure having the properties $\mathbf{n}(X_{0+}>0)=0$ and $\mathbf{n}(1-e^{-T_0})=1$. See subsection 2.1 for the definitions.

The first aim of this paper is to provide a more explicit description of the excursion measure $\mathbf{n}$ and its associated entrance law $(\mathbf{n}_s, s>0)$. To this end, we shall mimic a well known construction of the Brownian excursion measure via the Bessel(3) process that we next sketch for ease of reference. Let P (respectively R) be a probability measure on $(\mathbb{D}^+,\mathcal{G}_\infty)$ under which the coordinate process is a Brownian motion killed at 0 (respectively a Bessel(3) process). The probability measure R appears as the law of the Brownian motion conditioned to never hit 0. More precisely, for $u>0, x>0$

$$\lim_{t\to\infty}P_x(A\mid T_0>t)=R_x(A),$$

for any $A\in\mathcal{G}_u$, see e.g. McKean [23]. Moreover, the function $h(x)=x^{-1}, x>0$ is excessive for the semigroup of the Bessel(3) process and its h–transform is the semigroup of the Brownian motion killed at 0. Let n be the h–transform of R_0 via the function $h(x)=x^{-1}$, i.e. n is the unique measure in $(\mathbb{D}^+,\mathcal{G}_\infty)$ with support on $\{0<T_0<\infty\}$ such that under n the coordinate process is Markovian with semigroup that of Brownian motion killed at 0, and for every $\mathcal{G}_t$–stopping time T and any $\mathcal{G}_T$–measurable variable F_T,

$$\mathrm{n}(F_T, T < T_0) = R_0(F_T \, \frac{1}{X_T}).$$

Then the measure n is a multiple of the Itô's excursion measure for Brownian motion, see e.g. Imhof [20] § 4.

In order to carry out this program, through this chapter, unless otherwise stated, we will assume that ξ is a Lévy process with infinite lifetime that satisfies the following hypotheses

(H2-a) ξ is not arithmetic, i.e. the state space is not a subgroup of $k\mathbb{Z}$ for any real number k;

(H2-b) There exists $\theta > 0$ such that $\mathbf{E}(e^{\theta\xi_1}) = 1$;

(H2-c) $\mathbf{E}(\xi_1^+ e^{\theta\xi_1}) < \infty$, with $a^+ = a \vee 0$.

The condition (H2-c) can be stated in terms of the Lévy measure Π of ξ as

(H2-c') $\int_{\{x>1\}} xe^{\theta x}\Pi(dx) < \infty$;

cf. Sato [32] Theorem 25.3. Such hypotheses are satisfied by a wide class of Lévy processes, in particular by those associated with self–similar diffusions and stable processes with no negative jumps. In the sequel we will refer to these hypotheses as (H2) hypotheses.

The condition (H2-b) is called *Cramér's condition* for the Lévy process ξ and force ξ to drift to $-\infty$ or equivalently $\mathbf{E}(\xi_1) < 0$. Cramér's condition enables us to construct a law $\mathbf{P}^\natural$ on $\mathbb{D}$, such that under $\mathbf{P}^\natural$ the coordinate process $\xi^\natural$ is a Lévy process that drifts to ∞ and $\mathbf{P}^\natural|_{\mathcal{D}_t} = e^{\theta\xi_t}\,\mathbf{P}|_{\mathcal{D}_t}$. Then, we will show that the self–similar Markov process $X^\natural$ associated to the Lévy process $\xi^\natural$ plays the rôle of a Bessel(3) process in our construction of the excursion measure $\mathbf{n}$.

The rest of this paper is organized as follows. In Subsection 2.1 we recall the Itô's program as established by Blumenthal [7]. The excursion measure $\mathbf{n}$ that interests us is the unique (up to a multiplicative constant) excursion measure having the property $\mathbf{n}(X_{0+} > 0) = 0$. Nevertheless, this is not the only excursion measure compatible with the semigroup of the minimal process, which is why in Subsection 2.2 we review some properties that should be satisfied by any excursion measure corresponding to a self–similar extension of the minimal process. There we also obtain necessary and sufficient conditions for the existence of an excursion measure n^j such that $n^j(X_{0+} = 0) = 0$, which are valid for any self–similar Markov process having 0 as a trap, regardeless if it hits 0 continuously or by a jump. In Subsection 2.3 we construct a self–similar Markov process $X^\natural$ which is related to (X, T_0) in an analogous way to that in which the Bessel(3) process is related to Brownian motion killed at 0. We also prove that the conditions (H1) are satisfied under the hypothesis (H2), give a more explicit expression for the limit in equation (3) and show that the hypotheses (H1) imply the conditions (H2-b,c). Next, in Section 3 we give our main description of the excursion measure $\mathbf{n}$ and give an answer to the question raised by Lamperti that can be sketched as follows: given a Lévy process ξ satisfying the hypotheses (H2), then an α–self–similar Markov process X associated to ξ admits a recurrent extension that leaves 0 continuously a.s. if and only if $0 < \alpha\theta < 1$. The purpose of Section 4 is to give an alternative description of the measure $\mathbf{n}$ by determining the law of the excursion process conditioned by its length (for Brownian motion this corresponds to the description of the Itô excursion measure via the law of a Bessel(3) bridge). In Section 5 we study some duality relations for the minimal process and in particular we determine the image under time reversal of $\mathbf{n}$. Finally, in

Appendix A we establish that the extensions of any two minimal processes which are in weak duality are still in weak duality as might be expected.

Last, the development of this work uses the theory of h–transforms of Doob, we refer to Sharpe [33] or Walsh [35] for background.

2 Preliminaries and first results

This section contains several parts. In the first one, we recall the Itô's program and the results in Blumenthal [7]. The purpose of Subsection 2.2 is study the excursion measures compatible with the semigroup of the minimal process (X, T_0). Finally, in Subsection 2.3 we establish the existence of a self–similar Markov process $X^\natural$ which bears the same relation to the minimal process (X, T_0) as the Bessel(3) process does to Brownian motion killed at 0. The results in Subsections 2.1 and 2.2 do not require hypotheses (H2).

2.1 Some general facts on recurrent extensions of Markov processes

A measure n on $(\mathbb{D}^+, \mathcal{G}_\infty)$ having infinite mass is called a *pseudo excursion measure* compatible with the semigroup P_t if the following are satisfied:

(i) n is carried by
$$\{\omega \in \mathbb{D}^+ \mid 0 < T_0(\omega) < \infty \text{ and } X_t(\omega) = 0, \forall t \geq T_0\};$$

(ii) for every bounded $\mathcal{G}_\infty$–measurable H and each $t > 0$ and $\Lambda \in \mathcal{G}_t$
$$n(H \circ \theta_t, \Lambda \cap \{t < T_0\}) = n(\mathbb{E}_{X_t}(H), \Lambda \cap \{t < T_0\}),$$
where θ_t denotes the shift operator.

If moreover

(iii) $n(1 - e^{-T_0}) < \infty$,

we will say that n is an *excursion measure.* A normalized excursion measure n is an excursion measure n such that $n(1 - e^{-T_0}) = 1$. The rôle played by condition (iii) will be explained below.

The entrance law associated to a pseudo excursion measure n is defined by
$$n_s(dy) := n(X_s \in dy, s < T_0), \quad s > 0.$$

A partial converse holds: given an entrance law $(n_s, s > 0)$ such that
$$\int_0^\infty (1 - e^{-s}) dn_s 1 < \infty,$$

there exists a unique excursion measure n such that its associated entrance law is $(n_s, s > 0)$, see e.g. [7].

It is well known in the theory of Markov processes that one way to construct recurrent extensions of a Markov process is the Itô's program or pathwise approach that can be described as follows. Assume that there exists an excursion measure n compatible with the semigroup of the minimal process P_t. Realize a Poisson point process $\Delta = (\Delta_s, s > 0)$ on $\mathbb{D}^+$ with characteristic measure n. Thus each atom Δ_s is a path and $T_0(\Delta_s)$ denotes its lifetime, i.e.

$$T_0(\Delta_s) = \inf\{t > 0 : \Delta_s(t) = 0\}.$$

Set

$$\sigma_t = \sum_{s \le t} T_0(\Delta_s), \qquad t \ge 0.$$

Since $n(1 - e^{-T_0}) < \infty$, $\sigma_t < \infty$ a.s. for every $t > 0$. It follows that the process $\sigma = (\sigma_t, t \ge 0)$ is an increasing càdlàg process with stationary and independent increments, i.e. a subordinator. Its law is characterized by its Laplace exponent ϕ, defined by

$$\mathbf{E}(e^{-\lambda\sigma_1}) = e^{-\phi(\lambda)}, \qquad \lambda > 0,$$

and $\phi(\lambda)$ can be expressed thanks to the Lévy–Khintchine formula as

$$\phi(\lambda) = \int_{]0,\infty[} (1 - e^{-\lambda s})\nu(ds),$$

with ν a measure such that $\int s \wedge 1 \; \nu(ds) < \infty$, called the Lévy measure of σ; see e.g. Bertoin [1] § 3 for background. An application of the exponential formula for Poisson point processes gives

$$\mathbf{E}(e^{-\lambda\sigma_1}) = e^{-n(1-e^{-\lambda T_0})}, \qquad \lambda > 0,$$

i.e. $\phi(\lambda) = n(1 - e^{-\lambda T_0})$ and the tail of the Lévy measure is given by

$$\nu[s, \infty[= n(s < T_0) = n_s 1, \qquad s > 0.$$

Observe that if we assume $\phi(1) = n(1 - e^{-T_0}) = 1$ then ϕ is uniquely determined. Since n has infinite mass, σ_t is strictly increasing in t. Let L_t be the local time at 0, i.e. the continuous inverse of σ

$$L_t = \inf\{r > 0 : \sigma_r > t\} = \inf\{r > 0 : \sigma_r \ge t\}.$$

Define a process $(\widetilde{X}_t, t \ge 0)$ as follows. For $t \ge 0$, let $L_t = s$, then $\sigma_{s-} \le t \le \sigma_s$, set

$$\widetilde{X}_t = \begin{cases} \Delta_s(t - \sigma_{s-}) & \text{if} \quad \sigma_{s-} < \sigma_s \\ 0 & \text{if} \quad \sigma_{s-} = \sigma_s \text{ or } s = 0. \end{cases} \tag{4}$$

That the process so constructed is a Markov process has been established in full generality by Salisbury [30, 31] and under some regularity hypotheses on the semigroup of the minimal process by Blumenthal [7]. See also Rogers [29] for its analytical counterpart. In our setting, the hypotheses in [7] are satisfied as is shown by the following lemma.

Lemma 1. *Let $C_0]0, \infty[$, be the space of continuous functions on $]0, \infty[$ vanishing at 0 and ∞.*

(i) if $f \in C_0]0, \infty[$, then $P_t f \in C_0]0, \infty[$ and $P_t f \to f$ uniformly as $t \to 0$.

(ii) $\mathbb{E}_x(e^{-qT_0})$ is continuous in x for each $q > 0$ and

$$\lim_{x \to 0} \mathbb{E}_x(e^{-T_0}) = 1 \quad \textit{and} \quad \lim_{x \to \infty} \mathbb{E}_x(e^{-T_0}) = 0.$$

This Lemma is an easy consequence of Lamperti's transformation. Alternatively a proof can be found in [34] pp. 549–550. Therefore we have from [7] that $\widetilde{X}$ is a Markov process with Feller semigroup and its resolvent $\{U_q, q > 0\}$ satisfies

$$U_q f(x) = V_q f(x) + \mathbb{E}_x(e^{-qT_0})U_q f(0), \quad x > 0,$$

for $f \in C_b(\mathbb{R}^+) = \{f : \mathbb{R}^+ \to \mathbb{R}, \text{ continuous and bounded}\}$. That is $\widetilde{X}$ is an extension of the minimal process. Furthermore, if $\{X'_t, t \geq 0\}$ is a Markov process extending the minimal one with Itô excursion measure n and local time at 0, say $\{L'_t, t \geq 0\}$, such that

$$\mathbb{E}'_0(\int_0^\infty e^{-s} dL'_s) = 1,$$

where $\mathbb{E}'$ is the law for X'. Then the process $\widetilde{X}$ and X' are equivalent and the Itô's excursion measure for $\widetilde{X}$ is n.

Thus, the results in [7] establish a one to one correspondence between excursion measures and recurrent extensions of Markov processes. Given an excursion measure n we will say that the associated extension of the minimal process leaves 0 continuously a.s. if $n(X_{0+} > 0) = 0$ or, equivalently, in terms of its entrance law, $\lim_{s\to 0} n_s(B^c) = 0$ for every neighborhood B of 0, see e.g. [7]; if n is such that $n(X_{0+} = 0) = 0$, we will say that the extension leaves 0 by jumps a.s. The latter condition on n is equivalent to the existence of a jumping–in measure η, that is η is a σ–finite measure on $]0, \infty[$ such that the entrance law associated to n can be expressed as

$$n_s f = n(f(X_s), s < T_0) = \int_{]0,\infty[} \eta(dx) P_s f(x), s > 0,$$

for every $f \in C_b(\mathbb{R}^+)$, cf. Meyer [25].

Finally, observe that if n is a pseudo excursion measure that does not satisfy the condition (iii), we can still realize a Poisson point process of excursions on $(\mathbb{D}^+, \mathcal{G}_\infty)$ with characteristic measure n but we cannot form a process extending the minimal one by sticking together the excursions because the sum of lengths $\sum_{s\leq t} T_0(\Delta_s)$, is infinite $\mathbb{P}$-a.s. for every $t > 0$.

2.2 Some properties of excursion measures for self–similar Markov processes

Next, we deduce necessary and sufficient conditions that must be satisfied by an excursion measure in order that the associated recurrent extension of the minimal process is self–similar. For $c \in \mathbb{R}$, let H_c be the dilatation $H_c f(x) = f(cx)$.

Lemma 2. *Let n be an excursion measure and $\widetilde{X}$ the associated recurrent extension of the minimal process. The following are equivalent*

(i) The process $\widetilde{X}$ has the scaling property

(ii) There exists $\gamma \in]0, 1[$ such that for any $c > 0$,

$$n(\int_0^{T_0} e^{-qs} f(X_s) ds) = c^{(1-\gamma)/\alpha} n(\int_0^{T_0} e^{-(qc^{1/\alpha}s)} H_c f(X_s) ds),$$

for $f \in C_b(\mathbb{R}^+)$.

(iii) There exists $\gamma \in]0,1[$ such that for any $c > 0$,

$$n_s f = c^{-\gamma/\alpha} n_{s/c^{1/\alpha}} H_c f \quad \textit{for all} \quad s > 0,$$

for $f \in C_b(\mathbb{R}^+)$.

Remark If one of the conditions (i–iii) in the preceding Lemma holds, then the subordinator σ which is the inverse local time of $\widetilde{X}$ is a stable subordinator of parameter γ, where γ is determined in the condition (ii) or (iii).

Proof. (ii) $\iff$ (iii) is straightforward.

(i) $\Rightarrow$ (ii). Suppose that there exists an excursion measure n such that the associated recurrent extension $\widetilde{X}$ has the scaling property (1). Let $\mathcal{M}$ be the random set of zeros of the process $\widetilde{X}$, i.e. $\mathcal{M} = \{t \geq 0 | \widetilde{X}(t) = 0\}$. By construction $\mathcal{M}$ is the closed range of the subordinator $\sigma = (\sigma_t, t \geq 0)$, that is $\mathcal{M}$ is a regenerative set. The recurrence of $\widetilde{X}$ implies that $\mathcal{M}$ is unbounded a.s. By the scaling property for $\widetilde{X}$ we have that

$$\mathcal{M} \stackrel{d}{=} c\mathcal{M}, \qquad \text{for each} \qquad c > 0,$$

that is $\mathcal{M}$ is self–similar. Thus the subordinator should have the scaling property and since the only Lévy processes that have the scaling property are the stable processes it follows that σ is a stable subordinator of parameter γ for some $\gamma \in]0,1[$ or, in terms of its Laplace exponent $\phi(\lambda) = n(1 - e^{-\lambda T_0}) = \lambda^{\gamma}, \lambda > 0$. Recall that the scaling property for the extension can be stated in terms of its resolvent by saying that for any $c > 0$,

$$U_q f(x) = c^{1/\alpha} U_{qc^{1/\alpha}} H_c f(x/c), \quad \text{for all} \quad x \geq 0, \tag{5}$$

for $f \in C_b(\mathbb{R}^+)$. Using the compensation formula for Poisson point processes we get that

$$U_q f(0) = \frac{n(\int_0^{T_0} e^{-qs} f(X_s) ds)}{n(1 - e^{-qT_0})}, \tag{6}$$

From equation (5) we have that the measure n should be such that

$$\frac{n(\int_0^{T_0} e^{-qs} f(X_s) ds)}{n(1 - e^{-qT_0})} = c^{1/\alpha} \frac{n(\int_0^{T_0} e^{-qc^{1/\alpha} s} H_c f(X_s) ds)}{n(1 - e^{-qc^{1/\alpha} T_0})},$$

and therefore we conclude that

$$n(\int_0^{T_0} e^{-qs} f(X_s) ds) = c^{(1-\gamma)/\alpha} n(\int_0^{T_0} e^{-(qc^{1/\alpha} s)} H_c f(X_s) ds).$$

(ii) $\Rightarrow$ (i). The scaling property of $\widetilde{X}$ is obtained by means of (5). In fact, the only thing that should be verified is that equation (5) holds for $x = 0$, since we have the identity

$$U_q f(x) = V_q f(x) + \mathbb{E}_x(e^{-qT_0}) U_q f(0), \quad x > 0,$$

and the scaling property of the minimal process stated in terms of its resolvent V_q, i.e.

$$V_q f(x) = c^{1/\alpha} V_{qc^{1/\alpha}} H_c f(x/c), \quad x > 0, c > 0, q > 0.$$

Indeed, by construction it follows that the formula (6) holds and the hypothesis (ii) implies that $n(1 - e^{-qT_0}) = q^{\gamma},\ q > 0$; the conclusion is immediate. □

In the following proposition we give a description of the sojourn measure of $\widetilde{X}$ and a necessary condition for the existence of a excursion measure n such that one of the conditions in Lemma 2 holds.

Lemma 3. *Let n be a normalized excursion measure and $\widetilde{X}$ the associated extension of the minimal process (X, T_0). Assume that one of the conditions (i–iii) in Lemma 2 holds. Then*

$$n(\int_0^{T_0} 1_{\{X_s \in dy\}} ds) = C_{\alpha,\gamma} y^{(1-\alpha-\gamma)/\alpha} dy, \quad y > 0,$$

with γ determined in (ii) of Lemma 2 and $C_{\alpha,\gamma} \in]0, \infty[$ a constant. As a consequence, $\mathbf{E}(I^{-(1-\gamma)}) < \infty$ and $C_{\alpha,\gamma} = (\alpha\, \mathbf{E}(I^{-(1-\gamma)})\Gamma(1-\gamma))^{-1}$, where I denotes the exponential functional (2).

Proof. Recall that the sojourn measure

$$n(\int_0^{T_0} 1_{\{X_s \in dy\}} ds) = \int_0^\infty n_s(dy) ds,$$

is a σ–finite measure on $]0, \infty[$ and is the unique excessive measure for the semigroup of the process $\widetilde{X}$, see e.g. Dellacherie et al. [12] XIX.46. Next, using the result (iii) in Lemma 2 and the Fubini's Theorem we obtain the following representation of the sojourn measure, for $f \geq 0$ measurable

$$\begin{aligned}\int_0^\infty n_s f ds &= \int_0^\infty s^{-\gamma} n_1(H_{s^\alpha} f) ds \\ &= \int n_1(dz) \int_0^\infty s^{-\gamma} f(s^\alpha z) ds \\ &= C_{\alpha,\gamma} \int_0^\infty u^{(1-\alpha-\gamma)/\alpha} f(u) du,\end{aligned}$$

with $0 < C_{\alpha,\gamma} = \alpha^{-1} \int n_1(dz) z^{-(1-\gamma)/\alpha} < \infty$. This proves the first part of the claimed result. We now prove that $\mathbf{E}(I^{-(1-\gamma)}) < \infty$. On the one hand, the function $\varphi(x) = \mathbb{E}_x(e^{-T_0})$ is integrable with respect to the sojourn measure. To see this, use the Markov property under n, to obtain

$$\begin{aligned}n(\int_0^{T_0} \varphi(X_s) ds) &= \int_0^\infty n(\varphi(X_s), s < T_0) ds \\ &= \int_0^\infty n(e^{-T_0} \circ \theta_s, s < T_0) ds \\ &= \int_0^\infty n(e^{-(T_0 - s)}, s < T_0) ds \\ &= n(1 - e^{-T_0}) = 1.\end{aligned}$$

On the other hand, using the representation of the sojourn measure, Fubini's Theorem and the scaling property we have that

$$\begin{aligned}C_{\alpha,\gamma} \int_0^\infty \mathbb{E}_y(e^{-T_0}) y^{(1-\alpha-\gamma)/\alpha} dy &= C_{\alpha,\gamma} \int_0^\infty \mathbf{E}(e^{-y^{1/\alpha} I}) y^{(1-\alpha-\gamma)/\alpha} dy \\ &= C_{\alpha,\gamma} \alpha\, \mathbf{E}(I^{-(1-\gamma)}) \Gamma(1-\gamma).\end{aligned}$$

Therefore, $\mathbf{E}(I^{-(1-\gamma)}) < \infty$ and $C_{\alpha,\gamma} = (\alpha\, \mathbf{E}(I^{-(1-\gamma)})\Gamma(1-\gamma))^{-1}$. □

We next study the extensions $\widetilde{X}$ that leave 0 a.s. by jumps. Using only the scaling property (1) it can be verified that the only possible jumping–in measures such that the associated excursion measure satisfies (ii) in Lemma 2 should be of the type

$$\eta(dx) = b_{\alpha,\beta} x^{-(1+\beta)}dx, \quad x > 0,\ 0 < \alpha\beta < 1,$$

with a constant $b_{\alpha,\beta} > 0$, depending on α and β, cf. [34]. This being said we can state an elementary but satisfactory result on the existence of extensions of the minimal process that leaves 0 by jumps a.s.

Proposition 1. *Let $\beta \in]0, 1/\alpha[$. The following are equivalent*

(i) $\mathbf{E}(I^{\alpha\beta}) < \infty$;

(ii) The pseudo excursion measure $n^j = \mathbb{P}^{\eta}$, based on the jumping–in measure

$$\eta(dx) = x^{-(1+\beta)}dx, \qquad x > 0,$$

is an excursion measure;

(iii) The minimal process (X, T_0) admits an extension $\widetilde{X}$, that is a self–similar recurrent Markov process and leaves 0 by jumps a.s. according to the jumping–in measure $\eta(dx) = b_{\alpha,\beta}x^{-(1+\beta)}dx$, with $b_{\alpha,\beta} = \beta/\,\mathbf{E}(I^{\alpha\beta})\Gamma(1-\alpha\beta)$.

If one of these conditions holds then γ in (ii) in Lemma 2 is equal to $\alpha\beta$.

The condition (i) in Proposition 1 is easily verified under weak technical assumptions. Namely, if we assume the hypothesis (H2) the aforementioned condition is verified for every $\beta \in]0, (1/\alpha) \wedge \theta[$; this will be deduced from Lemma 4 below. On the other hand, the condition is verified in other settings, as can be seen in the following example.

Example 1 (Generalized self–similar saw tooth processes). Let $\alpha > 0$, ζ a subordinator such that $\mathbf{E}(\zeta_1) < \infty$, and X the α–self–similar process associated to the Lévy process $\xi = -\zeta$. Then ξ drifts to $-\infty$, X has a finite lifetime T_0 and X decreases from its starting point until the time T_0, when it is absorbed at 0. Furthermore, it was proved by Carmona et al. [10] that the Lévy exponential functional $I = \int_0^\infty \exp\{-\zeta_s/\alpha\}ds$, has finite integral moments of all orders. It follows that the condition (i) in Proposition 1 is satisfied by every $\beta \in]0, 1/\alpha[$. Thus for each $\beta \in]0, 1/\alpha[$ the α–self–similar extension $\widetilde{X}$ that leaves 0 by jumps according to the jumping–in measure in (iii) of Proposition 1, is a process having sample paths that looks like a saw with "rough" teeth. These are all the possible extensions of X, that is, it is impossible to construct an excursion measure such that its associated extension of (X, T_0) leaves 0 continuously a.s. since we know that the process X decreases to 0.

Proof of Proposition 1. Let $\eta(dx) = x^{-(1+\beta)}dx$, $x > 0$ and n^j be the pseudo excursion measure $n^j = \mathbb{P}^{\eta}$. By definition the entrance law associated to n^j is

$$n^j_s f = \int_0^\infty dx\ x^{-(1+\beta)}P_s f(x),\ s > 0.$$

Thus, for n^j to be an excursion measure, the only condition it needs to satisfy is $n^j(1-e^{-T_0})<\infty$. This follows from the elementary calculation

$$\begin{aligned}\int_0^\infty dx\; x^{-(1+\beta)}\,\mathbb{E}_x(1-e^{-T_0}) &= \int_0^\infty dx\; x^{-(1+\beta)}\,\mathbf{E}(1-e^{-x^{1/\alpha}I})\\ &= \alpha\,\mathbf{E}\Big(\int dy\; y^{-\alpha\beta-1}(1-e^{-yI})\Big)\\ &= \mathbf{E}(I^{\alpha\beta})\frac{\Gamma(1-\alpha\beta)}{\beta}.\end{aligned}$$

That is, $n^j(1-e^{-T_0})<\infty$ if and only if $\mathbf{E}(I^{\alpha\beta})<\infty$, which proves the equivalence between the assertions in (i) and (ii). If (ii) holds it follows from the results in [7] and Lemma 2 that associated to the normalized excursion measure $n^{j'}=b_{\alpha,\beta}\mathbb{P}^{\eta}$ there exists a unique extension of the minimal process (X,T_0) which is a self–similar Markov process and which leaves 0 by jumps according to the jumping–in measure $b_{\alpha,\beta}x^{-(1+\beta)}dx, x>0$, which establishes (iii). Conversely, if (iii) holds the Itô's excursion measure of $\widetilde{X}$ is $n^{j'}=b_{\alpha,\beta}\mathbb{P}^{\eta}$ and the statement in (ii) follows. □

2.3 The process $X^\natural$ analogous to the Bessel(3) process

Here we shall establish the existence of a self–similar Markov process $X^\natural$ that can be viewed as the self–similar Markov process (X,T_0) conditioned to never hit 0. In the case where (X,T_0) is a Brownian motion killed at 0, $X^\natural$ corresponds to the Bessel(3) process. To this end, we next recall some facts on Lévy processes and density transformations and deduce some consequences for self–similar Markov processes. We assume henceforth (H2).

The law of a Lévy process ξ, is characterized by a function $\Psi:\mathbb{R}\to\mathbb{C}$, defined by the relation

$$\mathbf{E}(e^{iu\xi_1})=\exp\{-\Psi(u)\},\qquad u\in\mathbb{R}\,.$$

The function Ψ is called the characteristic exponent of the Lévy process ξ and can be expressed thank to the Lévy–Khintchine formula as

$$\Psi(u)=iau+\frac{\sigma^2u^2}{2}+\int_{\mathbb{R}}(1-e^{iux}+iux1_{\{|x|<1\}})\Pi(dx),$$

where Π is a measure on $\mathbb{R}\setminus\{0\}$ such that $\int(|x|^2\wedge 1)\Pi(dx)<\infty$. The measure Π is called the Lévy measure, a the drift and σ^2 the Gaussian coefficient of ξ. Conditions (H2-b,c) imply that the Lévy exponent of ξ admits an analytic extension to the complex strip $\Im(z)\in[-\theta,0]$. Thus we can define a function $\psi:[0,\theta]\to\mathbb{R}$ by

$$\mathbf{E}(e^{\lambda\xi_1})=e^{\psi(\lambda)}\quad\text{and}\quad\psi(\lambda)=-\Psi(-i\lambda),\quad 0\le\lambda\le\theta.$$

Hölder's inequality implies that ψ is a convex function and that θ is the unique solution to the equation $\psi(\lambda)=0$ for $\lambda>0$. Furthermore, the function $h(x)=e^{\theta x}$ is invariant for the semigroup of ξ. Let $\mathbf{P}^\natural$ be the h–transform of $\mathbf{P}$ via the invariant function $h(x)=e^{\theta x}$. That is, the measure $\mathbf{P}^\natural$ is the unique measure on $(\mathbb{D},\mathcal{D})$ such that for every finite $\mathcal{D}_t$-stopping time T and each $A\in\mathcal{D}_T$

$$\mathbf{P}^\natural(A)=\mathbf{P}(e^{\theta\xi_T}A).$$

Under $\mathbf{P}^\natural$ the process $(\xi_t,t\ge 0)$ still is a Lévy process, with characteristic exponent

$$\Psi^\natural(u)=\Psi(u-i\theta),\qquad u\in\mathbb{R},$$

and drifts to ∞, more precisely,

$$0 < m^{\natural} := \mathbf{E}^{\natural}(\xi_1) = \psi'(\theta-) < \infty.$$

See e.g. Sato [32] § 33, for a proof of these facts and more about this change of measure.

Let ${\mathbb{P}^{\natural}}_x$ denote the law on $\mathbb{D}^+$ of the self–similar Markov process started at $x > 0$ associated to the Lévy process $\xi^{\natural}$ via Lamperti's transformation. In the sequel it will be implicit that the superscript $\natural$ refers to the measure $\mathbb{P}^{\natural}$ or $\mathbf{P}^{\natural}$. We now establish a relation between the probability measures $\mathbb{P}$ and $\mathbb{P}^{\natural}$ analogous to that between the law of a Brownian motion killed at 0 and the law of a Bessel(3) process, see e.g. McKean [23]. Informally, the law ${\mathbb{P}^{\natural}}_x$ can be interpreted as the law under $\mathbb{P}_x$ of X conditioned to never hit 0.

Proposition 2. *(i) Let $x > 0$ be arbitrary. Then we have that ${\mathbb{P}^{\natural}}_x$ is the unique measure such that for every $\mathcal{G}_t$–stopping time T we have*

$${\mathbb{P}^{\natural}}_x(A) = x^{-\theta}\mathbb{P}_x(A\ X_T^{\theta}, T < T_0),$$

for any $A \in \mathcal{G}_T$. In particular, the function $h^* : [0,\infty[\to [0,\infty[$ defined by $h^*(x) = x^{\theta}$ is invariant for the semigroup P_t.

(ii) For every $x > 0$ and $t > 0$ we have

$${\mathbb{P}^{\natural}}_x(A) = \lim_{s\to\infty} \mathbb{P}_x(A \mid T_0 > s),$$

for any $A \in \mathcal{G}_t$.

The proof of (i) in Proposition 2 is a straightforward consequence of the fact that $\mathbf{P}^{\natural}$ is the h–transform of $\mathbf{P}$ and that for every $\mathcal{G}_t$–stopping time T we have that $\tau(T)$ is a $\mathcal{F}_t$–stopping time. To prove (ii) in Proposition 2 we need the following lemma that provides us with a tail estimate for the law of the Lévy exponential functional I associated to ξ as defined in (2).

Lemma 4. *Under the conditions (H2) we have that*

$$\lim_{t\to\infty} t^{\alpha\theta}\,\mathbf{P}(I > t) = C,$$

where

$$0 < C = \frac{\alpha}{m^{\natural}}\int t^{\alpha\theta-1}(\mathbf{P}(I > t) - \mathbf{P}(e^{\xi_1'}I > t))dt < \infty,$$

with $\xi_1' =^d \xi_1$ and independent of I. If $0 < \alpha\theta < 1$, then

$$C = \frac{\alpha}{m^{\natural}}\,\mathbf{E}(I^{-(1-\alpha\theta)}).$$

Two proofs of this result have been given in a slightly restricted setting by Mejane [24]. However, one of these proofs can be extended to our case and in fact it is an easy consequence of a result on random equations originally due to Kesten [21], who in turn uses a difficult result on random matrices. A simpler proof of Kesten's result was given in Goldie [19].

Sketch of proof of Lemma 4. It is straightforward that the Lévy exponential functional I satisfies the equation in law

$$I =^d \int_0^1 e^{\xi_s/\alpha}ds + e^{\xi_1/\alpha}I' = Q + MI',$$

with I' the Lévy exponential functional associated to $\xi' = \{\xi'_t = \xi_{1+t} - \xi_1, t \geq 0\}$, a Lévy process independent of $\mathcal{F}_1$ and with the same distribution as ξ. Thus, according to [21] if the conditions (i–iv) below are satisfied then there exists a strictly positive constant C such that

$$\lim_{t\to\infty} t^{\alpha\theta}\,\mathbf{P}(I > t) = C.$$

The hypotheses of Kesten's Theorem are

(i) M is not arithmetic

(ii) $\mathbf{E}(M^{\alpha\theta}) = 1$,

(iii) $\mathbf{E}(M^{\alpha\theta}\ln^+(M)) < \infty$,

(iv) $\mathbf{E}(Q^{\alpha\theta}) < \infty$.

Assuming the conditions (H2) the only thing that needs to be verified is that (iv) holds. Indeed,

$$\begin{aligned}\mathbf{E}(Q^{\alpha\theta}) &\leq \mathbf{E}\left(\sup\{e^{\theta\xi_s} : s \in [0,1]\}\right)\\ &\leq \frac{e}{e-1}\left(1 + \theta\sup\{\mathbf{E}(\xi_s^+ e^{\theta\xi_s}) : s \in [0,1]\}\right) < \infty.\end{aligned}$$

The second inequality is obtained using the fact that $(e^{\theta\xi_t}, t \geq 0)$ is a positive martingale and a Doob's inequality. The first formula for the value of the limit, $C = \lim_{t\to\infty} t^{\alpha\theta}\,\mathbf{P}(I > t)$ is a consequence of Lemma 2.2 and Theorem 4.1 in Goldie [19]. That the latter limit exists implies that $\mathbf{E}(I^a) < \infty$, for all $0 < a < \alpha\theta$. Now, to obtain the expression for C when $0 < \alpha\theta < 1$, we will use the following formula for the moments of I,

$$\mathbf{E}(I^a) = \frac{a}{-\psi(a/\alpha)}\,\mathbf{E}(I^{a-1}), \quad \text{for} \quad 0 < a < \alpha\theta, \tag{7}$$

which can be proved with arguments similar to that given by Bertoin and Yor [5] Proposition 2. We will also use the well known identity

$$\lambda^a = \frac{a}{\Gamma(1-a)}\int_0^\infty (1 - e^{-\lambda x})x^{-(1+a)}dx, \quad \lambda > 0, a \in]0,1[.$$

On the one hand, since $0 < \alpha\theta < 1$, Corollary 8.1.7 in Bingham et al. [6] implies

$$\lim_{s\to 0}\frac{\mathbf{E}(1 - e^{-sI})}{s^{\alpha\theta}} = C\Gamma(1-\alpha\theta).$$

On the other hand, by equation (7) we have

$$\begin{aligned}\mathbf{E}(I^{-(1-\alpha\theta)})\alpha\theta &= \lim_{a\uparrow\alpha\theta}\mathbf{E}(I^{a-1})a\\ &= \frac{\alpha\theta}{\Gamma(1-\alpha\theta)}\lim_{a\uparrow\alpha\theta}(-\psi(a/\alpha))\int_0^\infty s^{-(1+a)}\,\mathbf{E}\left(1 - e^{-sI}\right)ds\\ &= C\alpha\theta\lim_{a\uparrow\alpha\theta}\frac{-\psi(a/\alpha)}{\alpha\theta - a}\\ &= C\theta\psi'(\theta-).\end{aligned} \tag{8}$$

Indeed, write

$$\mathbf{E}(I^{a-1}) = \mathbf{E}(I^{a-1}1_{\{I\geq 1\}}) + \mathbf{E}(I^{a-1}1_{\{I<1\}}).$$

The first term tends to $\mathbf{E}(I^{\alpha\theta-1}1_{\{I\geq 1\}})$ as $a\uparrow\alpha\theta$, by dominated convergence. A consequence of equation (7) is that $\mathbf{E}(I^{a-1})<\infty$ for every $0<a<\alpha\theta$. Then by monotone convergence the second term tends to $\mathbf{E}(I^{\alpha\theta-1}1_{\{I<1\}})$. Then $\lim_{a\uparrow\alpha\theta}\mathbf{E}(I^{a-1}) = \mathbf{E}(I^{\alpha\theta-1})$. Next, using that the Stieltjes measure $q_{\alpha\theta-a}$ over $[0,\infty[$ defined by $q_{\alpha\theta-a}[0,s[= s^{\alpha\theta-a}$, $s>0$ converges weakly to the Dirac mass at 0 as $a\uparrow\alpha\theta$ we obtain that

$$\lim_{a\uparrow\alpha\theta}(\alpha\theta-a)\int_0^\infty s^{-(1+a)}\,\mathbf{E}\left(1-e^{-sI}\right)ds = \lim_{a\uparrow\alpha\theta}\int_0^\infty \frac{\mathbf{E}\left(1-e^{-sI}\right)}{s^{\alpha\theta}}q_{\alpha\theta-a}(ds) = C\Gamma(1-\alpha\theta)$$

and the claim in equation (8) follows. □

The proof of Proposition 2 follows from standard arguments.

Proof of (ii) in Proposition 2. Recall that the law of T_0 under $\mathbb{P}_x$ is that of $x^{1/\alpha}I$ under $\mathbf{P}$. Thus we deduce from Lemma 4 that for every $x>0$,

$$\lim_{s\to\infty} s^{\alpha\theta}\mathbb{P}_x(T_0>s) = x^\theta C.$$

Using the Markov property and a dominated convergence argument, we obtain that

$$\begin{aligned}\mathbb{P}_x(A\mid T_0>s) &= \mathbb{P}_x(A\ 1_{\{t<T_0\}}\mathbb{P}_{X_t}(T_0>s-t)/\mathbb{P}_x(T_0>s))\\ &\xrightarrow[s\to\infty]{} x^{-\theta}\mathbb{P}_x(A\ X_t^\theta\ 1_{\{t<T_0\}}).\end{aligned}$$

□

By Proposition 2, the semigroup of X under $\mathbb{P}^\natural{}_x$ is given by

$$P_s^\natural f(x) := \mathbb{E}^\natural{}_x(f(X_s)) = x^{-\theta}\,\mathbb{E}_x(f(X_s)X_s^\theta 1_{\{s<T_0\}}), \qquad \text{for} \quad x>0,$$

with f a positive or bounded measurable function. Let J be the Lévy exponential functional associated to the process $\xi^\natural$, i.e.

$$J = \int_0^\infty \exp\{-\xi_s^\natural/\alpha\}ds, \tag{9}$$

which is finite $\mathbf{P}^\natural$–a.s. since $\xi^\natural$ drifts to ∞. Now, since under $\mathbf{P}^\natural$ the process $(\xi_s^\natural, s\geq 0)$ is a non arithmetic Lévy process with $0<m^\natural<\infty$, Theorem 1 in Bertoin and Yor [4] ensures that the measure $\mathbb{P}^\natural{}_x$ converges in the sense of finite dimensional distributions to a probability measure $\mathbb{P}^\natural{}_{0+}$ as $x\to 0$. Moreover, the law of X_s under $\mathbb{P}^\natural{}_{0+}$ is an entrance law for the semigroup $P_t^\natural$ and is related to the law of the Lévy exponential functional J under $\mathbf{P}^\natural$ by the formula

$$\mathbb{E}^\natural{}_{0+}(f(X_s^{1/\alpha})) = \frac{\alpha}{m^\natural}\,\mathbf{E}^\natural(f(s/J)/J), \quad s>0, \tag{10}$$

for f measurable and positive. Recall also that $m^\natural/\alpha = \mathbf{E}^\natural(1/J)<\infty$, cf. [4] for a proof of these facts.

The next result states that under the hypotheses (H2) the conditions (H1) hold and gives a first description of the entrance law $(\mathbf{n}_s, s>0)$.

Proposition 3. *Assume the hypotheses (H2).*

(i) If $0 < \alpha\theta < 1$, then the hypotheses (H1) hold for $\kappa = \theta$. Furthermore, the q–potential of the entrance law $(\mathbf{n}_s, s > 0)$, admits the representation

$$\int_0^\infty dse^{-qs}\,\mathbf{n}_s\,f = \gamma_{\alpha,\theta}\int_0^\infty f(y)\,\mathbf{E}^\natural(\exp\{-qy^{1/\alpha}J\})y^{(1-\alpha-\alpha\theta)/\alpha}dy,$$

where

$$\gamma_{\alpha,\theta} = \left(\alpha\,\mathbf{E}(I^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)\right)^{-1},$$

for every $f \in C_b(\mathbb{R}^+)$.

(ii) If $\alpha\theta \geq 1$, then either the hypothesis (H1-a) or (H1-b) fails to hold.

Proof. (i) That the hypothesis (H1-a) holds is easily proved. Indeed, since $0 < \alpha\theta < 1$ the Corollary 8.1.7. in Bingham et al. [6] implies that the result in Lemma 4 is equivalent to

$$\lim_{x\to 0}\frac{\mathbb{E}_x(1-e^{-T_0})}{x^\theta} = \lim_{x\to 0}\frac{\mathbf{E}(1-e^{-x^{1/\alpha}I})}{x^\theta} = \Gamma(1-\alpha\theta)\frac{\alpha\,\mathbf{E}(I^{-(1-\alpha\theta)})}{m^\natural}. \tag{11}$$

To prove (H1-b) we recall the identity,

$$\frac{V_qf(x)}{x^\theta} = V_q^\natural(f/h^*)(x),$$

where $V_q^\natural$ is the resolvent of the semigroup $P_t^\natural$ and $h^*(x) = x^\theta, x > 0$. As was already pointed out, the results in [4] are applicable in our setting to the self–similar process $X^\natural$. In particular, formula (4) op. cit. states that

$$\lim_{x\to 0} V_q^\natural g(x) = \frac{\alpha}{m^\natural}\int_0^\infty g(y^\alpha)\,\mathbf{E}^\natural(e^{-qyJ})dy,$$

for every function $g \in C_b(\mathbb{R}^+)$. Therefore,

$$\begin{aligned}\lim_{x\to 0}\frac{V_qf(x)}{x^\theta} &= \lim_{x\to 0} V_q^\natural(f/h^*)(x)\\ &= \frac{\alpha}{m^\natural}\int_0^\infty f(y^\alpha)y^{-\alpha\theta}\,\mathbf{E}^\natural(e^{-qyJ})dy,\\ &= \frac{1}{m^\natural}\int_0^\infty f(y)\,\mathbf{E}^\natural(e^{-qy^{1/\alpha}J})y^{(1-\alpha-\alpha\theta)/\alpha}dy\end{aligned} \tag{12}$$

for every $f \in C_K]0,\infty[$. Thus we have verified the hypotheses (H1) and the expression of the q–resolvent of the entrance law $(\mathbf{n}_s, s > 0)$ follows from the identity (3) using the calculations in equation (11) and (12).
(ii) If $\alpha\theta \geq 1$, the Fatou's lemma and the scaling property imply

$$\liminf_{x\to 0}\frac{\mathbb{E}_x(1-e^{-T_0})}{x^\theta} \geq \int_0^\infty e^{-s}s^{-\alpha\theta}\left(\liminf_{t\to\infty} t^{\alpha\theta}\,\mathbf{P}(I > t)\right)ds = \infty.$$

But from the proof of (i) we know that the limit

$$\lim_{x\to 0}\frac{V_qf(x)}{x^\theta}, \quad q > 0,$$

still exists and is not 0 for every non–negative function $f \in C_K]0,\infty[$ and, indeed, $f > 0$ in a set of positive Lebesgue measure. As a consequence, even if there exists $\kappa < \theta$, such that the limit $\lim_{x\to 0} x^{-\kappa}\,\mathbb{E}_x(1-e^{-T_0})$, exists and is positive, the limit $\lim_{x\to 0} x^{-\kappa}V_qf(x)$ is equal to zero for every function continuous f with bounded support on $]0,\infty[$. □

Proposition 3 proves that the hypotheses (H2) and $0 < \alpha\theta < 1$ imply the hypotheses (H1). In the next Proposition we establish a partial converse.

Proposition 4. *Assume that there exists a $\kappa > 0$ such that the hypothesis (H1) hold. Then*

(i) $0 < \alpha\kappa < 1$,

(ii) the hypotheses (H2-b) and (H2-c) are satisfied with $\theta = \kappa$.

Proof. To prove (i) we recall that under the hypotheses (H1) Theorem 2.1 in [34] states that the q–resolvent of the entrance law $(\mathbf{n}_s, s > 0)$ is characterized by the equation (3). Next, it is easily verified using the self–similarity of the minimal process (X, T_0), that for every $q > 0$, $c > 0$

$$\lim_{x\to 0} \frac{V_q f(x)}{\mathbb{E}_x(1 - e^{-T_0})} = c^{(1-\alpha\kappa)/\alpha} \lim_{x\to 0} \frac{V_{qc^{1/\alpha}} H_c f(x)}{\mathbb{E}_x(1 - e^{-T_0})}.$$

Then the excursion measure $\mathbf{n}$ is such that for every $c > 0$

$$\mathbf{n}(\int_0^{T_0} e^{-qs} f(X_s)ds) = c^{(1-\alpha\kappa)/\alpha}\, \mathbf{n}(\int_0^{T_0} e^{-qc^{1/\alpha}s} H_c f(X_s)ds).$$

The latter fact implies that (ii) in Lemma 2 is satisfied with $\gamma = \alpha\kappa$ and $0 < \alpha\kappa < 1$. Next we prove (ii). We first prove that under the hypothesis (H1) the process $(X_t^\kappa, t > 0)$ is a martingale for $\mathbb{P}_x$, which implies Cramér's condition (H2-b). Indeed, since the hypothesis (H1-a) holds we have that

$$\lim_{x\to 0} \frac{\mathbb{E}_x(1 - e^{-T_0})}{x^\kappa} = B \in]0, \infty[,$$

and, given that $0 < \alpha\kappa < 1$, the existence of this limit is equivalent to the existence of the limit

$$\lim_{s\to\infty} s^{\alpha\kappa} \mathbb{P}_x(T_0 > s) = x^\kappa B/\Gamma(1 - \alpha\kappa).$$

This fact suffices to prove that for every $x > 0$ and $t > 0$

$$\lim_{s\to\infty} \mathbb{P}_x(A|T_0 > s) = x^{-\kappa} \mathbb{P}_x(X_t^\kappa, A \cap \{t < T_0\}),$$

for any $A \in \mathcal{G}_t$. To see this just repeat the arguments in the proof of (ii) in Proposition 2. In particular, we have that for every $x > 0$ and $t > 0$, $x^\kappa = \mathbb{E}_x(X_t^\kappa, t < T_0)$. Using the Markov property we obtain that for every $x > 0$, under $\mathbb{P}_x$ the process X^κ is a martingale and as a consequence Cramér's condition follows. Moreover, the Lévy process ξ associated to X via Lamperti's transformation has a characteristic exponent Ψ which admits an analytic extension to the complex strip $\Im(z) \in [-\kappa, 0[$ defined by $\psi(z) = -\Psi(-iz)$, see the survey at the beginning of this subsection. Now to prove that the hypothesis (H2-c) is satisfied, we recall that under the hypotheses (H1) we have that

$$\lim_{s\to\infty} s^{\alpha\kappa}\, \mathbf{P}(I > s) = x^{-\kappa} \lim_{s\to\infty} s^{\alpha\kappa} \mathbb{P}_x(T_0 > s) = B/\Gamma(1 - \alpha\kappa),$$

and that $\mathbf{E}(I^{-(1-\alpha\kappa)}) < \infty$, the latter being a consequence of Lemma 3. Repeating the arguments in the calculation of the constant in the proof of Lemma 4 we obtain that

$$\mathbf{E}(I^{-(1-\alpha\kappa)}) = B\psi'(\theta-)/\Gamma(1 - \alpha\kappa) < \infty,$$

that is the exponent ψ of ξ has a left derivative at κ which is equivalent to

$$\mathbf{E}(\xi_1 e^{\kappa\xi_1}) < \infty.$$

Using the elementary relation

$$0 \le (\xi_1 \exp\{\kappa\xi_1\})^- = \xi_1^- \exp\{\kappa\xi_1\} = \xi_1^- \exp\{-\kappa\xi_1^-\} \le \kappa^{-1}$$

with $a^- = (-a) \vee 0$, we obtain that $0 \le \mathbf{E}((\xi_1 e^{\kappa\xi_1})^-) < 1/\kappa$. Therefore, $\mathbf{E}(\xi_1 e^{\kappa\xi_1}) < \infty$ if and only if $\mathbf{E}(\xi_1^+ e^{\kappa\xi_1}) < \infty$, which ends the proof. $\square$

Remark

1. If $0 < \alpha\theta < 1$ we have the following equality

$$\mathbf{E}(I^{-(1-\alpha\theta)}) = \mathbf{E}^\natural(J^{-(1-\alpha\theta)}).$$

This can be seen by making elementary calculations to obtain that

$$\int e^{-s}\,\mathbf{n}_s\,1 ds = \gamma_{\alpha,\theta} \int_0^\infty \mathbf{E}^\natural(e^{-y^{1/\alpha}J}) y^{(1-\alpha-\alpha\theta)} dy = \frac{\mathbf{E}^\natural(J^{-(1-\alpha\theta)})}{\mathbf{E}(I^{-(1-\alpha\theta)})},$$

and comparing this with the fact that $\int e^{-s}\,\mathbf{n}_s\,1 ds = 1$ gives the equality

2. A consequence of Lemma (4) is that

$$\mathbf{E}(I^{\beta\alpha}) < \infty \quad \text{for every} \quad 0 < \beta < \theta$$

and that $\mathbf{E}(I^{\alpha\theta}) = \infty$. Then under the hypotheses (H2) any extension which leaves 0 by jumps a.s. has a jumping–in measure $\eta(dx) = b_{\alpha,\beta} x^{-(1+\beta)} dx$, $x > 0$, with $0 < \beta < \theta \wedge 1/\alpha$ and $b_{\alpha,\beta}$ as defined in Proposition 1.

3 Existence of recurrent extensions that leaves 0 continuously

We next study the excursion measure such that the related extension leaves 0 continuously. To this end, we suppose throughout the rest of this section that the hypotheses (H2) holds.

Theorem 1. *There exists a pseudo excursion measure* $\mathbf{n}'$ *such that* $\mathbf{n}'(X_{0+} > 0) = 0$. *Its associated entrance law* $(\mathbf{n}'_s, s > 0)$ *is given by*

$$\mathbf{n}'_s f = \mathbb{E}^\natural{}_{0+}(f(X_s) X_s^{-\theta}), \qquad s > 0.$$

We have that $\mathbf{n}'$ *is an excursion measure if and only if* $0 < \alpha\theta < 1$. *Assume that this condition holds and let*

$$a_{\alpha,\theta} = \alpha\, \mathbf{E}^\natural(J^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)/m^\natural.$$

Then the measure $(a_{\alpha,\theta})^{-1}\,\mathbf{n}'$, *is the normalized excursion measure* $\mathbf{n}$.

Proof. We know from Proposition 2 that the function $h(x) = x^{-\theta}$ is excessive for the semigroup $P_t^{\natural}$ and that the corresponding h–transform is P_t. Let $\mathbf{n}'$ be the h–transform of $\mathbb{E}^{\natural}{}_{0+}$ by means of $h(x) = x^{-\theta}$. That is, $\mathbf{n}'$ is the unique measure in $\mathbb{D}^+$ that is carried by $\{T_0 > 0\}$, such that under $\mathbf{n}'$ the coordinate process is Markovian with semigroup P_t and for every $\mathcal{G}_t$–stopping time T and any $A_T \in \mathcal{G}_T$

$$\mathbf{n}'(A_T, T < T_0) = \mathbb{E}^{\natural}{}_{0+}(A_T, X_T^{-\theta}).$$

Therefore, $\mathbf{n}'$ is a pseudo excursion measure such that $\mathbf{n}'(X_{0+} > 0) = 0$ and the entrance law associated to $\mathbf{n}'$ is defined by

$$\mathbf{n}'_s f := \mathbf{n}'(f(X_s), s < T_0) = \mathbb{E}^{\natural}{}_{0+}(f(X_s)X_s^{-\theta}), \qquad s > 0, \tag{13}$$

for $f : \mathbb{R}^+ \to \mathbb{R}^+$ measurable. This proves the existence of a pseudo excursion measure such that $\mathbf{n}'(X_{0+} > 0) = 0$. To determine when $\mathbf{n}'$ is in fact an excursion measure we have to specify when $\mathbf{n}'(1 - e^{-T_0})$ is finite. Using standard arguments we obtain that

$$\begin{aligned}
\mathbf{n}'(1 - e^{-T_0}) &= \int_0^\infty ds e^{-s}\, \mathbf{n}'(T_0 > s) \\
&= \int_0^\infty ds e^{-s}\, \mathbb{E}^{\natural}{}_{0+}(X_s^{-\theta}) \\
&= \begin{cases} \alpha\, \mathbf{E}^{\natural}(J^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)/m^{\natural} & \text{if} \quad \alpha\theta < 1 \\ \infty & \text{if} \quad \alpha\theta \geq 1, \end{cases}
\end{aligned}$$

the third equality is obtained from (10). If $0 < \alpha\theta < 1$, then $\mathbf{E}^{\natural}(J^{-(1-\alpha\theta)}) < \infty$ since $\mathbf{E}^{\natural}(J^{-1}) < \infty$. As a consequence $\mathbf{n}'(1 - e^{-T_0}) < \infty$, if and only if $0 < \alpha\theta < 1$. If we assume that $0 < \alpha\theta < 1$, it follows that the measure $a^{-1}_{\alpha,\theta}\mathbf{n}'$ is a normalized excursion measure compatible with the semigroup P_t. Furthermore, it is straightforward to check that $a^{-1}_{\alpha,\theta}\mathbf{n}'$ satisfies the condition (ii) in Lemma 2 for $\gamma = \alpha\theta$. The normalized excursion measure $a^{-1}_{\alpha,\theta}\mathbf{n}'$ is equal to the measure $\mathbf{n}$ since this is the unique normalized excursion measure having the property $\mathbf{n}(X_{0+} > 0) = 0$. □

In the following theorem we give a simple criterion to determine, in terms of the Lévy process ξ, whether there exists a self–similar recurrent extension of (X, T_0) that leaves 0 continuously. Furthermore, with this result we give a complete solution to the problem posed by Lamperti since we have already established the existence of self–similar recurrent extensions of the minimal process that leave 0 by jumps.

Theorem 2. *(i) Assume $0 < \alpha\theta < 1$. The minimal process admits a unique self–similar recurrent extension $\widetilde{X} = (\widetilde{X}_t, t \geq 0)$ that leaves 0 continuously a.s. The resolvent of $\widetilde{X}$ is determined by*

$$U_q f(0) = \frac{\gamma_{\alpha,\theta}}{q^{\alpha\theta}} \int_0^\infty f(y)\, \mathbf{E}^{\natural}(e^{-qy^{1/\alpha}J}) y^{(1-\alpha-\alpha\theta)/\alpha} dy,$$

with $\gamma_{\alpha,\theta}$ as defined in Proposition 3 and

$$U_q f(x) = V_q f(x) + \mathbb{E}_x(e^{-qT_0}) U_q f(0), \qquad x > 0,$$

for $f \in C_b(\mathbb{R}^+)$. The resolvent U_q is Fellerian.

(ii) If $\alpha\theta \geq 1$, there does not exist a self-similar recurrent extension that leaves 0 continuously.

Proof. To obtain (i) we use the Lemma 1. This enables us to apply the results in Blumenthal [7] to ensure that associated to the excursion measure $\mathbf{n}$ described in Theorem 1 there exists a Markov process $\widetilde{X}$ having a Feller resolvent that is an extension of the minimal process. The self–similarity of $\widetilde{X}$ follows from Lemma 2. The only thing that needs a justification is the expression for the q–resolvent of the extension. Using the compensation formula for Poisson point processes we obtain that

$$U_q f(0) = \mathbf{n}\left(\int_0^{T_0} e^{-qs} f(X_s)ds\right) / \,\mathbf{n}(1 - e^{-qT_0}),$$

for every $f \in C_b(\mathbb{R}^+)$. From Lemma 2 we deduce that $\mathbf{n}(1 - e^{-qT_0}) = q^{\alpha\theta}$. The expression of $U_q f(0)$ is then obtained from Proposition 3. The proof of (ii) is a straightforward consequence of Lemma 5 below. □

The next lemma states that if $\alpha\theta \geq 1$, the only excursion measures compatible with (X, T_0) which satisfy (ii) in Lemma 2 are those associated to a jumping–in measure as in (ii) in Proposition 1.

Lemma 5. *Assume that $\alpha\theta \geq 1$. If there exists a normalized excursion measure $\mathbf{m}$ compatible with the minimal process such that conditions (ii) and (iii) in Lemma 2 are satisfied, then $\mathbf{m}(X_{0+} = 0) = 0$.*

Sketch of Proof. We recall from the proof of Proposition 3 that if $\alpha\theta \geq 1$ we have that

$$\liminf_{x\to 0} \frac{\mathbb{E}_x(1 - e^{-T_0})}{x^\theta} = \infty,$$

and that

$$\lim_{x\to 0} \frac{V_q f(x)}{x^\theta}, \quad q > 0,$$

exists in $\mathbb{R}$ for every function $f \in C_K]0, \infty[$. Therefore,

$$\lim_{x\to 0} \frac{V_q f(x)}{\mathbb{E}_x\left(1 - e^{-T_0}\right)} = 0,$$

for every function $f \in C_K]0, \infty[$. Now, we may simply repeat the arguments in the proof of Lemma 1.1 in [34] to prove that for $q > 0$

$$\mathbf{m}(\int_0^{T_0} e^{-qs} f(X_s)ds) = b\int_0^\infty V_q f(x) x^{-(1+\beta)}dx,$$

for some $\beta \in]0, 1/\alpha[$ and a constant $b \in]0, \infty[$. The result follows. □

Corollary 1. *Assume $0 < \alpha\theta < 1$.*

(i) The law of T_0 under $\mathbf{n}$ is

$$\mathbf{n}(T_0 \in ds) = \frac{\alpha\theta}{\Gamma(1 - \alpha\theta)}\, s^{-(1+\alpha\theta)}ds.$$

(ii) Under $\mathbf{n}$ the law of the height of the excursion, say $H := \sup_{0\leq t\leq T_0} X_s$, is given by

$$\mathbf{n}(H > z) = p_{\alpha,\theta} z^{-\theta}, \qquad z > 0.$$

with $p_{\alpha,\theta} = p\left(\alpha\theta\, \mathbf{E}^\natural(J^{-(1-\alpha\theta)})\Gamma(1 - \alpha\theta)\right)^{-1}$, and $p \in]0, 1]$ a constant that depends on the law of ξ.

Proof. The result in (i) follows from the fact that the subordinator σ which is the inverse local time of $\widetilde{X}$ is a stable subordinator of parameter $\alpha\theta$, cf. Lemma 2. The main ingredient in the proof of (ii) is that the tail distribution of the random variable $S_\infty = \sup_{r>0} \xi_r$ is such that

$$\lim_{s\to\infty} e^{\theta s}\, \mathbf{P}(S_\infty > s) = p/m^\natural \theta,$$

for a constant $p \in]0,1]$, cf. Bertoin and Doney [3] for a proof of this fact and an expression of the constant p. We deduce from this a tail estimate for the behavior of the supremum of the minimal process (X, T_0) as the initial point tends to 0. More precisely, defining $S^X_\infty := \sup_{0\leq r\leq T_0} X_r$, we have

$$\lim_{x\to 0} x^{-\theta}\mathbb{P}_x(S^X_\infty > z) = z^{-\theta}(p/m^\natural\theta), \quad z > 0.$$

Let $H_t = \sup_{t\leq s\leq T_0} X_s,\ t > 0$. Besides, we have that for any $z > 0$

$$\lim_{t\to 0+} \mathbf{n}(H_t > z, t < T_0) = \mathbf{n}(H > z),$$

and that for any $\epsilon, \delta > 0$, there exists a $t_0 > 0$ such that

$$\mathbf{n}(X_t \in (\epsilon, \infty), t < T_0) \leq \delta, \quad \forall t < t_0.$$

Therefore,

$$\mathbf{n}(X_t \in]0,\epsilon[, H_t > z, t < T_0) \leq \mathbf{n}(H_t > z, t < T_0) \leq \delta + \mathbf{n}(X_t \in]0,\epsilon[, H_t > z, t < T_0),$$

and by the Markov property under $\mathbf{n}$, we get that

$$\begin{aligned}\mathbf{n}(X_t \in]0,\epsilon[, H_t > z, t < T_0) &= (a_{\alpha,\theta})^{-1}\, \mathbb{E}^\natural{}_{0+}(X_t \in]0,\epsilon[, X_t^{-\theta}\, \mathbb{E}_{X_t}(S^X_\infty > z)) \\ &\sim p_{\alpha,\theta} z^{-\theta}\, \mathbb{E}^\natural{}_{0+}(X_t \in]0,\epsilon[) \\ &\sim p_{\alpha,\theta} z^{-\theta},\end{aligned}$$

for t small enough. Thus,

$$p_{\alpha,\theta} z^{-\theta} \leq \mathbf{n}(H > z) \leq \delta + p_{\alpha,\theta} z^{-\theta},$$

and the result follows by letting $\delta \to 0$. □

If $0 < \alpha\theta < 1$, it was shown by Vuolle-Apiala that given an excursion measure, the extension $\widetilde{X}$ associated to this excursion measure either leaves 0 continuously or by jumps. This fact is natural when we observe that the excursions that leave 0 continuously have different duration that those leaving 0 by jumps. Indeed, the duration of the former has distribution

$$\mathbf{n}(T_0 > t) = t^{-\alpha\theta}(\Gamma(1-\alpha\theta))^{-1},$$

and for the latter

$$n^j(T_0 > t) = t^{-\alpha\beta}(\Gamma(1-\alpha\beta))^{-1}, \qquad 0 < \beta < \theta.$$

In the case when the Lévy process ξ is a Brownian motion with a negative drift, the criterion in Theorem 2 coincides with the classification from Feller's diffusion theory for 0 to be a regular or an exit boundary point, as is explained in Example 2 below. By analogy, one can say that 0 is a regular boundary point for $\widetilde{X}$ if $0 < \alpha\theta < 1$ and an exit boundary point if $1 \leq \alpha\theta$. Even in the case $\alpha\theta < 0$, which is not considered in this chapter, it is easy to see that if $\theta < 0$ in Cramér's condition then the Lévy process ξ drifts to ∞. The only way to extend a self–similar Markov process X associated to a Lévy process that drifts to ∞ is by making 0 an entrance boundary point. This possibility is considered by Bertoin and Caballero [2], Bertoin and Yor [4, 5] and Caballero and Chaumont [9].

4 Excursions conditioned by their durations

It is well known that the excursion measure for the Brownian motion can be described using the law of the excursion process conditioned to return to 0 at time 1, i.e. the law of a Bessel(3) bridge of length 1, see e.g. McKean [23] or Revuz and Yor [27] §XII.4. In this section we follow this idea to describe the law under the excursion measure **n** defined in Theorem 1 of the excursion process conditioned to return to zero at a given time. We then give an alternative description of the excursion measure **n**. To that end, we will make the additional hypotheses

(H2-d) $\mathbf{E}(\xi_1) > -\infty$ and the distribution of the Lévy exponential functional I has a continuous density on $[0,\infty[$, say ρ, with respect to Lebesgue measure.

The condition that the law of the exponential functional I has a continuous density is satisfied by a wide variety of Lévy processes, cf. Carmona et al. [10] Proposition 2.1.

We next introduce another self–similar process. Denote by $\widehat{\xi} = (-\xi_s, s > 0)$ the dual Lévy process and by $\widehat{\mathbf{P}}$, and $\widehat{\mathbf{E}}$, its probability and expectation. Then define $(\widehat{\mathbb{P}}_x, x > 0)$ to be the distribution on $\mathbb{D}^+$ of the α–self–similar process associated to the Lévy process with law $\widehat{\mathbf{P}}$. The process $\widehat{X}$ is usually called the dual α–self–similar process; the term dual is justified by the relation

$$\int_0^\infty g(x)V_q f(x)x^{(1-\alpha)/\alpha}dx = \int_0^\infty f(x)\widehat{V}_q g(x)x^{(1-\alpha)/\alpha}dx, \tag{14}$$

for every $f, g :]0,\infty[\to \mathbb{R}^+$ measurable, see e.g. Lemma 2 in [4]. By hypothesis (H2-d) we have that $0 < m := |\psi'(0^+)| = \widehat{\mathbf{E}}(\xi_1) < \infty$. Let $\widehat{\mathbb{P}}_{0+}$ be the limit in the sense of finite dimensional marginals of $\widehat{\mathbb{P}}_x$ as $x \to 0$, whose existence is ensured by Theorem 1 in [4]. The latter theorem also establishes that for every $t > 0$ and for $f : \mathbb{R}^+ \to \mathbb{R}^+$, measurable we have

$$\widehat{\mathbb{E}}_{0+}(f(X_t)) = \frac{\alpha}{m}\,\mathbf{E}(f((t/I)^\alpha)/I), \tag{15}$$

where I is defined in (2). Hypothesis (H2-d) implies that for any $t > 0$ the law of X_t under $\widehat{\mathbb{P}}_{0+}$ has a density with respect to the measure $\upsilon(dy) = y^{(1-\alpha)/\alpha}dy, y > 0$, given by the formula

$$\frac{\widehat{\mathbb{P}}_{0+}(X_t \in dy)}{\upsilon(dy)} = m^{-1}y^{-1/\alpha}\rho(ty^{-1/\alpha}) := \widehat{p}_t(y), \qquad y > 0.$$

Let $(\mu_s(dy) = \widehat{\mathbb{P}}_{0+}(X_s \in dy), s > 0)$. A consequence of the duality relation (14) is that the relation $\mu_s \widehat{P}_{t-s} = \mu_t$ for $s < t$ can be shifted to the semigroup of the minimal process P_t as $\widehat{p}_t = P_s\widehat{p}_{t-s}$ υ–a.s. It was proved in Rivero [28] section 4, that these densities can be used to construct a regular version of the family of probability measures $(\mathbb{P}_x(\cdot|T_0 = r), r > 0)$ when the underlying Lévy process is a subordinator. Morever, the same argument applies to any Lévy process assuming only (H2-d). Here the densities $(\widehat{p}_t, t \geq 0)$ will be used to construct a bridge for the coordinate process under $\mathbb{E}^\natural{}_{0+}$; the techniques here used are reminiscent of those in Fitzsimmons et al. [15].

Recall that the semigroup $(P^\natural_t, t \geq 0)$ is the h–transformation of the semigroup $(P_t, t \geq 0)$ via the invariant function $h(x) = x^\theta, x > 0$. Using the fact that for every $t > s > 0$, the equality $\widehat{p}_t = P_s\widehat{p}_{t-s}$ υ–a.s. holds, we obtain that for $r > 0$ arbitrary, the function

$$h^{\natural r}(s,x) = \widehat{p}_{r-s}(x)x^{-\theta}1_{\{s<r\}}, \quad x > 0, s > 0,$$

is excessive for the semigroup $(\pi_t \otimes P^\natural_t, t \geq 0)$ of the space-time process. Let $\overline{\Lambda}^r$ be the h–transform of the measure $\mathbb{E}^\natural{}_{0+}$ by means of the space–time excessive function $h^{\natural r}(s,x)$. Then under $\overline{\Lambda}^r$ the space process $(X_t, t > 0)$ is an inhomogeneous Markov process with entrance law

$$\overline{\Lambda}^r_s f = \mathbb{E}^\natural{}_{0+}(f(X_s)\widehat{p}_{r-s}(X_s)X_s^{-\theta}), \quad 0 < s < r,$$

for $f : \mathbb{R}^+ \to \mathbb{R}^+$ measurable, and inhomogeneous semigroup

$$K^r_{t,t+s}(x,dy) = \frac{P^\natural_s(x,dy)h^{\natural r}(t+s,y)}{h^{\natural r}(t,x)} = \frac{P_s(x,dy)\widehat{p}_{r-(t+s)}(y)}{\widehat{p}_{r-t}(x)}, \quad y > 0; \quad t, t+s < r.$$

Observe that the inhomogeneous semigroup $K^r_{t,t+s}$ is that of X conditioned to die at 0 at time r, cf. [28] Lemma 7. Moreover, using the fact that $\overline{\Lambda}^r$ is a h–transform of the measure $\mathbb{E}^\natural{}_{0+}$ it is easily verified that the measure $\overline{\Lambda}^r$ has the property

$$\overline{\Lambda}^r(F(X_s, 0 \leq s < r)) = r^{-(1+\alpha\theta)}\overline{\Lambda}^1(F(r^\alpha X_s, 0 \leq s < 1)),$$

for every positive measurable F. In particular, the total mass of $\overline{\Lambda}^r$ is determined by

$$b_r := \overline{\Lambda}^r(1) = r^{-(1+\alpha\theta)}\overline{\Lambda}^1(1),$$

and it will be shown below that

$$\overline{\Lambda}^1(1) = \frac{\alpha^2\theta\, \mathbf{E}^\natural(J^{-(1-\alpha\theta)})}{m^\natural m} < \infty. \tag{16}$$

Therefore, assuming the hypotheses (H2-a,b,c,d) and $\overline{\Lambda}^1(1) < \infty$, we can define a probability measure on $\mathcal{G}_\infty$ by $\Lambda^r = b_r^{-1}\overline{\Lambda}^r$. The distribution under Λ^r of the lifetime T_0 is the Dirac distribution at r i.e. $\Lambda^r(T_0 = r) = 1$, cf. [28] Lemma 7. We can now state the main result of this section.

Proposition 5 (Itô's description of the measure n). *Assume hypotheses (H2-a,b,c,d) holds and $0 < \alpha\theta < 1$. Then $\overline{\Lambda}^1(1) < \infty$. Let $\mathbf{n}$ be the unique normalized excursion measure such that $\mathbf{n}(X_{0+} > 0) = 0$. For $F \in \mathcal{G}_\infty$,*

$$\mathbf{n}(F) = \frac{\alpha\theta}{\Gamma(1-\alpha\theta)}\int_0^\infty \Lambda^r(F \cap \{T_0 = r\})\frac{dr}{r^{1+\alpha\theta}}.$$

The proof of this proposition is similar to that given in [27] Theorem XII.4.2 for the analogous result for Brownian excursion measure.

Proof. We first show that

$$\mathbf{n}(F) = \frac{m}{a_{\alpha,\theta}}\int_0^\infty \overline{\Lambda}^r(F \cap \{T_0 = r\})dr, \tag{17}$$

with $a_{\alpha,\theta}$ as defined in Theorem 1. We deduce from this that

$$\overline{\Lambda}^1(1) = \frac{\alpha^2\theta\, \mathbf{E}^\natural(J^{-(1-\alpha\theta)})}{m^\natural m}.$$

Indeed, by the monotone class theorem it is enough to prove the assertion for sets F of the form

$$F = \bigcap_1^n \{X(t_i) \in B_i\},$$

with $0 < t_1 < t_2 < \cdots < t_n$ and Borel sets $B_i \subset]0,\infty[, i \in \{1,\ldots,n\}$. On the one hand, according to Theorem 1 we have

$$\mathbf{n}(F) = \int_{B_1} \mathbf{n}_{t_1}(dx_1) \int_{B_2} P_{t_2-t_1}(x_1, dx_2) \cdots \int_{B_n} P_{t_n-t_{n-1}}(x_{n-1}, dx_n),$$

On the other hand, using that $F \cap \{T_0 < t_n\} = \emptyset$ we have that the right hand term in (17) can be written as

$$\frac{m}{a_{\alpha,\theta}} \int_{t_n}^{\infty} dr \int_{B_1} \overline{\Lambda}^r_{t_1}(dx_1) \int_{B_2} K_{t_1,t_2}(x_1, dx_2) \cdots \int_{B_n} K_{t_{n-1},t_n}(x_{n-1}, dx_n). \tag{18}$$

Recall from Theorem 1 that

$$\overline{\Lambda}^r_{t_1}(dx_1) = \mathbb{P}^{\natural}{}_{0^+}(X_{t_1} \in dx_1)\widehat{p}_{r-t_1}(x_1)x_1^{-\theta} = a_{\alpha,\theta}\, \mathbf{n}_{t_1}(dx_1)\widehat{p}_{r-t_1}(x_1).$$

Using this identity and the expression of the transition probabilities $K_{t_i,t_{i+1}}$ we get that (18) is equal to

$$m \int_{t_n}^{\infty} dr \int_{B_1} \mathbf{n}_{t_1}(dx_1) \int_{B_2} P_{t_2-t_1}(x_1, dx_2) \cdots \int_{B_n} P_{t_n-t_{n-1}}(x_{n-1}, dx_n)\widehat{p}_{r-t_n}(x_n).$$

Finally, using

$$m \int_s^{\infty} \widehat{p}_{r-s}(x)dr = \int_s^{\infty} \rho((r-s)x^{-1/\alpha})\frac{dr}{x^{1/\alpha}} = 1,$$

for all $x > 0$, we conclude that both expressions in (17) for $\mathbf{n}(F)$ coincide. In particular, if $F = 1 - e^{-T_0}$ we have that

$$1 = \mathbf{n}(1 - e^{-T_0}) = \frac{m}{a_{\alpha,\theta}} \int_0^{\infty} \overline{\Lambda}^r(1)(1 - e^{-r})dr = \frac{\overline{\Lambda}^1(1)m}{a_{\alpha,\theta}} \left(\frac{\Gamma(1-\alpha\theta)}{\alpha\theta}\right).$$

The value of $\overline{\Lambda}^1(1)$ in (16) is obtained by using the expression for $a_{\alpha,\theta}$ and we derive from (17) that

$$\mathbf{n}(F) = \frac{m\overline{\Lambda}^1(1)}{a_{\alpha,\theta}} \int_0^{\infty} \Lambda^r(F \cap \{T_0 = r\})\frac{dr}{r^{1+\alpha\theta}},$$

and the result follows. □

Remark A result equivalent to that in Proposition 5 can be obtained for the excursion measure n^j obtained via the jumping-in measure $\eta(dx) = b_{\alpha,\beta}x^{-(1+\beta)}dx$. The method is similar and we leave the details to the interested reader.

5 Duality

In this section we will construct a self–similar Markov process which is in weak duality with the process $\widetilde{X}$ and whose excursion measure is the image under time reversal of $\mathbf{n}$. This will be given under the hypotheses (H2) and

(H2-e) $\mathbf{E}(\xi_1^-) < \infty$, with $a^- = (-a) \vee 0$.

Next we introduce some notation. Let $\xi^{\natural}$ be a Lévy process with law $\mathbf{P}^{\natural}$ and $\widehat{\xi^{\natural}}$ its dual, i.e. $\widehat{\xi^{\natural}} = -\xi^{\natural}$. Denote by $\widehat{\mathbf{P}^{\natural}}$ and $\widehat{\mathbf{E}^{\natural}}$ the probability and expectation for $\widehat{\xi^{\natural}}$. The process $\widehat{\xi^{\natural}}$ drifts to $-\infty$ and satisfies the hypotheses (H2-a,b,c). Indeed, that (H2-b) holds follows from

$$\widehat{\mathbf{E}^{\natural}}(e^{\theta\xi_1}) = \mathbf{E}^{\natural}(e^{-\theta\xi_1}) = \mathbf{E}(e^{-\theta\xi_1}e^{\theta\xi_1}) = 1,$$

in the same way is verified that (H2-c) holds,

$$\widehat{\mathbf{E}^{\natural}}(\xi_1^+e^{\theta\xi_1}) = \mathbf{E}^{\natural}((-\xi_1)^+e^{-\theta\xi_1}) = \mathbf{E}(\xi_1^-) < \infty.$$

Let $(\widehat{\mathbb{P}^{\natural}}_x, x \geq 0)$ be the law on $\mathbb{D}^+$ of the α–self–similar process $\widehat{X}^{\natural} = (\widehat{X}^{\natural}_t, t \geq 0)$ associated by Lamperti's transformation to the Lévy process with law $\widehat{\mathbf{P}^{\natural}}$. The process $\widehat{X}^{\natural}$ has a lifetime $\widehat{T}_0 = \inf\{t > 0 : \widehat{X}^{\natural}_t = 0\}$ which is finite $\widehat{\mathbb{P}^{\natural}}_x$–a.s. for all $x \geq 0$. Denote by $(\widehat{P}^{\natural}_t, t \geq 0)$ and $(\widehat{V}^{\natural}_q, q > 0)$ the semigroup and resolvent of the minimal process for $\widehat{X}^{\natural}$, i.e.

$$\widehat{P}^{\natural}_t f(x) = \widehat{\mathbb{P}^{\natural}}_x(f(X_t), t < T_0), \quad t \geq 0,$$

and

$$\widehat{V}^{\natural}_q f(x) = \int e^{-qt}\widehat{P}^{\natural}_t f(x)dt, \quad q > 0.$$

By the duality relation (14), the resolvents $V^{\natural}_q$ and $\widehat{V}^{\natural}_q$ are in weak duality with respect to the measure $\upsilon(dx) = x^{(1-\alpha)/\alpha}dx,\ x > 0$. Furthermore, it follows that the resolvents V_q and $\widehat{V}^{\natural}_q$, are in weak duality with respect to the measure $\zeta(dx) = x^{(1-\alpha-\alpha\theta)/\alpha}dx,\ x > 0$.

We assume henceforth that $0 < \alpha\theta < 1$. The results in section 3 can be applied to the minimal process $(\widehat{X}^{\natural}, \widehat{T}_0)$ to ensure that there exists a unique normalized excursion measure $\widehat{\mathbf{n}}$, compatible with the semigroup $(\widehat{P}^{\natural}_t, t \geq 0)$ and its associated entrance law admits the representation

$$\widehat{\mathbf{n}}_s f = (\widehat{a}_{\alpha,\theta})^{-1}\widehat{\mathbb{E}}_{0+}(f(X_s)X_s^{-\theta}), \qquad s > 0,$$

where $\widehat{a}_{\alpha,\theta} = \alpha\,\mathbf{E}(I^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)/m$, for f continuous and bounded. To see this it should be verified that the measure $\widehat{\mathbf{P}^{\natural}}^{\natural}$, obtained by h–transformation of the law $\widehat{\mathbf{P}^{\natural}}$ by means of the function $h(x) = e^{\theta x}$ is $\widehat{P}$. To that end, it suffices to prove that both probability measures have the same 1–dimensional marginals. Indeed,

$$\widehat{\mathbf{P}^{\natural}}^{\natural}(f(\xi_s)) = \widehat{\mathbf{P}^{\natural}}(f(\xi_s)e^{\theta\xi_s}) = \mathbf{P}^{\natural}(f(-\xi_s)e^{-\theta\xi_s}) = \mathbf{P}(f(-\xi_s)) = \widehat{\mathbf{P}}(f(\xi_s)),$$

for every f continuous and bounded. Then the α–self–similar Markov process associated to the Lévy process with law $\widehat{\mathbf{P}^{\natural}}^{\natural}$ is equivalent to that associated to the Lévy process with law $\widehat{\mathbf{P}}$. Remark that the law of J under $\widehat{\mathbf{P}^{\natural}}^{\natural}$ is the same as that of I under $\mathbf{P}$.

Then the minimal process $(\widehat{X}^{\natural}, \widehat{T}_0)$ admits a unique extension $(\widetilde{Z}_t, t \geq 0)$, that leaves 0 continuously a.s. Let $(\widehat{U}_q, q > 0)$ denote the resolvent of the process $\widetilde{Z}$. Because of the weak duality relation between the resolvents V_q, and $\widehat{V}^{\natural}_q$ it is natural to ask if this property is inherited by the resolvents U_q and $\widehat{U}_q$. That is the content of the following result.

Lemma 6. *The resolvents $(U_q, q > 0)$ and $\widehat{U}_q$ are in weak duality with respect to the measure $\zeta(dx) = x^{(1-\alpha-\alpha\theta)/\alpha}dx, x > 0$.*

Proof. From Proposition 3 we have that the resolvent at 0 of $\widetilde{Z}$ is determined by the expression

$$\widehat{U}_q f(0) = \frac{\widehat{\gamma}_{\alpha,\theta}}{q^{\alpha\theta}} \int_0^\infty f(y)\, \mathbf{E}(e^{-qy^{1/\alpha} I}) y^{(1-\alpha-\alpha\theta)/\alpha} dy,$$

with $\widehat{\gamma}_{\alpha,\theta} = (\widehat{a}_{\alpha,\theta} m)^{-1}$. Recall that the resolvent at 0 of $\widetilde{X}$ is given by

$$U_q f(0) = \frac{\gamma_{\alpha,\theta}}{q^{\alpha\theta}} \int_0^\infty f(y)\, \mathbf{E}^\natural(e^{-qy^{1/\alpha} J}) y^{(1-\alpha-\alpha\theta)/\alpha} dy.$$

On the other hand, for any $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ we have

$$\begin{aligned}
\int_0^\infty \zeta(dy) g(y) U_q f(y) &= \int_0^\infty \zeta(dy) g(y) V_q f(y) + U_q f(0) \int_0^\infty \zeta(dy) g(y)\, \mathbb{E}_y(e^{-qT_0}) \\
&= \int_0^\infty \zeta(dy) f(y) \widehat{V}_q^\natural g(y) + U_q f(0) \int_0^\infty \zeta(dy) g(y)\, \mathbf{E}(e^{-qy^{1/\alpha} I}) \\
&= \int_0^\infty \zeta(dy) f(y) \widehat{V}_q^\natural g(y) \\
&\quad + \frac{\widehat{a}_{\alpha,\theta} m}{a_{\alpha,\theta} m^\natural} \widehat{U}_q g(0) \int_0^\infty \zeta(dx) f(x)\, \mathbf{E}^\natural(e^{-qx^{1/\alpha} J}) \\
&= \int_0^\infty \zeta(dy) f(y) \widehat{U}_q g(y),
\end{aligned}$$

where the last equality follows from the fact that the constants $\gamma_{\alpha,\theta}$ and $\widehat{\gamma}_{\alpha,\theta}$ are equal. To see this recall that $\mathbf{E}(I^{-(1-\alpha\theta)}) = \mathbf{E}^\natural(J^{-(1-\alpha\theta)})$, as remarked after Proposition 3. □

Some results on time reversal can be derived from the preceding facts. To give a precise statement we introduce some notation. Let ϱ denote the operator of time reversal at time T_0, that is

$$(\varrho X(\omega))(t) = \begin{cases} X_{(T_0 - t)^-}(\omega) & \text{if} \quad 0 \le t < T_0 < \infty \\ 0 & \text{otherwise} \end{cases}$$

and let $\varrho\,\mathbf{n}$ denote the image under time reversal at time T_0 of $\mathbf{n}$. Recall that L is a return time if

$$L \circ \theta_t = (L - t)^+, \quad \text{a.s. for all} \quad t \ge 0.$$

The first part of the following result is an extension for self–similar process of the celebrated result on time reversal of Williams [36]: a three dimensional Bessel process starting from 0 and reversed at its last exit time from $x > 0$, is identical in law to a Brownian motion killed at its first hitting time of 0. In the second part we determine $\varrho\,\mathbf{n}$.

Proposition 6. *(i) If L is a finite return time then under* $\mathbb{E}^\natural{}_{0+}$ *the reversed process* $(X_{(L-t)-}, 0 \le t < L)$ *is Markovian and has semigroup* $(\widehat{P}_t^\natural, t \ge 0)$.

(ii) We have that $\varrho\,\mathbf{n} = \widehat{\mathbf{n}}$.

Proof. (i) The potential of the measure $\mathbb{E}^\natural{}_{0+}$ is determined by

$$\begin{aligned}
\mathbb{E}^\natural{}_{0+}\Big(\int_0^\infty ds f(X_s)\Big) &= a_{\alpha,\theta} \int_0^\infty ds\, \mathbf{n}_s(fh^*) \\
&= a_{\alpha,\theta} \int f(y) y^{(1-\alpha)/\alpha} dy,
\end{aligned}$$

with the notation of Sections 2.3 and 3. Because of the weak duality between the resolvents $V^{\natural}_{\lambda}$ and $\widehat{V}^{\natural}_{\lambda}$ with respect to the measure $y^{(1-\alpha)/\alpha}dy, y>0$, the statement in (i) is a direct consequence of a result of Nagasawa on time reversal. A general version of Nagasawa's result can be found in Dellacherie et al. [12]§ XVIII.46.
(ii) Assuming that the excursion of $\widetilde{X}$ from 0 starts and ends at 0 and using the weak duality in Lemma 6 it follows from a result due to Mitro [26] § 4 that $\varrho\,\mathbf{n}=\widehat{\mathbf{n}}$. To see that under our hypotheses the excursions of the self–similar process $\widetilde{X}$ from 0 starts and ends at 0, it should be verified that $\widetilde{X}_{g_t}=0$ and $\widetilde{X}_{D^-_{g_t}}=0$ for all t a.s. with $g_t=\sup\{s\leq t:\widetilde{X}_s=0\}$ and $D_t=\inf\{t\leq s:\widetilde{X}_s=0\}$, see e.g. Getoor and Sharpe [18] § 9. In fact, since we already know that $\mathbf{n}(X_{0+}>0)=0$, it suffices to verify that $\mathbf{n}(X_{T_0-}>0)=0$. The latter is a straightforward consequence of the Markov property and that $\mathbb{P}_x(X_{T_0-}>0)=0$ for all $x>0$, since X is a self–similar Markov process associated to a Lévy process that drifts to $-\infty$, see e.g. [22] Theorem 4.1. □

6 Examples

Example 2 (Self–similar diffusions). Here we consider the case when the Lévy process is a Brownian motion with negative drift. Let $(\xi_t=\varepsilon B_t-\mu t, t\geq 0)$ with $(B_t,t\geq 0)$ a Brownian motion and $\varepsilon,\mu>0$. The hypotheses (H2) are satisfied with $\theta=2\mu/\varepsilon^2$ and under $\mathbf{P}^{\natural}$ the law of $\xi^{\natural}$ is that of $\varepsilon B_t+\mu t$. Then the α–self–similar Markov process X associated to ξ has continuous paths and has an infinitesimal generator of the form

$$Lf(x)=(\varepsilon^2/2-\mu)x^{1-1/\alpha}f'(x)+\varepsilon^2/2x^{2-1/\alpha}f''(x),\quad x>0.$$

Then for $\alpha>0$ we have that $0<\alpha\theta<1$ if and only if $0<\mu<\varepsilon^2/2\alpha$. This corresponds to the case when the point 0 is a regular boundary point for the self–similar diffusion associated to the infinitesimal generator L just described; in the case $1\leq\alpha\theta$, or equivalently $\varepsilon^2/2\alpha\leq\mu$, 0 is an exit boundary point, see e.g. Lamperti [22] Theorem 5.1 and Vuolle-Apiala [34] Theorem 3.1 for a related discussion. If $0<\mu<\varepsilon^2/2\alpha$ holds, the process X admits a unique extension that is continuous and is characterized by Theorem 2. Furthermore, using the fact that the law of J under $\mathbf{E}^{\natural}$ is that of $2\alpha^2/(\varepsilon^2 Z_{\alpha\theta})$, with $Z_{\alpha\theta}$ a random variable of law gamma of parameter $\alpha\theta$, (see e.g. Dufresne [13]), we deduce that the entrance law in Theorem 1 has a density w.r.t. Lebesgue measure

$$\frac{\mathbf{n}_s(dy)}{dy}=c_{\alpha\theta}s^{-2(1-\alpha\theta)-1}y^{2(1-\alpha\theta)/\alpha-1}\exp(-y^{1/\alpha}s^{-1}d_{\varepsilon,\alpha})\qquad y>0,$$

with

$$c_{\alpha\theta}=\frac{(1-\alpha\theta)\alpha}{\Gamma(1-\alpha\theta)\mu^2}\left(\frac{\varepsilon^2}{2\alpha^2}\right)^{\alpha\theta}\quad\text{and}\quad d_{\varepsilon,\alpha}=\frac{2\alpha^2}{\varepsilon^2}.$$

Example 3 (Reflected stable processes). Let Y be a stable process of parameter $a\in]0,2[$ and $(\underline{\mathbb{P}}_x,x\geq 0)$ its law. Assume that Y has no negative jumps and $|Y|$ is not a subordinator. Define $\rho=\underline{\mathbb{P}}(Y_1>0)$ and

$$X'_t=\begin{cases}Y_t-\inf_{0\leq s\leq t}Y_s & \text{if } t\geq T_{]-\infty,0]}\\ Y_t & \text{if } t<T_{]-\infty,0]}\end{cases}$$

with $T_{]-\infty,0]}$ the first hitting time of $]-\infty,0]$ by Y. Then $\rho\in]0,1[$ and 0 is a regular recurrent state for X'. (We refer to Bertoin [1] § VIII and Chaumont [11] for background on stable processes and its excursion theory.) We denote by (X, T_0) the process X' killed at $T_{]-\infty,0]}$; this process is $1/a$–self–similar. The hypotheses on Y imply that

$$\underline{\mathbb{P}}_x(T_0 < \infty, X_{T_0-} = 0) = 1, \quad x > 0.$$

Let ξ be the Lévy process associated to (X, T_0) via Lamperti's transformation (see Caballero and Chaumont [9] for a precise description of ξ). We claim that the hypothesis (H2) are satisfied for $\theta = a(1-\rho)$. This can be viewed either by barehand calculations using the results in [9] or by the following arguments.

It is known that the function $h(x) = x^{a(1-\rho)}, x > 0$ is, up to a multiplicative constant, the only invariant function for the semigroup of the process (X, T_0). Then Cramér's condition (H2-b) for ξ, is satisfied with $\theta = a(1-\rho)$. A consequence of this fact and Proposition 3.1 in[24] is that the Lévy exponential functional $I = \int_0^\infty \exp\{a\xi_s\}ds$, has finite moments

$$\mathbf{E}(I^{\beta/a}) < \infty \quad \text{for every} \quad 0 < \beta < a(1-\rho).$$

The excursion measure for X' away from 0, say $\underline{\mathbf{n}}$, is an excursion measure compatible with the minimal process (X, T_0) such that its entrance law satisfies (iii) in Lemma 2 with $\gamma = 1-\rho$, and $\underline{\mathbf{n}}(X_{0+} > 0) = 0$ (see [11] and the reference therein). Thus $\mathbf{E}(I^{-\rho}) < \infty$, by Lemma 3. Therefore, it is easily verified by repeating the arguments in the proof of Proposition 4 that the condition (H2-c) is satisfied.

Finally, the excursion measure $\mathbf{n}$ defined in Theorem 1 is equal to $\underline{\mathbf{n}}$ and the recurrent extension in Theorem 2 associated to $\mathbf{n}$ is equivalent to X'.

Example 4. Let ξ be a non–arithmetic Lévy process with no positive jumps such that ξ derives to $-\infty$. We assume that ξ is neither the negative of a subordinator nor a deterministic drift. The case of the negative of a subordinator was discussed in example 1 and the case of a deterministic drift can be treated in the same way. From the theory of Lévy processes with no positive jumps we know that $\mathbf{E}(e^{\lambda\xi_1}) < \infty$, for all $\lambda > 0$. Then the convex function $\psi(\lambda) : \mathbb{R}^+ \to \mathbb{R}$, defined by $\mathbf{E}(e^{\lambda\xi_1}) = e^{\psi(\lambda)}$, is such that $\psi(0) = 0$, and $\lim_{\lambda\to\infty}\psi(\lambda) = \infty$. Since ξ drifts to $-\infty$ there exists a unique $\theta > 0$, such that $\psi(\theta) = 0$. It follows that ξ satisfies the conditions (H2). Let $0 < \alpha < 1/\theta$, and let (X, T_0) be the α–self–similar minimal process associated to ξ. Owing to the absence of positive jumps, we have that $X_{T_{[z,\infty[}} = z$ whenever $T_{[z,\infty[} < T_0$, with $T_{[z,\infty[} = \inf\{t > 0 : X_t \geq z\}$. The excursion measure $\mathbf{n}$ compatible with the process (X, T_0) defined in Theorem 1 has the property:

> Under the probability measure on $\mathbb{D}^+$, $\mathbf{n}\,|(T_{[z,\infty[} < T_0)$, the processes $(X_t, t \leq T_{[z,\infty[})$ and $(X_{T_z+t}, t \leq T_0 - T_{[z,\infty[})$, are independent. The law of the former is $\mathbb{E}^\natural{}_{0+}$ killed at $T_{[z,\infty[}$ and of the latter is that of (X, T_0) started at z.

Here $\mathbf{n}\,|(T_{[z,\infty[} < T_0)$ means $\mathbf{n}(A \cap \{T_{[z,\infty[} < T_0\})/\,\mathbf{n}(\{T_{[z,\infty[} < T_0\})$ for $A \in \mathcal{G}_\infty$. This claim is easily verified using the fact that the measure $\mathbf{n}$ is a multiple of the h–transform of $\mathbb{E}^\natural{}_{0+}$ via the excessive function $h^*(x) = x^{-\theta}, x > 0$. Moreover, the law of the Lévy exponential functional $I = \int_0^\infty \exp\{\xi_s/\alpha\}ds$, associated to ξ is self–decomposable and as a consequence the law of I has a continuous density, cf. [28] Proposition 4. Therefore, to apply the results in Sections 4 & 5, the only hypothesis that should be made on ξ is that $\mathbb{E}(\xi_1) > -\infty$.

A On dual extensions

This section is motivated by Section 5, where we proved that given two minimal process X and $\widehat{X}$ which are self–similar and that are in weak duality, there exist Markov processes $\widetilde{X}$ and $\widetilde{Z}$ extending

(X, T_0) and $(\widehat{X}, \widehat{T}_0)$ respectively, which still are in weak duality. The purpose of this section is to give a generalization of this fact under the hypotheses of Blumenthal. The result given here is of independent interest and to make the section self–contained, we next introduce some notation. Let $(Y_t, t \geq 0)$ and $(\widehat{Y}_t, t \geq 0)$ be Markov processes having 0 as a trap. Denote by P, E, (resp. $\widehat{\mathrm{P}}, \widehat{\mathrm{E}}$) the probabilities and expectation for Y, (resp. $\widehat{Y}$) and by T_0 (resp. $\widehat{T}_0$) the first hitting time of 0 for Y (resp. for $\widehat{Y}$), i.e. $T_0 = \inf\{t > 0 : Y_t = 0\}$. Assume $\mathrm{P}_x(T_0 < \infty) = \widehat{\mathrm{P}}_x(T_0 < \infty) = 1$ for any $x > 0$. Let Q_t^0, and W_λ^0, (resp. $\widehat{Q}_t^0$, $\widehat{W}_\lambda^0$) denote the semigroup and λ–resolvent for Y killed at 0, (resp. $\widehat{Y}$). For $\lambda > 0$, define the functions $\varphi_\lambda, \widehat{\varphi}_\lambda : \mathbb{R}^+ \to [0, 1]$, by

$$\varphi_\lambda(x) = \mathrm{E}_x(e^{-\lambda T_0}); \qquad \widehat{\varphi}_\lambda = \widehat{\mathrm{E}}_x(e^{-\lambda T_0}), \qquad x > 0.$$

The main assumptions of this section are

(H3-a) $Y, \widehat{Y}$, both satisfy the basic hypotheses in [7];

(H3-b) the resolvents W_λ^0 and $\widehat{W}_\lambda^0$ are in weak duality with respect to a σ–finite measure $\zeta(dx)$ on $]0, \infty[$;

(H3-c) We have

$$\int_{]0,\infty[} \zeta(dx)\varphi_\lambda(x) < \infty; \quad \int_{]0,\infty[} \zeta(dx)\widehat{\varphi}_\lambda(x) < \infty, \quad \text{for all} \quad \lambda > 0.$$

Theorem 3. *Assume hypotheses (H3). Then there exist excursion measures m and $\widehat{m}$ compatible with the semigroups $(Q_t^0, t \geq 0)$ and $(\widehat{Q}_t^0, t \geq 0)$ respectively. The Laplace transforms of the entrance laws $(m_s, s > 0)$ and $(\widehat{m}_s, s > 0)$ associated to m and $\widehat{m}$ respectively, are determined by*

$$\int_0^\infty e^{-\lambda s} m_s f ds = \int_{]0,\infty[} \zeta(dx) f(x)\widehat{\varphi}_\lambda(x); \quad \int_0^\infty e^{-\lambda s} \widehat{m}_s f ds = \int_{]0,\infty[} \zeta(dx) f(x)\varphi_\lambda(x),$$

for $\lambda > 0$, and f continuous and bounded. Furthermore, associated to these excursion measures there exist Markov processes Y^* and $\widehat{Y}^*$ which are extensions for Y and $\widehat{Y}$ respectively and which are still in weak duality with respect to the measure $\zeta(dx)$.

The proof of this theorem will be given via three lemmas. The first of them ensures the existence of the excursion measures.

Lemma 7. *The family of finite measures $M_\lambda f = \int_{]0,\infty[} \zeta(dx) f(x)\widehat{\varphi}_\lambda(x)$, $\lambda > 0$, is such that*

(i) $\lim_{\lambda \to \infty} M_\lambda 1 = 0$

(ii) For $\mu, \lambda > 0$, $\mu \neq \lambda$

$$(\mu - \lambda) M_\lambda W_\mu^0 f = M_\lambda f - M_\mu f,$$

for f continuous and bounded.

Proof. That $M_\lambda \longrightarrow 0$, as $\lambda \to \infty$, follows from the monotone convergence theorem. Using the weak duality for the resolvents W_λ^0 and $\widehat{W}_\lambda^0$, we get

$$\begin{aligned} M_\lambda W_\mu^0 f &= \int_{]0,\infty[} \zeta(dx) W_\mu^0 f(x) \widehat{\varphi}_\lambda(x) \\ &= \int_{]0,\infty[} \zeta(dx) f(x) \widehat{W}_\mu^0 \widehat{\varphi}_\lambda(x). \end{aligned}$$

The result is then obtained from the elementary identity

$$\widehat{W}^0_\mu\widehat{\varphi}_\lambda(x) = \frac{\widehat{\mathrm{E}}_x(e^{-\lambda T_0} - e^{-\mu T_0})}{\mu - \lambda}.$$

□

From Lemma 7 and Theorem 6.9 of Getoor and Sharpe [17], there exists a unique entrance law $(m_t, t > 0)$, for the semigroup $(Q_t, t \geq 0)$, such that for each $\lambda > 0$

$$M_\lambda f = \int_0^\infty e^{-\lambda t} m_t f dt,$$

for f measurable and bounded, and

$$\int_0^1 m_t 1 dt < \infty.$$

According to Blumenthal [7], for an entrance law $(m_s, s > 0)$ there exists a unique excursion measure m, such that its entrance law is $(m_s, s > 0)$. The same method ensures the existence of an excursion measure $\widehat{m}$ and an entrance law $(\widehat{m}_t, t > 0)$, for the semigroup $(\widehat{Q}_t, t \geq 0)$.

Using the results in [7] we obtain that associated to the excursion measure m (resp. to $\widehat{m}$) there exists a unique Markov process Y^* extending Y (resp. $\widehat{Y}^*$ extends $\widehat{Y}$) and the λ–resolvent of Y^* is determined by

$$W_\lambda f(0) = \frac{M_\lambda f}{\lambda M_\lambda 1}; \quad W_\lambda f(x) = W^0_\lambda f(x) + \varphi_\lambda(x) W_\lambda f(0), \quad x > 0,$$

for f measurable and bounded; the λ–resolvent for $\widehat{Y}^*$, say $\widehat{W}_\lambda$, is defined in a similar way. To establish weak duality with respect to the σ–finite measure $\zeta(dx)$ for the resolvents W_λ and $\widehat{W}_\lambda$ we will need the following technical result.

Lemma 8. *For every $\lambda > 0$, we have that $\lambda M_\lambda 1 = \lambda \widehat{M}_\lambda 1$.*

Proof. This result is a consequence of the following identity, for $\lambda, \mu > 0$

$$\lambda M_\lambda 1 - \mu M_\mu 1 = \lambda \widehat{M}_\lambda 1 - \mu \widehat{M}_\mu 1;$$

and the fact that

$$\lim_{\mu\to\infty} \mu M_\mu 1 = 0,$$

since $m(1 - e^{-\mu T_0}) = \mu M_\mu 1$, with m the excursion measure associated to the entrance law $(m_s, s > 0)$. Thus, to end the proof we just have to prove the former identity. Indeed, this follows from the fact that

$$M_\lambda \varphi_\mu = \int_{]0,\infty[} \zeta(dx)\widehat{\varphi}_\lambda(x)\varphi_\mu(x) = \widehat{M}_\mu \widehat{\varphi}_\lambda,$$

and the following elementary identities: for $\lambda, \mu > 0$

$$(\lambda - \mu) M_\lambda \varphi_\mu = \lambda M_\lambda 1 - \mu M_\mu 1, \quad \text{and} \quad (\lambda - \mu)\widehat{M}_\lambda \widehat{\varphi}_\mu = \lambda \widehat{M}_\lambda 1 - \mu \widehat{M}_\mu 1.$$

□

Finally, the following lemma establishes weak duality for the resolvents W_λ and $\widehat{W}_\lambda$.

Lemma 9. *For every $\lambda > 0$ and every measurable functions $f, g : [0,\infty[\to \mathbb{R}^+$, we have*

$$\int_{]0,\infty[} \zeta(dy) g(y) W_\lambda f(y) = \int_{]0,\infty[} \zeta(dy) f(y) \widehat{W}_\lambda g(y).$$

The proof of this lemma is a straightforward consequence of Lemma 8 and the construction of W_λ and $\widehat{W}_\lambda$; see the proof of Lemma 6.

Remarks

1. Observe that
$$\lim_{\lambda \to 0} \int_0^\infty ds e^{-\lambda s} m_s f = \int_0^\infty ds m_s f = \int_{]0,\infty[} \zeta(dy) f(y).$$
By the weak duality relation in Lemma 9 we have that $\zeta(dy)$ is invariant for the semigroup of Y^* and, since 0 is a recurrent state for Y^*, $\zeta(dy)$ is in fact the unique (up to a multiplicative constant) excessive measure for this semigroup, see e.g. Dellacherie et al. [12] XIX.46.

2. We have not considered here the possibility of a *stickiness* parameter in the construction of the processes Y^* and $\widehat{Y}^*$; that is constructing Y^* and $\widehat{Y}^*$ via the subordinators
$$\sigma_t = \mathrm{d}t + \sum_{s \le t} T_0(\Delta_s); \quad \widehat{\sigma}_t = \widehat{\mathrm{d}}t + \sum_{s \le t} \widehat{T}_0(\Delta_s), \quad t > 0,$$
for some $\mathrm{d}, \widehat{\mathrm{d}} > 0$ (see section 2.1 for the notation or Blumenthal [8] § 5 for an account). In such a case, the λ–resolvent for Y^* (resp. $\widehat{Y}^*$) at 0 is given by
$$W_\lambda f(0) = \frac{\mathrm{d} f(0) + M_\lambda f}{\lambda \mathrm{d} + \lambda M_\lambda 1}; \quad W_\lambda f(0) = \frac{\widehat{\mathrm{d}} f(0) + \widehat{M}_\lambda f}{\lambda \widehat{\mathrm{d}} + \lambda \widehat{M}_\lambda 1},$$
for f continuous and bounded, and, if $\mathrm{d} = \widehat{\mathrm{d}}$, then the resolvents W_λ and $\widehat{W}_\lambda$ are still in weak duality but this time with respect to the measure $\zeta^d(dx) = d\delta_0(dx) + \zeta(dx)$.

3. Assume moreover that for every $x > 0$, $\widehat{\mathrm{P}}_x(T_0 \in dt)$ is absolutely continuous with respect to Lebesgue measure, having a density
$$a(x,t) = \frac{\widehat{\mathrm{P}}_x(T_0 \in dt)}{dt}, \quad x, t > 0,$$
which is jointly Borel measurable. Then for $\lambda > 0$,
$$\int_0^\infty ds e^{-\lambda s} m_s f = \int_{]0,\infty[} \zeta(dx) \widehat{\varphi}_\lambda(x) f(x) = \int_0^\infty ds e^{-\lambda s} \int_{]0,\infty[} \zeta(dx) a(x,s) f(x),$$
for f continuous and bounded. The second equality is a consequence of Fubini's theorem. By inverting the Laplace transform we obtain that for $s > 0$,
$$m_s f = \int_{]0,\infty[} \zeta(dx) a(x,s) f(x).$$
A similar result was obtained by Getoor in [16] Proposition 10.10 in a different setting.

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Chapitre IV

Recurrent extensions of self–similar Markov processes and Cramér's condition II

Abstract

This chapter is a continuation of Chapter III. We indicate how the methods used there can be extended to study the recurrent extensions of a positive self–similar Markov process that makes a jump to 0. The unique excursion measure $\overline{\mathbf{n}}$ under which the excursion process leaves 0 continuously is constructed as well as its associated self–similar recurrent extension. The image under time reversal of $\overline{\mathbf{n}}$ is determined and we construct a dual self–similar recurrent Markov process associated to it. We make explicit the law of the meander process and that of the excursion process conditioned to have a given length. We construct a self–similar Markov process conditioned to hit 0 continuously.
Key words. Self–similar Markov process, description of excursion measures, weak duality, Lévy processes.
A.M.S. Classification. 60 J 25 (60 G 18).

1 Introduction

Let $(\mathbb{Q}_x, x \geq 0)$ be the law of a $\mathbb{R}^+$–valued self-similar Markov process Y started at $x \geq 0$. Assume that Y hits 0 at some finite time and then dies. We will refer to $(Y, \mathbb{Q})$ as the minimal process. This chapter is the companion of Chapter III. There we studied the excursion measures and the recurrent extensions of a self–similar Markov process Y that hits 0 continuously, i.e.

$$\mathbb{Q}_x(T_0 < \infty, Y_{T_0 -} = 0) = 1 \qquad \text{for all } x > 0,$$

where $T_0 = \inf\{t > 0 : Y_{t-} = 0 \text{ or } Y_t = 0\}$. Here we are interested in the same problem but for a self–similar Markov process that hits 0 by a jump a.s.

$$\mathbb{Q}_x(T_0 < \infty, Y_{T_0 -} > 0) = 1 \qquad \text{for all } x > 0. \tag{1}$$

As was proved by Lamperti [21], the former corresponds to a self–similar Markov process associated to a Lévy process ξ with an infinite lifetime and which drifts to $-\infty$, $\lim_{t\to\infty} \xi_s = -\infty$ a.s., while the latter corresponds to one associated to a Lévy process killed at an independent exponential time (i.e. jumps to $-\infty$ with some strictly positive rate).

In this Chapter, instead of using Brownian motion as a thread for introducing our main results, as we did in Chapter III, we prefer to use stable processes with negative jumps, that is Lévy processes which are self–similar. This choice is more appropiate to the present framework since it is well known that stable processes with negative jumps hit $]-\infty, 0[$ a.s. by a jump. As a consequence a stable process killed at its first hitting time of $]-\infty, 0[$ is a positive self–similar Markov process that satisfies the property (1). With this in mind we next briefly recall some known results on stable processes. We refer to Chaumont [8, 10] for an account of stable processes and their excursion theory.

Let (X, P) be an a–stable Lévy process for $a \in]0, 2[$, i.e. a real–valued Lévy process that is $1/a$–self–similar, and we assume that X has negative jumps and that $|X|$ is not a subordinator. We denote by P^0 the law of the process X killed at its first entrance into $]-\infty, 0[$ and take 0 as a cemetery point. Since X has negative jumps we have that X hits $]-\infty, 0]$ by a jump and

$$P^0_x(X_{T_0-} > 0,\ T_0 < \infty) = 1, \qquad \forall x > 0,$$

where $T_0 = \inf\{t \geq 0 : X^0_t = 0\}$. We denote $(\xi, \mathbf{Q})$ the Lévy process associated to (X^0, P^0) via Lamperti's [21] transformation. According to Lamperti, under our assumptions the real–valued Lévy process $(\xi, \mathbf{Q})$ is a Lévy process killed at an independent exponential time. A consequence of the results of Silverstein [25] is that the function

$$h_\rho(x) = x^{a(1-\rho)}, x \geq 0, \qquad \rho = P(X_1 \geq 0),$$

is, up to a multiplicative constant, the unique invariant function for P^0, i.e. for any $t > 0$

$$P^0_x(h_\rho(X_t)) = h_\rho(x), \qquad \text{for all } x \geq 0.$$

It follows that the function $h(x) = e^{a(1-\rho)x}, x \in \mathbb{R}$, is an invariant function for the process $(\xi, \mathbf{Q})$. Next, let $P^\natural$ be the h–transform of P^0 via the invariant function h_ρ. The probability measure $P^\natural$ is the law of a positive $1/a$–self–similar Markov process such that

$$P^\natural_x(\lim_{t\to\infty} X_t = \infty, T_0 = \infty) = 1 \qquad x \geq 0.$$

It is not hard to see that the Lévy process associated to $(X^\natural, P^\natural)$ via Lamperti's transformation is in fact the process $(\xi, \mathbf{Q})$ h–transformed via the function $h(x) = e^{a(1-\rho)x}, x \in \mathbb{R}$, and can be interpreted as $(\xi, \mathbf{Q})$ conditioned to drift to ∞. Furthermore, Chaumont [8, 10] showed that the measures P and $P^\natural$ are related in the same way as the law of a Brownian motion killed at 0 is related to that of a Bessel(3) process, see e.g. [22]. Using this fact Chaumont obtains a description of the unique excursion measure n compatible with the law of (X^0, P^0) such that $n(X_{0+} > 0) = 0$ and $n(1 - e^{-T_0}) = 1$, which is reminiscent of Imhof's [18] description of Itô's excursion measure for the Brownian motion using the law of a Bessel(3) process. The measure n is the Itô's excursion measure of X reflected at its infimum, that is $((X_t - \inf_{s \leq t} X_s, t \geq 0), P)$. In section 2 we obtain, under some hypotheses, results that are analogous to those above and then are used to construct the unique excursion measure, say $\overline{\mathbf{n}}$, compatible with $(Y, \mathbb{Q})$ and such that $\overline{\mathbf{n}}(Y_{0+} > 0) = 0$ and $\overline{\mathbf{n}}(1 - e^{-T_0}) = 1$. Associated to this excursion measure there is a unique self–similar recurrent extension of the process $(Y, \mathbb{Q})$, say $(\widetilde{Y}, \widetilde{\mathbb{Q}})$, which, in the case of the stable process corresponds to the stable process reflected at its infimum.

We noted above that (X^0, P^0) hits 0 by a jump and, by the Markov property, it follows that $n(X_{T_0-} = 0) = 0$, i.e. the excursions end by a jump a.s. Chaumont [8] Corollaire 1 proved that conditionally on the value of X_{T_0-} the image under time reversal of n is equal to the law, say $P^{*\downarrow}$, of the dual stable process, $(X^*, P^*) = (-X, P)$, killed at its first hitting time of $]-\infty, 0]$ and conditioned to hit 0 continuously. In part (ii) of Theorem 2 we determine the image under time reversal of $\overline{\mathbf{n}}$ and we deduce therefrom that a similar property for the image under time reversal of $\overline{\mathbf{n}}$ conditioned on the value of Y_{T_0-} still holds. In part(iii) of Theorem 2 we construct a process Z_θ whose excursion measure from 0 is the image under time reversal of $\overline{\mathbf{n}}$ and which is in weak duality with the process $\widetilde{Y}$. Then we prove that the process Z_θ started at 0 is equal in law to the process obtained by time reversing one by one the excursions from 0 of the process $\widetilde{Y}$ started at 0. The latter result is reminiscent of Theorem 4.8 of Getoor and Sharpe [16]. In the stable process setting one can use the result of Doney [12] to interpret the process Z_θ as the process (X^*, P^*) conditioned to stay positive and reflected at its future infimum. Doney gives a pathwise construction of a Lévy process conditioned to stay positive by using Tanaka's [26] method.

Section 4 is devoted to the construction of the law under $\overline{\mathbf{n}}$ of the excursion process conditioned by its length and to establishing an absolute continuity relation between this law and that of the meander process. This relation between the law of the excursion process conditioned by its length and that of the meander process was established by Chaumont [10] Théorème 2 for stable processes with negative jumps.

In addition to $(X^\natural, P^\natural)$ there is another process, say $(X^\downarrow, P^\downarrow)$, associated to (X^0, P^0) which plays an important rôle in the understanding of n. This is process can be thought of as (X^0, P^0) conditioned to hit 0 continuously. More precisely, Silverstein's [25] results imply that the function $h'_\rho(x) = x^{a(1-\rho)-1}, x \geq 0$ is excessive for (X^0, P^0). Using this, Chaumont [8] Section 1.3 constructs a process $(X^\downarrow, P^\downarrow)$ as a h–transform of (X^0, P^0) via the function h'_ρ and shows that this is a self–similar Markov process that hits 0 continuously. Actually, the function $h^\downarrow(x) = \exp\{(a(1-\rho)-1)x\}, x \in \mathbb{R}$, is invariant for the Lévy process $(\xi, \mathbf{Q})$ and the corresponding h–transform can be thought of as $(\xi, \mathbf{Q})$ conditioned to tend to $-\infty$ as the time tends to ∞. The purpose of Section 5 is, under supplementary hypotheses, to provide a construction of a self–similar Markov process $Y^\downarrow$ that can be thought ofas $(Y, \mathbb{Q})$ conditioned to hit 0 continuously. The results of Section III.3 can be applied to this process to ensure the existence of an excursion measure $\mathbf{n}^\downarrow$, such that $\mathbf{n}^\downarrow(X_{0+} > 0) = 0$ and $\mathbf{n}^\downarrow(X_{T_0-} > 0) = 0$. Furthermore this excursion measure is absolutely continuous w.r.t. the excursion measure $\overline{\mathbf{n}}$.

In Section 6.1 we verify that stable processes with negative jumps satisfy our hypotheses and we go into more detail about the results recalled above. Moreover, with the aim of establishing further connections with the results in Chapter III in Section 6.2, we work in the framework of Section III.5 to determine the weak dual of a self–similar Markov process that leaves 0 by a jump and hits 0 continuously.

2 Settings and first results

Our first purpose is to establish the analogues of Propositions III.2 and III.3 and Theorems III.1 and III.2 for the class of self–similar Markov processes that hit 0 by a jump. With this aim we recall that the techniques used in the proofs of those results are based essentially on two facts which are deduced from the hypothesis that the underlying Lévy process satisfies Cramér's condition. Under this assumption we can ensure that there exists a $\theta > 0$ such that the function $h(x) = x^\theta, x \geq 0$ is invariant for the semi-group of the process Y, and that the law ${\mathbb{Q}^\natural}_x$, which is the h–transform of $\mathbb{Q}_x$ via $h(x) = x^\theta$, has a limit ${\mathbb{Q}^\natural}_{0+}$ as x goes to 0 in the sense of finite dimensional laws. The probability measure ${\mathbb{Q}^\natural}_x$ can be viewed as the law of the process Y conditioned to never hit 0. Therefore, in order to establish the main results of sections III.2 and III.3 in the present case, we just have to ensure that the latter facts still hold and the same proofs will still be valid. We devote this section to this task.

Let $\mathbf{Q}'$ be a measure on the space $(\mathbb{D}, \mathcal{D})$, of càdlàg trajectories with values in $\mathbb{R}$ endowed with the σ–algebra generated by the coordinate maps and $(\mathcal{D}'_t, t \geq 0)$ the natural filtration. Assume that under $\mathbf{Q}'$ the canonical process is a Lévy process and that the convex set

$$C = \{\lambda \in \mathbb{R} : \mathbf{Q}'(e^{\lambda\xi_1}) < \infty\},$$

contains a point different from 0, $C \setminus \{0\} \neq \emptyset$. Then the characteristic exponent of ξ, i.e. $\Psi : \mathbb{R} \to \mathbb{C}$, defined by

$$\mathbf{Q}'(e^{i\lambda\xi_1}) = e^{-t\Psi(\lambda)} \quad \lambda \in \mathbb{R},$$

admits an analytic extension to the complex strip $-\Im(z) \in C$. Thus we can define the Laplace exponent $\psi : C \to \mathbb{R}$ of $\mathbf{Q}'$ by

$$\mathbf{Q}'(e^{\lambda\xi_1}) = e^{\psi(\lambda)}, \quad \text{with} \quad \psi(\lambda) = -\Psi(-i\lambda), \quad \lambda \in C.$$

Hölder's inequality implies that ψ is a convex function on C. Let $\mathbf{Q}$ be the law of the Lévy process ξ which is obtained by killing ξ' at a rate $\boldsymbol{k}$, that is ξ' is killed at an independent exponential random variable of parameter $\boldsymbol{k} > 0$. Then the Laplace exponent $\psi_{\boldsymbol{k}}$ of ξ under $\mathbf{Q}$ is

$$\mathbf{Q}(e^{\lambda\xi_1}) = e^{\psi_{\boldsymbol{k}}(\lambda)}, \quad \psi_{\boldsymbol{k}}(\lambda) = \psi(\lambda) - \boldsymbol{k}, \quad \lambda \in C.$$

We will denote by ζ the lifetime of ξ, by $(\mathcal{D}_t, t \geq 0)$ the filtration of the killed process, by Δ the cemetery point for ξ and, as usual we extend the functions $f : \mathbb{R} \to \mathbb{R}$ to $\mathbb{R} \cup \Delta$ by $f(\Delta) = 0$.

We assume henceforth

(HI–a) ξ is not arithmetic, i.e. the state space is not a subgroup of $c\mathbb{Z}$ for any real c;

(HI–b) there exists $\theta > 0$ such that $\mathbf{Q}(e^{\theta\xi_1}, 1 < \zeta) = 1$;

(HI–c) $\mathbf{Q}(\xi_1^+ e^{\theta\xi_1}, 1 < \zeta) < \infty$.

We will refer to (HI–b) as Cramér's condition by analogy with Chapter III. Condition (HI-b) holds if and only if we kill ξ' at an independent exponential time $\boldsymbol{k} = \psi(\theta)$ for some θ in $C \cap]0, \infty[$. A sufficient condition for (HI-c) is that θ belongs to the interior of C. Cramér's condition implies that the function $h(x) = e^{\theta x}, x \in \mathbb{R}$, is invariant for the semi-group of ξ under $\mathbf{Q}$. Let $\mathbf{Q}^\natural$ be the h–transform of $\mathbf{Q}$ via the function $h(x) = e^{\theta x}$. That is $\mathbf{Q}^\natural$ is the unique measure on the space of càdlàg trajectories with lifetime such that

$$\mathbf{Q}^\natural(F_T) = \mathbf{Q}(F_T e^{\theta\xi_T}, T < \zeta) \qquad \text{for every stopping time } T \text{ of } \mathcal{D}_t.$$

Moreover, under $\mathbf{Q}^\natural$ the canonical process still is a Lévy process but with infinite lifetime and finite mean $m^\natural = \psi'(\theta) > 0$, owing to (HI-c) and the convexity of $\psi_{\boldsymbol{k}}$. Thus $\xi^\natural$ drifts to ∞, $\lim_{s\to\infty} \xi_s = \infty$ $\mathbf{Q}^\natural$–a.s. The characteristic exponent of $\xi^\natural$ is given by $\Psi^\natural(\lambda) = \Psi(\lambda - i\theta) + \boldsymbol{k}$ for $\lambda \in \mathbb{R}$.

Hereafter we take an arbitrary fixed $\alpha > 0$. Next, let $(\mathbb{Q}_x, x > 0)$ be the law of the α–self–similar Markov process Y associated to $(\xi, \mathbf{Q})$ via Lamperti's transformation. That is, let

$$A_t = \int_0^t \exp\{(1/\alpha)\xi_s\} ds \qquad t \geq 0$$

and let $\tau(t)$ be its inverse,

$$\tau(t) = \inf\{s > 0 : A_s > t\},$$

with the convention $\inf\{\emptyset\} = \infty$. For $x > 0$, let $\mathbb{Q}_x$ be the law of the process

$$Y_t = x \exp\{\xi_{\tau(tx^{-1/\alpha})}\}, \qquad t > 0,$$

with the convention that the above quantity is 0 if $\tau(tx^{-1/\alpha}) = \infty$. The Volkonskii theorem ensures that the process Y is a strong Markov process in the filtration $(\mathcal{G}_t = \mathcal{D}_{\tau(t)}, t \geq 0)$. Furthermore, by construction the process Y has the scaling property: for every $c > 0$ the law of the process $(cY_{tc^{-1/\alpha}}, t \geq 0)$ under $\mathbb{Q}_x$ is $\mathbb{Q}_{cx}$. It follows that Y has a finite lifetime $T_0 = \inf\{t > 0 : Y_t = 0\}$ and that it has the same law under $\mathbb{Q}_x$ as $x^{1/\alpha}A_{\boldsymbol{e}}$ under $\mathbf{Q'}$ with

$$A_{\boldsymbol{e}} = \int_0^{\boldsymbol{e}} \exp\{(1/\alpha)\xi'_s\} ds, \tag{2}$$

with $\boldsymbol{e}$ an exponential random variable of parameter $\boldsymbol{k}$ independent of ξ'. Since ξ has a finite lifetime, Y hits 0 by a jump in finite time and then dies. We denote (Y, T_0) the process killed at 0 and by $(P_t, t \geq 0)$ and $(V_q, q > 0)$ its semi-group and resolvent respectively. Observe that the results of Section III 2.3 are still valid under the assumptions of this chapter since their proofs only use the property that the self–similar Markov process hits 0 in a finite time a.s.

Remark 1. The process Y is obtained by applying first an operation of killing and then a time change to the Lévy process. If the order of this construction is inverted, first time change and then killing according to a multiplicative functional, we obtain an equivalent self–similar Markov process. More precisely, given a Lévy process with law $\mathbf{Q'}$ and infinite lifetime, we construct a self–similar Markov process $(Y', \mathbb{Q}'_x, x \geq 0)$ via Lamperti's transformation of ξ'. This process either hits 0 continuously or never hits 0 a.s. Next we kill the process Y' according to the multiplicative functional

$$M_t = \exp\{-\boldsymbol{k}\varphi(t)\}, \quad \varphi(t) = \int_0^t (Y'_s)^{-1/\alpha} ds, \quad t < T'_0 = \inf\{r > 0 : Y'_r = 0\},$$

to obtain a self–similar Markov process Y''. See Lamperti [21] for a detailed study of the additive functional φ. The Feymann-Kac formula allows us to determine the infinitesimal generator of Y'', which is equal to that of Y. Thus the processes Y and Y'' are equivalent.

After this slight digression on the construction of Y we continue with our program. Let $(\mathbb{Q}^{\natural}{}_x, x > 0)$ be the law of the α–self–similar Markov process $Y^{\natural}$ associated to the Lévy process $\xi^{\natural}$ with law $\mathbf{Q}^{\natural}$ via Lamperti's transformation. Since $\xi^{\natural}$ drifts to ∞ we have that $Y^{\natural}$ never hits 0 and $\lim_{t\to\infty} Y^{\natural}_t = \infty$, $\mathbb{Q}^{\natural}{}_x$–a.s. for all $x > 0$. As in Section III.3, the process $Y^{\natural}$ can be thought of as the process Y conditioned never to hit 0, thanks to the following statements which are the analogues of Proposition III.2

(i) *Let $x > 0$ be arbitrary. We have that $\mathbb{Q}^{\natural}{}_x$ is the unique measure such that for every $\mathcal{G}_t$ stopping time, T, we have*

$$\mathbb{Q}^{\natural}{}_x(A) = x^{-\theta}\, \mathbb{Q}_x(A\ Y_T^{\theta}, T < T_0),$$

for any $A \in \mathcal{G}_T$. In particular, the function $h^* : [0,\infty[\to [0,\infty[$ defined by $h^*(x) = x^{\theta}$ is invariant for the semi-group P_t.

(ii) *For every $x > 0$ and $t > 0$ we have*

$$\mathbb{Q}^{\natural}{}_x(A) = \lim_{s\to\infty} \mathbb{Q}_x(A \mid T_0 > s),$$

for any $A \in \mathcal{G}_t$.

The proof of (i) is the same as (i) in Proposition III.2; the proof of (ii) needs a lemma just as in the proof of (ii) in Proposition III.2

Lemma 1. *Under the hypothesis (HI) we have that there exists a constant* $C \in]0,\infty[$ *such that*

$$\lim_{t\to\infty} t^{\alpha\theta}\,\mathbf{Q}'(A_{\boldsymbol{e}} > t) = C.$$

Moreover, if $0 < \alpha\theta < 1$ *then*

$$C = \frac{\alpha}{m^{\natural}}\,\mathbf{Q}'(A_{\boldsymbol{e}}^{-(1-\alpha\theta)}).$$

Proof. This proof, like that of the analogous result in Chapter III, is based on a result of Kesten [20] and Goldie [17] on random equations. We claim that $A_{\boldsymbol{e}}$ has the same law as $D + MA'_{\boldsymbol{e}'}$ with $D = \int_0^1 \exp\{\xi'_s\}1_{\{s<\boldsymbol{e}\}}ds$, $M = e^{(1/\alpha)\xi'_1}1_{\{1<\boldsymbol{e}\}}$ and $A'_{\boldsymbol{e}'}$ with the same law as $A_{\boldsymbol{e}}$ and independent of (D, M). Furthermore, $\mathbf{Q}'(D^{\alpha\theta}) < \infty$. These two facts enable us to apply the results of Kesten and Goldie to prove that

$$\lim_{t\to\infty} t^{\alpha\theta}\,\mathbf{Q}'(A_{\boldsymbol{e}} > t) = C,$$

for some $C \in]0,\infty[$ whose expression can be found in [17]. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable and bounded function and put $\widetilde{D} = \int_0^1 \exp\{(1/\alpha)\xi'_s\}ds$ and $\widetilde{M} = \exp\{(1/\alpha)\xi'_1\}$. Indeed, using the lack of memory of the exponential law we obtain

$$\begin{aligned}
\mathbf{Q}'(f(A_{\boldsymbol{e}})) &= \mathbf{Q}'(f(D)1_{\{\boldsymbol{e}<1\}}) + \mathbf{Q}'(1_{\{\boldsymbol{e}>1\}}f(\widetilde{D} + \widetilde{M}\int_0^{\boldsymbol{e}-1}\exp\{(1/\alpha)(\xi_{1+s}-\xi_1)\}ds))\\
&= \mathbf{Q}'(f(D + MA'_{\boldsymbol{e}'})1_{\{\boldsymbol{e}<1\}}) + e^{-\boldsymbol{k}}\,\mathbf{Q}'(f(\widetilde{D} + \widetilde{M}A'_{\boldsymbol{e}'}))\\
&= \mathbf{Q}'(f(D + MA'_{\boldsymbol{e}'})1_{\{\boldsymbol{e}<1\}}) + \mathbf{Q}'(f(D + MA'_{\boldsymbol{e}'})1_{\{\boldsymbol{e}>1\}})\\
&= \mathbf{Q}'(f(D + MA'_{\boldsymbol{e}'})),
\end{aligned} \tag{3}$$

where we observe that in the second equality the random variable $A'_{\boldsymbol{e}'}$ is independent of $\sigma(\xi'_s, s \le 1)$ and $\boldsymbol{e}$. We next prove that $\mathbf{Q}'(D^{\alpha\theta}) < \infty$.

$$\begin{aligned}
\mathbf{Q}'(D^{\alpha\theta}) &\le \mathbf{Q}'(\sup\left\{e^{\theta\xi_s}1_{\{s<\boldsymbol{e}\}}; s \le 1\right\})\\
&\le \frac{e}{e-1}\left(1 + \sup_{\{0\le s\le 1\}}\mathbf{Q}'(e^{\theta\xi_s}\log^+(e^{\theta\xi_s}1_{\{s<\boldsymbol{e}\}})1_{\{s<\boldsymbol{e}\}})\right)\\
&= \frac{e}{e-1}\left(1 + \theta\sup_{\{0\le s\le 1\}}\mathbf{Q}'(e^{\theta\xi_s}\xi_s^+1_{\{s<\boldsymbol{e}\}})\right) < \infty,
\end{aligned} \tag{4}$$

where the second inequality is due to the fact that the process $e^{\theta\xi_s}1_{\{s<\boldsymbol{e}\}}$ is a positive martingale for $\mathbf{Q}'$ and a so we can apply a Doob's inequality, with the convention $0\log^+(0) = 0$. The last right–hand term is finite due to assumption (HI–c).

In the case $0 < \alpha\theta < 1$, the value of the constant C is determined as in the proof of Lemma III.4 using the identity

$$\mathbf{Q}'(A_{\boldsymbol{e}}^{\alpha\beta}) = \frac{\alpha\beta}{-\psi_{\boldsymbol{k}}(\beta)}\,\mathbf{Q}'(A_{\boldsymbol{e}}^{\alpha\beta-1}), \quad \beta < \theta,$$

whose proof can be found in Carmona, Petit & Yor [7] Proposition 3.1.(i) $\square$

Corollary 1. *For each* $\beta \in]0, \theta \wedge (1/\alpha)[$ *there exists a self–similar recurrent extension of* (Y, T_0) *that leaves* 0 *a.s. by a jump according to the jump-in measure* $\eta^\beta(dx) = b_{\alpha,\beta} x^{-(1+\beta)} dx, x > 0$, *with* $b_{\alpha,\theta} = \beta / \mathbf{Q}'(A_{\boldsymbol{e}}^{\alpha\beta})\Gamma(1-\alpha\beta)$.

The proof of Corollary 1 is a straightforward consequence of Lemma 1 and Proposition III.1; see the remarks at the end of section III.2.

Furthermore, since the Lévy process $\xi^\natural$ has a strictly positive finite mean $\mathbf{Q}^\natural(\xi_1) = m^\natural$ we know from [1] that there exists a measure $\mathbb{Q}^\natural{}_{0+}$ which is the limit in the sense of finite dimensional laws of $\mathbb{Q}^\natural{}_x$ as $x \to 0+$. Under $\mathbb{Q}^\natural{}_{0+}$ the law of Y_s is an entrance law for the semi-group of $Y^\natural$ and is related to the law of the Lévy exponential functional $J = \int_0^\infty \exp\{-(1/\alpha)\xi^\natural_s\} ds$ by the formula

$$\mathbb{Q}^\natural{}_{0+}(f(Y_s^{1/\alpha})) = \frac{\alpha}{m^\natural} \mathbf{Q}^\natural(f(s/J)J^{-1}), \qquad s > 0, \tag{5}$$

for f measurable and positive, see [1]. Assume $0 < \alpha\theta < 1$. Then to construct an excursion measure $\overline{\mathbf{n}}$ compatible with the minimal process (Y, T_0) such that $\overline{\mathbf{n}}(Y_{0+} > 0) = 0$ and $\overline{\mathbf{n}}(1 - e^{-T_0}) = 1$, we can argue as in the proof of Theorem III.1. Indeed, this is an h–transform of $\mathbb{Q}^\natural{}_{0+}$ via the excessive function $x^{-\theta}, x > 0$. Furthermore, the proof of Proposition III.3 can also be extended to the present case to ensure that the measure $\overline{\mathbf{n}}$ is the unique excursion measure with these properties, that is compatible with the minimal process (Y, T_0). We have the following results.

Theorem 1. *Assume* $0 < \alpha\theta < 1$.

(i) The excursion measure $\overline{\mathbf{n}}$ *is such that for every* $\mathcal{G}_t$*–stopping time* T

$$\overline{\mathbf{n}}(A_T, T < T_0) = (a_{\alpha,\theta})^{-1} \mathbb{Q}^\natural{}_{0+}(A_T Y_T^{-\theta}), \qquad A_T \in \mathcal{G}_T,$$

with $a_{\alpha,\theta} = \alpha \mathbf{Q}^\natural(J^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)/m^\natural$.

(ii) The q*–potential of the entrance law* $(\overline{\mathbf{n}}_s, s > 0)$, *associated to* $\overline{\mathbf{n}}$, *admits the representation*

$$\int_0^\infty e^{-qs} \, \overline{\mathbf{n}}_s \, f ds = (m^\natural a_{\alpha,\theta})^{-1} \int_0^\infty f(y) \, \mathbf{Q}^\natural(e^{-y^{-1/\alpha}J}) y^{1/\alpha - 1 - \theta} dy,$$

for $f \in C_b(\mathbb{R}^+)$.

(iii) The minimal process (Y, T_0) *admits a unique self–similar recurrent extension* $\widetilde{Y}$ *that leaves* 0 *continuously a.s. The resolvent of* $\widetilde{Y}$ *is given by*

$$U_q f(0) = \frac{1}{(m^\natural a_{\alpha,\theta})q^{\alpha\theta}} \int_0^\infty f(y) \, \mathbf{Q}^\natural(e^{-y^{-1/\alpha}J}) y^{1/\alpha - 1 - \theta} dy$$

and $U_q f(x) = V_q f(x) + \mathbb{Q}_x(e^{-qT_0}) U_q f(0)$, *for* $x > 0$ *and* $f \in C_b(\mathbb{R}^+)$. *The resolvent* U_q *is Fellerian.*

The proof of (i) in Theorem 1 is the same as that of Theorem III.1.(i); (ii) in Theorem 1 is proved as Proposition III.3.(i); last, the proof of Theorem III.2.(i) applies to prove (iii) in Theorem 1.

Remark 2. We can deduce as in the proof of Proposition III.3 that

$$\mathbf{Q}'(A_{\boldsymbol{e}}^{-(1-\alpha\theta)}) = \mathbf{Q}^\natural(J^{-(1-\alpha\theta)}).$$

Remark 3. If $\alpha\theta \geq 1$, the arguments given in Theorem III.2 show that there does not exist an excursion measure compatible with the semigroup of Y such that the excursion process leaves 0 continuously.

3 Time reversed excursions

In this section we are interested in determining the image under time reversal of the unique excursion measure $\overline{\mathbf{n}}$ compatible with Y such that $\overline{\mathbf{n}}(Y_{0+}>0)=0$. Furthermore, we would like to determine whether the self–similar recurrent extension $\widetilde{Y}$ of Y admits a weak dual process and, if so, to identify it.

To this end, we recall that the process $Y^{\natural}$ has a weak dual that we denote by $\widehat{Y}^{\natural}$. The latter is the self–similar Markov process associated to $-\xi^{\natural}$, the dual of $\xi^{\natural}$. More precisely, let $V^{\natural}_q, \widehat{V}^{\natural}_q$ be the q–resolvents of $Y^{\natural}$ and $\widehat{Y}^{\natural}$ respectively. Then

$$\int_0^\infty dx x^{1/\alpha-1} f(x) V^{\natural}_q g(x) = \int_0^\infty dx x^{1/\alpha-1} g(x) \widehat{V}^{\natural}_q f(x),$$

for all measurable functions $f,g:\mathbb{R}^+\to\mathbb{R}^+$. See Bertoin and Yor [1]. Next, since the process Y is the h-transform of $Y^{\natural}$ via the excessive function $h(x)=x^{-\theta}, x>0$, we have that the q–resolvent V_q of Y is in weak duality with $\widehat{V}^{\natural}_q$ with respect to the measure $x^{1/\alpha-1-\theta}dx, x>0$.

Since the process $-\xi^{\natural}$ drifts to $-\infty$ it follows that

$$\widehat{\mathbb{Q}^{\natural}_x}(Y_{T_0-}=0, T_0<\infty)=1 \quad \text{for all } x>0.$$

Now, that Y hits 0 by a jump implies that the excursions of $\widetilde{Y}$ away from 0 terminate by a jump a.s., i.e. $\overline{\mathbf{n}}(Y_{T_0-}=0)=0$, and by the self–similarity it is easy to prove that $\overline{\mathbf{n}}(Y_{T_0-}\in dx)=x^{-(1+\gamma)}dx, x>0$ for some $\gamma>0$. These two statements allow us to guess that the candidate for a weak dual of $\widetilde{Y}$ should be a recurrent extension of $\widehat{Y}^{\natural}$ that leaves 0 by a jump a.s. We formalize this statement in the following theorem. Let $\varrho:\mathbb{D}^+\to\mathbb{D}^+$ be the operator of time–reversal at time T_0,

$$\varrho Y(t)=\begin{cases} Y_{(T_0-t)-} & \text{if} \quad 0\leq t<T_0<\infty \\ 0 & \text{otherwise,}\end{cases}$$

and $\varrho\,\overline{\mathbf{n}}$ the image under time reversal at T_0 of $\overline{\mathbf{n}}$.

Theorem 2. *(i) For each $\beta\in]0,\theta]$ the process $\widehat{Y}^{\natural}$ admits a self–similar recurrent extension $Z_\beta=(Z_{\beta,t}, t\geq 0)$ that leaves 0 by a jump according to the jumping-in measure*

$$\eta_\beta(dx)=b_{\alpha,\beta}x^{-(1+\beta)}dx, x>0,$$

with $b_{\alpha,\beta}=\beta/\Gamma(1-\alpha\beta)\widehat{\mathbf{Q}^{\natural}}(I^{\alpha\beta})$, and $I=\int_0^\infty \exp\{(1/\alpha)\widehat{\xi}^{\natural}_s\}ds$. The resolvent of Z_β is given by

$$\mathcal{U}_q f(0)=b_{\alpha,\beta}q^{-\alpha\beta}\int_0^\infty y^{-(1+\beta)}\widehat{V^{\natural}}_q f(y)dy; \quad \mathcal{U}_q f(x)=\widehat{V^{\natural}}_q f(x)+\widehat{\mathbb{Q}^{\natural}}_x(e^{-qT_0})\mathcal{U}_q f(0),$$

for $x>0$.

(ii) The image under time reversal of $\overline{\mathbf{n}}$, is given by

$$\varrho\,\overline{\mathbf{n}}(\cdot)=b_{\alpha,\theta}\int_0^\infty dx x^{-(1+\theta)}\widehat{\mathbb{Q}^{\natural}}_x(\cdot).$$

In particular, $\overline{\mathbf{n}}(Y_{T_0-}\in dx)=b_{\alpha,\theta}x^{-(1+\theta)}dx, x>0$ and $\varrho\,\overline{\mathbf{n}}(\cdot|Y_{T_0-}=x)=\widehat{\mathbb{Q}^{\natural}}_x(\cdot)$.

(iii) The process Z_θ is in weak duality with $\widetilde{Y}$ w.r.t. $x^{1/\alpha-1-\theta}dx, x>0$.

We have noted that in the stable processes setting, the self–similar process $\widetilde{Y}$ corresponds to a stable process reflected at its infimum and Z_θ is, as we will see later, the dual stable process conditioned to stay positive and reflected at its future infimum. Thus, in this case, (iii) in Theorem 2 establishes that these processes are in weak duality. We have said in the Introduction that Z_θ has the same law started at 0 as the process obtained by time reversing one by one the excursions from 0 of $\widetilde{Y}$ started from 0. This result still holds in a greater generality. To give a precise statement, in the sequel, we denote $\widetilde{\mathbb{Q}}_{\cdot}$ and $\widetilde{\mathbb{Q}}_{\cdot}^{\wedge}$ the law of the processes $\widetilde{Y}$ and Z_θ, respectively. We have the following corollary which is reminiscent of Theorem 4.8 of Getoor & Sharpe [16].

Corollary 2. *For any $t>0$, let $g_t=\sup\{s<t:\widetilde{Y}_s=0\}$, $d_t=\inf\{s>t:\widetilde{Y}_s=0\}$ and*

$$\overleftarrow{Y}_t=\begin{cases}Y_{(d_t-(t-g_t))-} & \text{if} \quad 0<g_t<d_t<\infty\\ Y_t & \text{otherwise.}\end{cases}$$

Then the process $\overleftarrow{Y}=(\overleftarrow{Y}_t, t\geq 0)$ has the same law under $\widetilde{\mathbb{Q}}_0$ as Z_θ under $\widetilde{\mathbb{Q}}_0^{\wedge}$.

We postpone the proof of Corollary 2 until subsection 3.1.

Proof of Theorem 2. (i) According to Proposition III.1 all that we have to verify in order to prove (i) is that $\widehat{\mathbf{Q}^{\natural}}(I^{\alpha\beta})<\infty$ for every $\beta\in]0,\theta]$. Indeed, due to (HI–c) we have that $-\widehat{\mathbf{Q}^{\natural}}(\xi_1)=m^{\natural}\in]0,\infty[$, and by the identity (5) that $\widehat{\mathbf{Q}^{\natural}}(I^{-1})=m^{\natural}/\alpha<\infty$ (observe that I under $\widehat{\mathbf{Q}^{\natural}}$ is equal to J under $\mathbf{Q}^{\natural}$). Therefore, for every $0<\alpha\beta\leq\alpha\theta<1$ we have that $\widehat{\mathbf{Q}^{\natural}}(I^{\alpha\beta-1})<\infty$. The claim follows using the identity

$$\widehat{\mathbf{Q}^{\natural}}(I^{\alpha\beta})=\frac{\alpha\beta}{-\boldsymbol{\psi}(\beta)}\widehat{\mathbf{Q}^{\natural}}(I^{\alpha\beta-1}) \quad \text{for } 0<\beta\leq\theta, \tag{6}$$

with $\boldsymbol{\psi}:[0,\theta]\to\mathbb{R}$ defined by

$$\widehat{\mathbf{Q}^{\natural}}(e^{\lambda\xi_1})=e^{\boldsymbol{\psi}(\lambda)}, \qquad 0\leq\lambda\leq\theta.$$

The identity (6) can be proved with arguments similar to those given by Bertoin & Yor [2]. Note that $\boldsymbol{\psi}(\lambda)=\psi_{\boldsymbol{k}}(\theta-\lambda)$, for every $0\leq\lambda\leq\theta$.

(ii) We first note that an application of Lemma III.3 proves that the entrance laws $(\overline{\mathbf{n}}_s(dy), s>0)$ and

$$N_s^{\theta}f=b_{\alpha,\theta}\int_0^\infty dx\, x^{-(1+\theta)}\widehat{P}_s^{\natural}f(x), \quad s>0,$$

for the semi-groups $(P_t, t\geq 0)$ and $(\widehat{P_s^{\natural}}, s\geq 0)$ respectively have the same potential

$$\int_0^\infty ds\,\overline{\mathbf{n}}_s\, f=C_{\alpha,\alpha\theta}\int_0^\infty f(x)x^{1/\alpha-1-\theta}dx=\int_0^\infty ds N_s^{\theta}f,$$

with $C_{\alpha,\alpha\theta}=(m^{\natural}a_{\alpha,\theta})^{-1}$. This enable us to use a result on time reversal of Kusnetzov measures established in Dellacherie, Maisonneuve & Meyer [11] XIX.33 to verify the claimed result.

(iii) We should prove that for any $q>0$ and f,g measurable positive functions

$$\int_0^\infty dx x^{1/\alpha-1-\theta}f(x)U_q g(x)=\int_0^\infty dx x^{1/\alpha-1-\theta}g(x)\mathcal{U}_q f(x),$$

with U_q the resolvent of $\widetilde{Y}$ defined in Theorem 1. Indeed, this is an elementary consequence of the identity (7) established in Lemma 2 below. Specifically,

$$\begin{aligned}
&\int_0^\infty y^{1/\alpha-1-\theta} f(y) U_q g(y) dy \\
&= \int_0^\infty y^{1/\alpha-1-\theta} f(y) V_q g(y) dy + U_q g(0) \int_0^\infty y^{1/\alpha-1-\theta} f(y)\, \mathbb{Q}_y(e^{-qT_0}) \\
&= \int_0^\infty y^{1/\alpha-1-\theta} g(y) \widehat{V^\natural}_q f(y) dy \\
&\quad + \left(\int_0^\infty x^{1/\alpha-1-\theta} g(x) \widehat{\mathbb{Q}^\natural}_x(e^{-qT_0}) dx \right) \left(b_{\alpha,\theta} q^{-\alpha\theta} \int_0^\infty y^{-(1+\theta)} \widehat{V^\natural}_q f(y) dy \right) \\
&= \int_0^\infty y^{1/\alpha-1-\theta} g(y) \widehat{V^\natural}_q f(y) dy + \mathcal{U}_q f(0) \int_0^\infty y^{1/\alpha-1-\theta} g(y) \widehat{\mathbb{Q}^\natural}_y(e^{-qT_0}) dy \\
&= \int_0^\infty y^{1/\alpha-1-\theta} g(y) \mathcal{U}_q g(y).
\end{aligned}$$

□

Lemma 2. *For every $q > 0$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ measurable*

$$b_{\alpha,\theta} \int_0^\infty y^{-(1+\theta)} \widehat{V^\natural}_q f(y) dy = C_{\alpha,\alpha\theta} \int_0^\infty y^{1/\alpha-1-\theta} f(y)\, \mathbb{Q}_y(e^{-qT_0}) dy, \tag{7}$$

with $C_{\alpha,\alpha\theta} = (m^\natural a_{\alpha,\theta})^{-1}$

We can prove Lemma 2 either by bare hands calculations or by using the following result proved by Carmona, Petit and Yor [6] Proposition 2.3.

Lemma 3. *The random variable $A_{\boldsymbol{e}}$ has a density $\rho(t) = \boldsymbol{k}\, \mathbb{Q}^\natural{}_1(Y_t^{-(1/\alpha)-\theta})$ for $t > 0$.*

Proof of Lemma 2. Let $W(x) = x^{-1/\alpha-\theta}, x > 0$. Using the fact that under $\mathbb{Q}_y$ the law of T_0 is that of $y^{1/\alpha} A_{\boldsymbol{e}}$ under $\mathbf{Q}'$, the self–similarity and the weak duality between the resolvents $V^\natural$ and $\widehat{V}^\natural$, we get

$$\begin{aligned}
&C_{\alpha,\alpha\theta} \int_0^\infty dy y^{1/\alpha-1-\theta} f(y)\, \mathbb{Q}_y(e^{-qT_0}) \\
&= C_{\alpha,\alpha\theta} \boldsymbol{k} \int_0^\infty dy y^{1/\alpha-1-\theta} f(y) \int_0^\infty dt\, \mathbb{Q}^\natural{}_1(Y_t^{-1/\alpha-\theta}) e^{-qy^{1/\alpha}t} \\
&= C_{\alpha,\alpha\theta} \boldsymbol{k} \int_0^\infty dy y^{1/\alpha-1-\theta} f(y) \int_0^\infty ds y^{-1/\alpha} y^{1/\alpha+\theta}\, \mathbb{Q}^\natural{}_y(Y_s^{-1/\alpha-\theta}) e^{-qs} \\
&= C_{\alpha,\alpha\theta} \boldsymbol{k} \int_0^\infty dy y^{1/\alpha-1} f(y) V_q^\natural W(y) \\
&= C_{\alpha,\alpha\theta} \boldsymbol{k} \int_0^\infty dy y^{1/\alpha-1} W(y) \widehat{V}_q^\natural f(y).
\end{aligned}$$

The claim follows since $b_{\alpha,\theta}/\boldsymbol{k} = C_{\alpha,\alpha\theta}$, due to identity (6), and $\boldsymbol{k} = \psi(\theta)$. □

Furthermore, Lemma 3 allows us to obtain a tail estimate for the law of T_0.

Lemma 4. *For any $x > 0$,*

$$\lim_{\epsilon \to 0+} \frac{\mathbb{Q}_x(T_0 \le \epsilon)}{\epsilon} = x^{-1/\alpha} \boldsymbol{k}.$$

Proof. First we prove that the limit exists. To this end we note that the function $s \mapsto f_s(\cdot) = \mathbb{Q}^{\natural}_{\cdot}(Y_s^{-(1/\alpha)-\theta}), s > 0$ is an exit law for the semigroup $(P^{\natural}_t, t \geq 0)$, i.e. for every $s > 0, t \geq 0$, $P^{\natural}_t f_s(x) = f_{t+s}(x), x > 0$. Thus the function $C_t(\cdot) = \int_0^t f_s(\cdot)ds$ is, in the terminology of potential theory, an "additive process"

$$C_{t+s}(\cdot) = C_t(\cdot) + P^{\natural}_t C_s(\cdot), \quad t, s \geq 0.$$

An ergodic local theorem due to Feyel [14], ensures that the limit $\lim_{t\to 0} C_t/t$, exists. In particular, the following limit exists

$$\lim_{\epsilon\to 0+} \frac{\mathbb{Q}_1(T_0 \leq \epsilon)}{\epsilon} = \lim_{\epsilon\to 0+} \frac{\mathbf{Q}'(A_{\boldsymbol{e}} \leq \epsilon)}{\epsilon} = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_0^{\epsilon} \boldsymbol{k}\, \mathbb{Q}^{\natural}{}_1(Y_s^{-(1/\alpha)-\theta})ds := a.$$

Using the self–similarity we have that under $\mathbb{Q}_x$ the law of T_0 is the same as that of $x^{1/\alpha}T_0$ under $\mathbb{Q}_1$; thus

$$\lim_{\epsilon\to 0+} \frac{\mathbb{Q}_x(T_0 \leq \epsilon)}{\epsilon} = x^{-1/\alpha}a.$$

We next prove that $a = \boldsymbol{k}$. On the one hand, we use Fatou's lemma twice to see that $a \geq \boldsymbol{k}$,

$$\begin{aligned}
a &= \liminf_{\epsilon\to 0} \frac{\boldsymbol{k}}{\epsilon}\int_0^{\epsilon} \mathbb{Q}^{\natural}{}_1(Y_s^{-(1/\alpha)-\theta})ds \\
&\geq \boldsymbol{k}\int_0^1 \liminf_{\epsilon\to 0} \mathbb{Q}^{\natural}{}_1(Y_{u\epsilon}^{-(1/\alpha)-\theta})du \\
&\geq \boldsymbol{k}\int_0^1 \mathbb{Q}^{\natural}{}_1(\liminf_{\epsilon\to 0}(Y_{u\epsilon}^{-(1/\alpha)-\theta}))du = \boldsymbol{k}.
\end{aligned}$$

On the other hand, Theorem 1 and Fatou's lemma imply that

$$\begin{aligned}
1 &= \liminf_{\epsilon\to 0} \frac{\overline{\mathbf{n}}(1 < T_0 \leq 1+\epsilon)}{\overline{\mathbf{n}}(1 < T_0 \leq 1+\epsilon)} \\
&= \liminf_{\epsilon\to 0} \frac{\epsilon}{\overline{\mathbf{n}}(1 < T_0 \leq 1+\epsilon)}\, \overline{\mathbf{n}}(\epsilon^{-1}\, \mathbb{Q}_{Y_1}(T_0 \leq \epsilon), 1 < T_0) \\
&= (cst) \liminf_{\epsilon\to 0+} \mathbb{Q}^{\natural}{}_{0+}(\epsilon^{-1}\, \mathbb{Q}_{Y_1}(T_0 \leq \epsilon) Y_1^{-\theta}) \\
&\geq (cst)\, \mathbb{Q}^{\natural}{}_{0+}(\liminf_{\epsilon\to 0+} \epsilon^{-1}\, \mathbb{Q}_{Y_1}(T_0 \leq \epsilon) Y_1^{-\theta})) \\
&\geq (cst) a\, \mathbb{Q}^{\natural}{}_{0+}(Y_1^{-(1+\alpha\theta)/\alpha)}),
\end{aligned}$$

where $cst = (\Gamma(1-\alpha\theta)/(\alpha\theta a_{\alpha,\theta}))$ and $a_{\alpha,\theta}$ is defined in Theorem 1. The rightmost hand term in the last inequality is equal to $(a/\boldsymbol{k})$, which proves $\boldsymbol{k} \geq a$. To see this we recall that $\mathbb{Q}^{\natural}{}_{0+}(Y_1^{-(1+\alpha\theta)/\alpha)}) = \frac{\alpha}{m^{\natural}}\, \mathbf{Q}^{\natural}(J^{\alpha\theta})$ by identity (5) and using (6) we get

$$(cst)\frac{\alpha}{m^{\natural}}\, \mathbf{Q}^{\natural}(J^{\alpha\theta}) = \frac{\mathbf{Q}^{\natural}(J^{\alpha\theta})}{\alpha\theta\, \mathbf{Q}^{\natural}(J^{-(1-\alpha\theta)})} = 1/\boldsymbol{k}.$$

□

Remark 4. It is interesting to observe that the preceding tail estimate is equivalent to

$$\lim_{\epsilon\to 0+} \frac{\mathbf{Q}'(A_{\boldsymbol{e}} \leq \epsilon)}{\mathbf{Q}'(\boldsymbol{e} \leq \epsilon)} = 1.$$

This a natural fact if the Lévy process ξ' does not drift to $-\infty$, $\limsup_{t\to\infty} \xi'_t = \infty$ $\mathbf{Q}$–p.s., since in this case $A_\infty = \infty$, $\mathbf{Q}'$–a.s. and therefore the small values of $A_{\boldsymbol{e}}$ should depend just on those of $\boldsymbol{e}$. Whereas, if ξ' drifts to $-\infty$ then $A_\infty < \infty$, $\mathbf{Q}'$–a.s. and it is easily deduced from Lemma 4 that $\mathbf{Q}'(A_\infty \leq \epsilon) = o(\epsilon)$.

3.1 Proof of Corollary 2

Getoor & Sharpe [16] Theorem 4.8 proved an analogous result for any Markov processes X and $\widehat{X}$ which are in duality, whose semi–groups have dual densities w.r.t. an invariant measure ζ and such that X leaves and hits continuously a recurrent regular state b. The proof of Getoor and Sharpe's result relies mainly on the fact (which they prove) that the excursion measure $\widehat{n}$ is the image under time reversal of n, with n and $\widehat{n}$ the excursion measures of X and $\widehat{X}$ from b, respectively. This relation between $\widehat{n}$ and n was proved by Mitro [23] assuming only that X and $\widehat{X}$ are weak duals and that the excursions from b start and end continuously. It follows that Theorem 4.8 in [16] is still true under these weaker hypotheses. Next, Kaspi [19] § 4 mentions that his results provide a tool to prove this result in a greater generality, namely when X does not enter or leave b continuously. However, for the sake of completeness we provide a sketch of the proof of Corollary 2.

First, we observe that versions of the processes Z_θ and $\widetilde{Y}$ can be constructed simultaneously using the same P.P.P. of excursions. More precisely, take a Poisson point process $\Delta = (\Delta_s, s \geq 0)$ with values in $\mathbb{D}^+$ and characteristic measure $\overline{\mathbf{n}}$. Thus each atom is a path and $T_0(\Delta_s)$ denotes its lifetime. We set $\sigma_t = \sum_{s \leq t} T_0(\Delta_s)$, for $t > 0$. This defines a subordinator with Laplace exponent $\phi(\lambda) = \overline{\mathbf{n}}(1 - e^{-\lambda T_0}), \lambda > 0$. Let L_t be the inverse of σ. On the one hand, the process $\widetilde{Y}$ is constructed, following [4], using this P.P.P. as we did in Chapter III.2. On the other hand, define a process $\underset{\leftarrow}{Y}$ as follows. For $t \geq 0$, let $s = L_t$, thus $\sigma_{s-} \leq t \leq \sigma_s$, and

$$\underset{\leftarrow}{Y}(t) = \begin{cases} \Delta_s((\sigma_s - t)-) & \text{if} \quad \sigma_{s-} < \sigma_s \\ 0 & \text{if} \quad \sigma_{s-} = \sigma_s \text{ or } s = 0. \end{cases}$$

Lemma 5. *The process $\underset{\leftarrow}{Y}$ is a self–similar recurrent extension of $\widehat{Y}^\natural$ and has the same law as Z_θ.*

Proof. Recall that ϱ is the function that time-reverses the trajectories at their lifetimes. The image under ϱ of Δ, say $\varrho\Delta$, still is a P.P.P. of excursions with characteristic measure $\varrho\,\overline{\mathbf{n}}$. We have that the subordinator $\widehat{\sigma}$ constructed as σ, but this time using $\varrho\Delta$, is equal to σ and

$$\underset{\leftarrow}{Y}(t) = \begin{cases} \varrho\Delta_s(t - \widehat{\sigma}_{s-}) & \text{if} \quad \widehat{\sigma}_{s-} < \widehat{\sigma}_s \\ 0 & \text{if} \quad \widehat{\sigma}_{s-} = \widehat{\sigma}_s \text{ or } s = 0. \end{cases}$$

Since $\varrho\,\overline{\mathbf{n}}$ is an excursion measure compatible with the law of $\widetilde{Y}^\natural$ we have from results in Blumenthal [4] that $\underset{\leftarrow}{Y}$ is the unique self–similar recurrent extension of $\widetilde{Y}^\natural$ whose excursion measure from 0 is $\varrho\,\overline{\mathbf{n}}$. □

Moreover, we have the equality between random sets

$$\{t > 0 : \widetilde{Y}(t)\} = \{t > 0 : \overleftarrow{Y}(t)\} = \{t > 0 : \underset{\leftarrow}{Y}(t)\},$$

and by construction it is easily seen that the processes $\overleftarrow{Y}$ and $\underset{\leftarrow}{Y}$ both started at 0, are identical. This ends the proof of Corollary 2.

4 Normalized excursion and meander for $\widetilde{Y}$

Motivated by the description of Itô's excursion measure for Brownian motion using the law of a Bessel(3) bridge, in Section III.4 we obtained a description of the excursion measure $\mathbf{n}$ of Theorem III.1

in terms the law of the excursion process conditioned to have a given length. The purpose of this section is to obtain an analogous result for the excursion measure of Theorem 1.

With the aim of giving a handy description of the excursion measure conditioned by its length, in the following proposition we construct a version of the conditional law $\overline{\mathbf{n}}(\,\cdot\,|T_0 = r)$. For any $r > 0$, define the function $h^{\natural r} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$h^{\natural r}(s, x) = \mathbb{Q}^{\natural}{}_x(Y_{r-s}^{-(1/\alpha)-\theta})1_{\{s<r\}}, \quad x > 0, s \geq 0,$$

and $h^{\natural r}(s, 0) = 0,\ s \geq 0$. Let b_r be the constant given by

$$b_r := \frac{\alpha^2\theta\, \mathbf{Q}^{\natural}(J^{-(1-\alpha\theta)})}{m^{\natural}\boldsymbol{k}} r^{-(1+\alpha\theta)} = \frac{\alpha\theta}{\boldsymbol{k}\Gamma(1-\alpha\theta)} a_{\alpha,\theta} r^{-(1+\alpha\theta)}, \tag{8}$$

with $a_{\alpha,\theta}$ defined in Theorem 1.

Proposition 1. *(i) For any $r > 0$, the function $h^{\natural r}$ is excessive for the semi–group of the space–time Markov process $((t, Y_t^{\natural}), t \geq 0)$.*

(ii) For any $r > 0$, the probability measure $\mathbf{\Lambda}^r$ over $\mathcal{G}_{r-}$ defined by

$$\mathbf{\Lambda}^r(F) = (b_r)^{-1}\, \mathbb{Q}^{\natural}{}_{0+}(F h^{\natural r}(t, Y_t)), \qquad F \in \mathcal{G}_t,\ t < r,$$

is such that for every $H \in \mathcal{G}$

$$\overline{\mathbf{n}}(H) = \frac{\alpha\theta}{\Gamma(1-\alpha\theta)} \int_0^{\infty} \mathbf{\Lambda}^r(H) \frac{dr}{r^{1+\alpha\theta}}.$$

Proof. The proof of (i) is a straightforward consequence of the Markov property.

For any $r > 0$, let $\overline{\mathbf{\Lambda}}^r$ be the h transform of the space-time process over $Y^{\natural}$ with law $\mathbb{Q}^{\natural}{}_{0+}$, via the excessive function $h^{\natural r}$. Then under $\overline{\mathbf{\Lambda}}^r$ the space process $Y^{\natural}$ is an inhomogeneous Markov process with entrance law

$$\overline{\mathbf{\Lambda}}^r_s(f) = \mathbb{Q}^{\natural}{}_{0+}(f(Y_s)h^{\natural r}(s, Y_s)), \qquad s > 0,$$

and for $s, t \geq 0$ its transition probabilities are given by

$$K^r_{t,t+s}(x, dy) = \frac{P^{\natural}_s(x, dy)h^{\natural r}(t + s, y)}{h^{\natural r}(t, x)}, \quad x > 0, y > 0,$$

where the quotient is taken to be 0 if the denominator is 0. The measure $\overline{\mathbf{\Lambda}}^r$ is a finite measure with total mass

$$\begin{aligned}
\overline{\mathbf{\Lambda}}^r(1) &= \lim_{s\to 0} \overline{\mathbf{\Lambda}}^r_s(1) \\
&= \lim_{s\to 0} \mathbb{Q}^{\natural}{}_{0+}(h^{\natural r}(s, Y_s)) \\
&= \mathbb{Q}^{\natural}{}_{0+}(Y_r^{-(1/\alpha)-\theta}) \\
&= b_r < \infty,
\end{aligned}$$

where the third equality is a consequence of the Markov property and the fourth follows from (5). To finish the proof we just have to prove that the probability measures $\mathbf{\Lambda}^r := (b_r)^{-1}\overline{\mathbf{\Lambda}}^r$ satisfy the identity in (ii) of Proposition 1. To that end it suffices to show the identity for any F_t of the form

$F_t = F \cap \{t < T_0\}, F \in \mathcal{G}_t, t > 0$. Indeed, recall from Theorem 1 that for every positive and $\mathcal{G}_t$–measurable H_t we have

$$\overline{\mathbf{n}}(H_t, t < T_0) = (a_{\alpha,\theta})^{-1}\,\mathbb{Q}^{\natural}{}_{0+}(H_t Y_t^{-\theta}),$$

and the expression for b_1 in (8). Therefore, using Fubini's theorem and that the law of T_0 under $\mathbb{Q}_x$ for $x > 0$ has a density

$$\mathbb{Q}_x(T_0 \in ds)/ds = \boldsymbol{k} x^{\theta}\,\mathbb{Q}^{\natural}{}_x(Y_s^{-(1/\alpha)-\theta}), \qquad x > 0$$

and $\mathbb{Q}_0(T_0 \in ds) = \delta_0(ds)$, we get that

$$\begin{aligned}
\overline{\mathbf{n}}(F \cap \{t < T_0\}) &= \overline{\mathbf{n}}(1_F \int_t^{\infty} \boldsymbol{k} Y_t^{\theta}\,\mathbb{Q}^{\natural}{}_{Y_t}(Y_{r-t}^{-(1/\alpha)-\theta}) dr) \\
&= \boldsymbol{k} \int_0^{\infty} \overline{\mathbf{n}}\left(1_F Y_t^{\theta}\,\mathbb{Q}^{\natural}{}_{Y_t}(Y_{r-t}^{-(1/\alpha)-\theta}) 1_{\{t<r\}}\right) dr \\
&= \frac{\alpha\theta}{\Gamma(1-\alpha\theta)} \int_0^{\infty} \frac{dr}{r^{1+\alpha\theta}} (b_r)^{-1}\,\mathbb{Q}^{\natural}{}_{0+}(1_F h^{\natural r}(t, Y_t)) \\
&= \int_0^{\infty} \frac{dr}{r^{1+\alpha\theta}} \boldsymbol{\Lambda}^r(F \cap \{t < T_0\}),
\end{aligned}$$

where the last equality holds due to the fact that $\boldsymbol{\Lambda}^r$ is an h–transform of $\mathbb{Q}^{\natural}{}_{0+}$. □

By an argument similar to that given in the previous proof it is proved that for any $x > 0$, $t > 0$ and positive measurable g,

$$\mathbb{Q}_x(F_t \cap \{t < T_0\} g(T_0)) = \int_0^{\infty} g(r) \boldsymbol{k} x^{\theta}\,\mathbb{Q}^{\natural}{}_x(Y_r^{-(1/\alpha)-\theta}) \frac{\mathbb{Q}^{\natural}{}_x(F_t h^{\natural r}(t, X_t))}{h^{\natural r}(0, x)} dr, \qquad F_t \in \mathcal{G}_t.$$

That is, the h–transform of the spac–time process $((t, Y_t^{\natural}), t \geq 0)$ started at $(0, x)$ via the excessive function $h^{\natural r}$ is a version of the conditional law

$$\mathbb{Q}_x(\cdot \mid T_0 = r).$$

As a consequence, the transition probabilities $K^r_{t,t+s}$ defined in the proof of Proposition 1 are those of Y conditioned to hit 0 at time r.

When the process $Y = X^0$ is a stable process X killed at its first hitting time of the set $]-\infty, 0]$, Chaumont [10] proved that the law of the excursion process conditioned to have a given length is absolutely continuous w.r.t. the law of the stable meander process. An analogous result still holds in our setting. To give a precise statement we next recall the definition of the law of the meander process. For any $r > 0$, the probability measure M^r defined over $\mathbb{D}^+([0, r])$ by

$$M^r(\cdot) := \overline{\mathbf{n}}(\cdot \circ k_r, T_0 > r)/\,\overline{\mathbf{n}}(T_0 > r),$$

with k_r the killing operator at time $r > 0$, is called the law of the meander process. This corresponds to the law of the process $(\widetilde{Y}_{g_t+s}, 0 \leq s \leq t - g_t)$ conditioned by $t - g_t = r$ for some $t > r$ and g_t the last hitting time of 0 before t, $g_t = \sup\{s \leq t : \widetilde{Y}_s = 0\}$, cf. Getoor [15].

We can now state a corollary to Proposition 1 which is the analogue of Theorem 3 in [10]:

Corollary 3. *For any $r > 0$, $t < r$ and $F \in \mathcal{G}_t$ we have that*

$$\boldsymbol{\Lambda}^r(F) = \frac{r\boldsymbol{k}}{\alpha\theta} M^r(F Y_r^{-1/\alpha}).$$

Proof. On the one hand, by the very definition of the law of the meander and Theorem 1 we have that

$$M^r(F) = \frac{r^{\alpha\theta}\Gamma(1-\alpha\theta)}{a_{\alpha,\theta}} \mathbb{Q}^{\natural}{}_{0+}(F\ Y_r^{-\theta}).$$

On the other hand, by Proposition 1 and the Markov property we have that

$$\Lambda^r(F) = (b_r)^{-1}\,\mathbb{Q}^{\natural}{}_{0+}(Fh^{\natural r}(t,Y_t)) = (b_r)^{-1}\,\mathbb{Q}^{\natural}{}_{0+}(F\ Y_r^{-(1/\alpha)-\theta}).$$

The result follows by identifying the constants. □

The law of the excursion process conditioned by its length Λ^r constructed in Chapter III.4 can be thought of as the law of a bridge for the process with law $\mathbb{E}^{\natural}_{0+}$ because the excursion hits 0 continuously. In fact, it can be proved that for every $t < r$ and $F \in \mathcal{G}_t$,

$$\Lambda^r(F) = \lim_{\epsilon\to 0} \mathbb{E}^{\natural}_{0+}(F|X_r \leq \epsilon).$$

The arguments used to prove a such result are similar to those given in [10] Lemme 2 and we omit them. An analogue result does not have meaning for the law $\mathbf{\Lambda}^r$ since the excursions are ended by a jump to 0 a.s. However, the following identity holds for any $r > 0$,

$$\mathbf{\Lambda}^r(\cdot) = \lim_{\epsilon\to 0} \overline{\mathbf{n}}(\cdot|r < T_0 \leq r+\epsilon). \tag{9}$$

This can be proved as in [10] or using the tail estimation in Lemma 4. Indeed, using the Markov property and a dominated convergence argument we have that for any $r > 0$, $t < r$ and $F \in \mathcal{G}_t$

$$\begin{aligned}\overline{\mathbf{n}}(F|r < T_0 \leq r+\epsilon) &= \frac{\epsilon}{\overline{\mathbf{n}}(r < T_0 \leq r+\epsilon)}\,\overline{\mathbf{n}}(F \cap \{r < T_0\}\left[\mathbb{Q}_{Y_r}(T_0 \leq \epsilon)/\epsilon\right])\\ &\sim (\boldsymbol{k}r^{1+\alpha\theta}\Gamma(1-\alpha\theta)/\alpha\theta)\,\overline{\mathbf{n}}(FY_r^{-1/\alpha}),\end{aligned}$$

as $\epsilon \longrightarrow 0$. By the Markov property and Proposition 1

$$(\boldsymbol{k}r^{1+\alpha\theta}\Gamma(1-\alpha\theta)/\alpha\theta)\,\overline{\mathbf{n}}(FY_r^{-1/\alpha}) = (cst)\,\mathbb{Q}^{\natural}{}_{0+}(Fh^{\natural r}(t,Y_t)) = \mathbf{\Lambda}^r(F),$$

with $cst = (\boldsymbol{k}r^{1+\alpha\theta}\Gamma(1-\alpha\theta)/\alpha\theta)(a_{\alpha,\theta})^{-1} = (b_r)^{-1}$.

Remark 5. The law of the excursion process conditioned by its length can be defined following Chaumont [10] since most of his arguments are easily generalized to any self–similar Markov process. The resulting measure is equal to the law $\mathbf{\Lambda}^r$ constructed here. We omit the details.

5 The process conditioned to hit 0 continuously

For the moment we leave aside hypotheses (HI-b,c) of Section 2 and work instead under hypotheses

(HI-d) there exists $\gamma < 0$ for which $\mathbf{Q}(e^{\gamma\xi_1}1_{\{1<\zeta\}}) = 1$.

(HI-e) $\mathbf{Q}(\xi_1^- e^{\gamma\xi_1}1_{\{1<\zeta\}}) < \infty$.

Under these hypotheses we will prove the existence of a self–similar Markov process $\mathrm{Y}^{\downarrow}$ that can be thought of as Y conditioned to hit 0 continuously.

The hypothesis (HI-d) implies that under $\mathbf{Q}$ the function $h^{\downarrow}(x) = e^{\gamma x}, x \in \mathbb{R}$ is an invariant function for the semigroup of the Lévy process with law $\mathbf{Q}$. Let $\mathbf{Q}^{\downarrow}$ be the h–transform of $\mathbf{Q}$ via the invariant function $h^{\downarrow}$. Under $\mathbf{Q}^{\downarrow}$ the canonical process is still a Lévy process with infinite lifetime that drifts to $-\infty$. Furthermore, by hypothesis (HI-e), we have that $m^{\downarrow} = \mathbf{Q}^{\downarrow}(\xi_1) \in]-\infty, 0[$. We will be interested in the self–similar Markov process $\mathrm{Y}^{\downarrow}$ of law $(\mathbb{Q}^{\downarrow}{}_x, x \geq 0)$, which is the Markov process associated to the Lévy process with law $\mathbf{Q}^{\downarrow}$ via Lamperti's transformation. Since the Lévy process $\xi^{\downarrow}$ drifts to $-\infty$ we have that $\mathrm{Y}^{\downarrow}$ hits 0 continuously at some finite time $\mathbb{Q}^{\downarrow}{}_x$ a.s. for every $x > 0$. As a consequence of the following result we will refer to $\mathrm{Y}^{\downarrow}$ as the process Y conditioned to hit 0 continuously.

Proposition 2. *(i) For any $x > 0$, we have that $\mathbb{Q}^{\downarrow}{}_x$ is the unique measure such that for every $\mathcal{G}_t$–stopping time T we have*

$$\mathbb{Q}^{\downarrow}{}_x(F_T, T < T_0) = x^{-\gamma}\,\mathbb{Q}_x(F_T Y_T^{\gamma}, T < T_0),$$

for every $F_T \in \mathcal{G}_T$.

(ii) For every $x > 0, t > 0$ we have

$$\lim_{\epsilon \to 0} \mathbb{Q}_x(F_t \cap \{t < T_0\} | Y_{T_0 -} \leq \epsilon) = \mathbb{Q}^{\downarrow}{}_x(F_t \cap \{t < T_0\}), \qquad F_t \in \mathcal{G}_t.$$

The proof of (i) in Proposition 2 is an immediate consequence of the fact that $\mathbf{Q}^{\downarrow}$ is an h–transform. To prove (ii) we will need the following Lemma in which we determine the tail distribution of a Lévy process at a given exponential time.

Lemma 6. *Let σ be a Lévy process of law P. Assume that σ is non–arithmetic and that there exists $\vartheta > 0$ for which $1 < E(e^{\vartheta\sigma_1}) < \infty$, and $E(\sigma_1^+ e^{\vartheta\sigma_1}) < \infty$. Let T_λ be a exponential random variable of parameter $\lambda = \log E(e^{\vartheta\sigma_1})$ and independent of σ. We have that*

$$\lim_{x \to \infty} e^{\vartheta x} P(\sigma_{T_\lambda} \geq x) = \frac{1 - e^{-\lambda} + \lambda}{\mu^{\natural}\vartheta},$$

with $\mu^{\natural} = \mathbf{Q}(\sigma_1 e^{\vartheta\sigma_1})$.

Lemma 6 is a consequence of the renewal theorem for real–valued random variables and an application of Cramér's method as explained by Feller [13] §XI.6.

Proof. The following three claims enable us to put Lemma 6 in a context similar to that of [13] XI.6. First, the function $Z(x) = P(\sigma_{T_\lambda} < x)$, satisfies a renewal equation. More precisely, for $z(x) = \int_0^1 dt \lambda e^{-\lambda t} P(\sigma_t < x)$ and $L(dy) = e^{-\lambda} P(\sigma_1 \in dy)$ we have that

$$Z(x) = z(x) + \int_{-\infty}^{\infty} L(dy) Z(x - y).$$

This is an elementary consequence of the fact that the process $(\sigma'_s = \sigma_{1+s} - \sigma_1, s \geq 0)$ is a Lévy process independent of $(\sigma_r, r \leq 1)$ with the same law as σ. Second, the measure L is a defective law, $L(\mathbb{R}) < 1$, such that

$$\int_{-\infty}^{\infty} e^{\vartheta y} L(dy) = e^{-\lambda} E(e^{\vartheta\sigma_1}) = 1; \quad \text{and} \quad \int_{-\infty}^{\infty} y e^{\vartheta y} L(dy) < \infty,$$

by hypothesis. Third, the function $z^{\natural}(x) = e^{\vartheta x}(z(\infty) - z(x))$ is directly Riemann integrable; with

$$z(\infty) = \lim_{x\to\infty} z(x) = \int_0^1 dt\lambda e^{-\lambda t} P(\sigma_t < \infty) = 1 - e^{-\lambda}.$$

The latter follows using the fact that $z^{\natural}(x) = e^{\vartheta x}\int_0^1 dt\lambda e^{-\lambda t} P(\sigma_t \geq x)$, is the product of an exponential function and a decreasing one and that $z^{\natural}$ is integrable. To see that $z^{\natural}$ is integrable, use Fubini's theorem to establish

$$\begin{aligned}\int_{-\infty}^{\infty} z^{\natural}(x)dx &= \int_0^1 dt\lambda e^{-\lambda t} E\left(\int_{-\infty}^{\infty} dx e^{\vartheta x} 1_{\{\sigma_t \geq x\}}\right)\\ &= \frac{1}{\vartheta}\int_0^1 dt\lambda e^{-\lambda t} E(e^{\vartheta \sigma_t})\\ &= \frac{\lambda}{\vartheta} < \infty.\end{aligned}$$

Therefore, we can repeat the arguments given in the proof of Theorem XI.6.2 in [13] but this time using the renewal theorem for real–valued random variables to prove that

$$\begin{aligned}\lim_{x\to\infty} e^{\vartheta x} P(\sigma_{T_\lambda} \geq x) &= \lim_{x\to\infty} e^{\vartheta x}(Z(\infty) - Z(x))\\ &= \frac{z(\infty)}{\mu^{\natural}\vartheta} + \frac{\int_{-\infty}^{\infty} z^{\natural}(x)dx}{\mu^{\natural}} = \frac{1 - e^{-\lambda} + \lambda}{\mu^{\natural}\vartheta}.\end{aligned}$$

□

Proof of Proposition 2 (ii). Observe that under $\mathbb{Q}_x$ the random variable Y_{T_0-} has the same law as $x\exp(\xi'_{\boldsymbol{e}})$ under $\mathbf{Q}'$, with $\boldsymbol{e}$ an exponential random variable of parameter $\boldsymbol{k} = \psi(\theta) = \psi(\gamma) > 0$, and independent of ξ'. Moreover, applying Lemma 6 to $-\xi'$ under $\mathbf{Q}'$ we obtain by hypotheses (HI-d) that

$$\lim_{y\to\infty} e^{-\gamma y}\,\mathbf{Q}'(\xi_{\boldsymbol{e}} \leq -y) = \frac{1 - e^{-\boldsymbol{k}} + \boldsymbol{k}}{\gamma\mu^{\downarrow}} := \mathrm{d}_{\boldsymbol{k}},$$

with $\mu^{\downarrow} = \mathbf{Q}'(\xi_1 e^{\gamma\xi_1}) \in]-\infty, 0[$, which is finite by hypothesis (HI-e). Thus, we have the following estimate of the left tail distribution of Y_{T_0-}:

$$\lim_{\epsilon\to 0} \epsilon^{\gamma}\,\mathbb{Q}_x(Y_{T_0-} \leq \epsilon) = x^{\gamma}\mathrm{d}_{\boldsymbol{k}}. \tag{10}$$

The proof of (ii) in Proposition 2 now follows by a standard application of the Markov property, estimate (10) and a dominated convergence argument. □

In the sequel, we will assume in addition that the hypotheses (HI-b,c) are satisfied. This implies in turn that the hypotheses (H2) of Chapter III are satisfied. Indeed, for $\hat{\theta} = \theta - \gamma$ we have that

$$\mathbf{Q}^{\downarrow}(e^{\hat{\theta}\xi_1}) = \mathbf{Q}(e^{\theta\xi_1} 1_{\{1<\zeta\}}) = 1,$$

and

$$\mathbf{Q}^{\downarrow}(\xi_1^+ e^{\hat{\theta}\xi_1}) = \mathbf{Q}(\xi_1^+ e^{\theta\xi_1} 1_{\{1<\zeta\}}) < \infty,$$

by hypotheses (HI-b) and (HI-c), respectively. By hypothesis (HI-e) we have that

$$\mathbf{Q}^{\downarrow}(\xi_1) = m^{\downarrow} \in]-\infty, 0[.$$

Since the measure $\mathbf{Q}^{\downarrow}$ satisfies the hypotheses (H2) of Chapter III we know from Theorem III.1 that if $0 < \alpha(\theta - \gamma) < 1$ then there exists a unique excursion measure $\mathbf{n}^{\downarrow}$ compatible with the semigroup of $(\mathrm{Y}^{\downarrow}, T_0)$, and associated to it a self–similar recurrent extension of $(\mathrm{Y}^{\downarrow}, T_0)$, say $\widetilde{\mathrm{Y}^{\downarrow}}$. The absolutely continuity relations in part (i)of Proposition 2 are inherited by the excursion measures $\overline{\mathbf{n}}$ and $\mathbf{n}^{\downarrow}$. More precisely, for every $\mathcal{G}_t$ stopping time, T, we have that

$$\mathbf{n}^{\downarrow}(F_T, T < T_0) = c_\gamma\, \overline{\mathbf{n}}(F_T Y_T^{\gamma}, T < T_0), \qquad F_T \in \mathcal{G}_T, \tag{11}$$

with $\overline{\mathbf{n}}$ the excursion measure of Theorem 1 and

$$c_\gamma = \frac{\mathbf{Q}^{\natural}(J^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)}{\mathbf{Q}^{\natural}(J^{-(1-\alpha\theta+\alpha\gamma)})\Gamma(1-\alpha\theta+\alpha\gamma)}.$$

To see this we just have to note that the measure $\mathbf{Q}^{\downarrow\natural}$ obtained by h-transforming $\mathbf{Q}^{\downarrow}$ via the invariant function $h_{\theta-\gamma}(x) = e^{(\theta-\gamma)x}, x \in \mathbb{R}$, is identical to the measure $\mathbf{Q}^{\natural}$ constructed in Section 2.

Furthermore, it is natural to hope that the conditioning on hitting 0 continuously should act just at the end of the excursions. This let us guess that the meander processes associated to $\widetilde{Y}$ and $\widetilde{\mathrm{Y}^{\downarrow}}$ should be related. This is indeed the case; a standard calculation shows that for every $r > 0$ the meander processes of length r, $(\widetilde{Y}, M^r)$ and $(\widetilde{\mathrm{Y}^{\downarrow}}, M^{\downarrow,r})$ are identical in law conditionally on their values at time r,

$$M^r(\cdot|Y_r = x) = M^{\downarrow,r}(\cdot|Y_r = x), \qquad x > 0,$$

in the obvious notation.

Regardless of the value of $\alpha(\theta - \gamma)$, we can always construct a pseudo–excursion measure $\mathbf{n}^{\downarrow}$ as an h–transform of $\overline{\mathbf{n}}$ via the excessive function $x^{\gamma}, x > 0$, and this pseudo–excursion measure is still compatible with the minimal process $(\mathrm{Y}^{\downarrow}, T_0)$. For us a pseudo–excursion measure has the same properties as an excursion measure except that it is possible that it does not integrate $1 - e^{-T_0}$. The latter holds if and only if $1 \leq \alpha(\theta - \gamma)$.

Remark 6. Observe that another consequence of Lemma 6 is that

$$\lim_{y\to\infty} y^{\theta}\, \mathbb{Q}_1(Y_{T_0-} \geq y) = \frac{1 - e^{-\boldsymbol{k}} + \boldsymbol{k}}{\theta m^{\natural}}.$$

6 Examples

6.1 Further details for stable processes

In the Introduction we noted (X, P) a real valued a–stable process with negative jumps and we assume that X is not the negative of a subordinator. Since X is a Lévy process its law is determined by its characteristic exponent which in turn can be described as

$$E(e^{i\lambda X_1}) = \exp\{-c|\lambda|(1 - i\beta \mathrm{sgn}(\lambda)\tan(a\pi/2))\} \qquad \lambda \in \mathbb{R}, c > 0, \beta \in [-1, 1[$$

(the case $\beta = 1$ is excluded since we assume that X has some negative jumps). The case where X does not have negative jumps enters in the setting considered in Chapter III. The Lévy measure of X has the form

$$\mathbf{\Pi}(dx) = C_+ x^{-1-a} 1_{\{x>0\}} + C_-|x|^{-1-a} 1_{\{x<0\}} dx,$$

for some constants $C_+, C_- \geq 0$ such that $\beta = C_+ - C_-/C_+ + C_-$. In a recent work, Caballero and Chaumont [5] determined explicitly the characteristics of the Lévy process $(\xi, \mathbf{Q})$ associated via Lamperti's transformation to the positive $1/a$–self–similar Markov process (X^0, P^0). The process $(\xi, \mathbf{Q})$ is a Lévy process whose characteristic exponent is given by

$$\Psi(\lambda) = \boldsymbol{k} + id\lambda + \int_{\mathbb{R}\setminus\{0\}} (e^{i\lambda x} - 1 - ix\lambda 1_{\{|x|<1\}})\Pi(dx), \lambda \in \mathbb{R},$$

where $\boldsymbol{k} = \lim_{s\to 0} s^{-1} P(T_{]-\infty,0[} \leq s)$ (this limit was calculated by Bingham [3]), $d \in \mathbb{R}$ is a drift coefficient whose value is not important for us here and

$$\Pi(dx) = C_+(e^x(e^x - 1)^{-1-a})1_{\{x>0\}} + C_-(e^x|e^x - 1|^{-1-a})1_{\{x<0\}}dx,$$

see [5] for the details. We have to verify that this Lévy process satisfies the conditions (HI) in order to apply our results to stable processes. Recall that to pass from the process X^0 to the process ξ we have to make the transformation

$$\xi_t = \log(X_{\varphi^{-1}(t)}), \quad \text{with } \varphi^{-1}(t) \text{ the inverse of } \varphi(t) = \int_0^t (X^0_s)^{-a} ds, \quad t < T_0.$$

Indeed, ξ is not arithmetic since the stable process is not. To verify that (HI-b) holds, we recall that the function $h_\rho(x) = x^{a(1-\rho)}x \geq 0$ is invariant for (X^0, P^0), see Silverstein [25] or Chaumont [10]. Since the measure $P^\natural$ is the h–transform of P^0 via the invariant function h_ρ we have that for every stopping time T in the filtration of X^0 we have

$$P^\natural_x(T < \infty) = x^{-a(1-\rho)} P^0_x(X_T^{a(1-\rho)} 1_{\{T<T_0\}}).$$

In particular, for $T = \varphi^{-1}(t)$ with $t > 0$, which is a stopping time for X^0, we have

$$P^\natural_x(1_{\{\varphi^{-1}(t)<\infty\}}) = x^{-a(1-\rho)} P^0_x(X^{a(1-\rho)}_{\varphi^{-1}(t)} 1_{\{\varphi^{-1}(t)<T_0\}}) = \mathbf{Q}(e^{a(1-\rho)\xi_t} 1_{\{t<\zeta\}}) = \mathbf{Q}(e^{a(1-\rho)\xi_t}),$$

and the leftmost term is equal to 1 since Lamperti [21] Lemma 3.1 proved that whenever the self–similar Markov process never hits 0 we have $\varphi(\infty) = \infty$ a.s. independently of the starting point, which is indeed the case under $P^\natural$. According to Sato [24] Theorem 25.3, the condition (HI-c) is equivalent to

$$\int_{\{x>1\}} xe^{a(1-\rho)x} \frac{e^x}{(e^x - 1)^{1+a}} dx < \infty,$$

and that the latter holds is straightforward. We have thus proved that the conditions (HI) are satisfied by the Lévy process associated to a stable process killed at $]-\infty, 0[$ with $\theta = a(1-\rho)$, and since the self–similarity index is $\alpha = 1/a$ we have that $0 < \alpha\theta = 1 - \rho < 1$. In this particular case most of the results in Section 2 are well known, see [10]. The recurrent extension of X^0 is exactly the process X reflected at its infimum $(X - \underline{X}, P)$, since it is a strong Markov process that leaves 0 continuously and its excursion measure n is the unique excursion measure compatible with the law P^0 such that $n(X^0_{0+} > 0) = 0$ and $n(1 - e^{-T_0}) < \infty$.

We will denote by (X^*, P^*) the dual stable process $(X^*, P^*) = (-X, P)$, by $(X^{*,0}, P^{*,0})$ the dual stable process killed at $]-\infty, 0[$ and by $\overline{X}^*_t = \sup_{s\leq t} X^*, t \geq 0$. One can construct the dual stable process conditioned to stay positive $(X^{*,\uparrow}, P^{*,\uparrow})$ analytically, as an h–transform of $P^{*,0}$ via the invariant function $x^{a\rho}, x \geq 0$, or pathwise, by using Tanaka's method [26]; that is $X^{*,\uparrow}$ is obtained by time–reversing one by one the excursions from 0 of the process X reflected at its supremum $(\overline{X}^* - X^*, P^*)$.

For details on the latter construction see the recent work of Doney [12]. From Doney's construction it is easily deduced that the process

$$R_t = \begin{cases} (\overline{X}^* - X^*)_{(d_t-(t-g_t))-} & \text{if } 0 < g_t \leq d_t < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where $g_t = \sup\{s < t : (\overline{X}^* - X^*)_s = 0\}$ and $d_t = \inf\{s > t : (\overline{X}^* - X^*_s) = 0\}$, has the same distribution under P^*_0 as the process $X^{*,\uparrow} - \underline{\underline{X}}^{*,\uparrow}$ under $P^{*,\uparrow}_{0+}$, where $\underline{\underline{X}}^{*,\uparrow}_t = \inf\{X^{*,\uparrow}_s, s \geq t\}$, and $P^{*,\uparrow}_{0+}$ is the limit in the Skorohod sense of $P^{*,\uparrow}_x$ as $x \to 0+$, see [9] Theorem 6. It follows that under P^*_0 the Poisson point process of excursions from 0 of R has the same law as that under $P^{*,\uparrow}_{0+}$ of $X^{*,\uparrow} - \underline{\underline{X}}^{*,\uparrow}$. Furthermore the former is the image under ϱ of the P.P.P. of excursions of $\overline{X}^* - X^*$ under P^*_0. Therefore, if $\underline{\underline{n}}$ is the excursion measure of $X^{*,\uparrow} - \underline{\underline{X}}^{*,\uparrow}$, we have that the image under time reversal of n is $\underline{\underline{n}}$. We borrow the following lemma from Chaumont [9] Theorem 5.

Lemma (Chaumont [9]). *Let $m = \sup\{t > 0 : X^{*,\uparrow}_t = \inf_{s \leq t} X^{*,\uparrow}_s\}$, and $X^{*,\uparrow}_m$ the absolute minimum. Under $P^{*,\uparrow}_x$, $x > 0$, the process $X^{*,\uparrow}$ reaches $X^{*,\uparrow}_m$ once only and the processes $(X^{*,\uparrow}_s - \underline{\underline{X}}^{*,\uparrow}_0, 0 \leq s \leq m)$ and $(X^{*,\uparrow}_s - \underline{\underline{X}}^{*,\uparrow}_s, m < s)$ are independent. Under $P^{*,\uparrow}_x$, conditionally on $\underline{X}^{*,\uparrow}_m = y$, $0 < y \leq x$, the law of the former is $P^{*,\downarrow}_{x-y}$ and the latter has the same law as $(X^{*,\uparrow}_s - \underline{\underline{X}}^{*,\uparrow}_s, s > 0)$ under $P^{*,\uparrow}_{0+}$.*

Moreover, under $\underline{\underline{n}}$ the excursion process is Markovian with semigroup

$$p^{*,\downarrow}_t(x, dy) = \frac{p^{*,0}(x, dy) y^{a\rho-1}}{x^{a\rho-1}},$$

that is, the h–transform of $(X^{0,*}, P^{*,0})$ via the excessive function $h'_{1-\rho}(x) = x^{a\rho-1}, x \geq 0$. We denote by $P^{*,\downarrow}$ the law of this h–transform.

The law $P^{*,\downarrow}$ is that of a self–similar Markov process that hits 0 continuously and then dies at 0. Thus, associated to $\underline{\underline{n}}$ and $P^{*,\downarrow}$ there is a self–similar Markov process Z that is a recurrent extension of $(X^{*,\downarrow}, T^{*,\downarrow}_0)$; this is, indeed, the process $Z_{a(1-\rho)}$ of Theorem 2 (iii). We denote its law by $\widetilde{\mathbb{Q}}^{\wedge}_{\cdot}$. We claim that under $P^{*,\uparrow}_x, x > 0$, conditionally on $X^{*,\uparrow}_m = y$, $0 < y \leq x$, the process $X^{*,\uparrow} - \underline{\underline{X}}^{*,\uparrow}$ has the same law as Z started at $x - y$. By a monotone class argument, to see this it suffices to prove that for all bounded measurable functionals F, G and all bounded measurable functions g we have that

$$\begin{aligned} &P^{*,\uparrow}_x(g(X_m)F(X^{*,\uparrow}_s - \underline{\underline{X}}^{*,\uparrow}_s, s \leq m)G(X^{*,\uparrow}_s - \underline{\underline{X}}^{*,\uparrow}_s, s > m)) \\ &= P^{*,\uparrow}_x(g(X_m)\,\widetilde{\mathbb{Q}}^{\wedge}_{x-X_m}(F(Z_s, s \leq T_0)G(Z_s, s > T_0))). \end{aligned}$$

Indeed, due to the preceding lemma, the left-hand side of the above equation is equal to

$$P^{*,\uparrow}_x(g(X_m)P^{*,\downarrow}_{x-X_m}(F(X_s, s \leq T_0))P^{*,\uparrow}_{0+}(G(X^{*,\uparrow}_s - \underline{\underline{X}}^{*,\uparrow}_s, s > 0))),$$

and, by the Markov property applied at time T_0, the right–hand side is equal to

$$P^{*,\uparrow}_x(g(X_m)\,\widetilde{\mathbb{Q}}^{\wedge}_{x-X_m}(F(Z_s, s \leq T_0))\,\widetilde{\mathbb{Q}}^{\wedge}_0(G(Z_s, s > 0))),$$

Using the fact that Z is a recurrent extension of $X^{*,\downarrow}$ we have that for any $z > 0$

$$\widetilde{\mathbb{Q}}^{\wedge}_z(F(Z_s, s \leq T_0)) = P^{*,\downarrow}_z(F(X_s, s \leq T_0)),$$

and, given that the processes Z and $X^{*,\uparrow} - \underline{\underline{X}}^{*,\uparrow}$ can be recovered from their respective Poisson point processes of excursions from 0, and that these have the same law since they have the same excursion measure, we get that

$$\widetilde{\mathbb{Q}}_0^{\wedge}(G(Z_s, s > 0)) = P_{0+}^{*,\uparrow}(G(X_s^{*,\uparrow} - \underline{\underline{X}}_s^{*,\uparrow}, s > 0)),$$

and the claim follows.

Another consequence of the results of Silverstein [25] is that the function $h_\rho^{'}(x) = x^{a(1-\rho)-1}, x > 0$ is excessive for the semigroup of (X^0, P^0). Then the process $(X^{\downarrow}, P^{\downarrow})$ which is the h–transform of (X^0, P^0) via the function $h_\rho^{'}$ is a self–similar Markov process that hits 0 continuously, see Chaumont [10]. Thus according to Lamperti [21], the Lévy process $(\xi^{\downarrow}, \mathbb{Q}^{\downarrow})$ associated to $(X^{\downarrow}, P^{\downarrow})$ is a Lévy process with infinite lifetime and that drifts to $-\infty$, namely it is the Lévy process $(\xi, \mathbf{Q})$ conditioned to drift to $-\infty$. To see this, we claim that the function $e^{(a(1-\rho)-1)x}, x \in \mathbb{R}$ is invariant for $(\xi, \mathbf{Q})$. Indeed, by properties of h-transformations we have for the stopping time $\varphi^{-1}(1)$ that

$$\begin{aligned} P_x^{\downarrow}(\varphi^{-1}(1) < T_0) &= P_x^0\left(X_{\varphi^{-1}(1)}^{a(1-\rho)-1}, \varphi^{-1}(1) < T_0\right)/h_\rho^{'}(x) \\ &= \mathbf{Q}(\exp\{(a(1-\rho)-1)\xi_1\}, 1 < \zeta) \\ &= \mathbf{Q}(\exp\{(a(1-\rho)-1)\xi_1\}). \end{aligned}$$

The leftmost term in the preceding equality is equal to $P_x^{\downarrow}(1 < \varphi(T_0)) = 1$ since $\varphi(T_0) = \infty$ $P_x^{\downarrow}$–a.s. for any $x > 0$, see [21] Lemma 3.3. Therefore, the law

$$\mathbf{Q}^{\downarrow}|_{D_t} = e^{(a(1-\rho)-1)\xi_t}\,\mathbf{Q}|_{D_t}, \qquad t \geq 0,$$

is that of a Lévy process with infinite lifetime. We also have that $\mathbf{Q}^{\downarrow}(e^{\xi_1}) = 1$, since $1 = a(1-\rho) - (a(1-\rho)-1)$, and as a consequence under $\mathbf{Q}^{\downarrow}$ the Lévy process $\xi^{\downarrow}$ drifts to $-\infty$. By arguments similar to those given in Section 2 we verify that the self–similar Markov process associated to $(\xi^{\downarrow}, P^{\downarrow})$ is equivalent to $(X^{\downarrow}, P^{\downarrow})$. Observe that, in general, $a(1-\rho)-1 < 0$ and thus $\gamma = a(1-\rho)-1$ is the only candidate to satisfy the hypotheses (HI–d,e) under $\mathbf{Q}$. We have already verified (HI–d) and using an argument similar to that used to verify that (HI-c) holds we get that (HI-e) holds. In this case the measure $\mathbf{n}^{\downarrow}$ constructed in Section 5 is equal to the one constructed by Chaumont [8] section 2.4 and plays an important rôle in obtaining pathwise transformations.

6.2 On the excursions that leave 0 by a jump and hit 0 continuously

Let $\mathbb{P}_x, x \geq 0$, be the law of a self–similar Markov process X such that under $\mathbb{P}_x$, X hits 0 continuously in a finite time:

$$\mathbb{P}_x(T_0 < \infty, X_{T_0-} = 0) = 1 \qquad \text{for all } x > 0,$$

and that 0 is a cemetery point. Assume that the Lévy process associated to X via Lamperti's transformation satisfies the hypothesis (H2) in Chapter III. Then in Chapter III we proved that the recurrent extension of (X, T_0) that leaves and hits 0 continuously admits a weak dual whose excursion measure is the image under time reversal of $\mathbf{n}$. A similar result can be established for the recurrent extensions that leave 0 by a jump. In order to give a precise statement we next recall and introduce some notation.

We will use the notation of Chapter III. We denote by $\mathbf{P}$ the law of the Lévy process ξ associated to X. We assume henceforth that $\mathbf{E}(\xi_1^-) < \infty$ and that the law $\mathbf{P}$ satisfies the hypotheses (H2) in Chapter III. We denote θ the Cramér exponent of $\mathbf{P}$ and by $\mathbf{P}^{\natural}$ the h–transform of $\mathbf{P}$ via the invariant function $h(x) = e^{\theta x}, x \in \mathbb{R}$. Let $\widehat{\mathbf{P}^{\natural}}$ be the law of $\widehat{\xi^{\natural}} = -\xi^{\natural}$ under $\mathbf{P}^{\natural}$. The probability measures

$\mathbb{P}_x, \widehat{\mathbb{P}}_x, \mathbb{P}^{\natural}_x$ and $\widehat{\mathbb{P}^{\natural}_x}$ are the laws of the self–similar Markov processes associated to the Lévy processes with laws $\mathbf{P}, \widehat{\mathbf{P}}, \mathbf{P}^{\natural}$ and $\widehat{\mathbf{P}^{\natural}}$ respectively.

By the hypotheses (H2) it follows that the measure $\widehat{\mathbf{P}^{\natural}}$ has some finite exponential moments; in fact

$$e^{\widehat{\psi^{\natural}}(\beta)} := \widehat{\mathbf{P}^{\natural}}(e^{\beta\xi_1}) \leq 1, \qquad \beta \in [0,\theta],$$

where the inequality is an equality only for $\beta = 0, \theta$. This implies that for any $\beta \in]0,\theta[$ the function $h_\beta(x) = e^{\beta x}, x \in \mathbb{R}$, is excessive for the semi–group of the process $\widehat{\xi^{\natural}}$. Thus the h–transform ${}^{\beta}\mathbf{Q}$, of $\widehat{\mathbf{P}^{\natural}}$ via the excessive function h_β is a probability measure over the space of càdlàg trajectories with a finite lifetime. Under ${}^{\beta}\mathbf{Q}$ the canonical process is a Lévy process with finite lifetime since ${}^{\beta}\mathbf{Q}(t < \zeta) = e^{t\widehat{\psi^{\natural}}(\beta)}$, $t > 0$ and, conditionally on $\{t < \zeta\}$, the increment $\xi_{t+s} - \xi_t$ is independent of $(\xi_r, r \leq t)$ and has the same law as ξ_s under ${}^{\beta}\mathbf{Q}$. Furthermore, we have constructed the measure ${}^{\beta}\mathbf{Q}$ in such way that it satisfies the hypotheses (HI). Indeed, under ${}^{\beta}\mathbf{Q}$ the canonical process is not an arithmetic process since by hypothesis it is not under $\mathbf{P}$. For $\theta_\beta = \theta - \beta$ we have that

$${}^{\beta}\mathbf{Q}(e^{\theta_\beta \xi_1}) = \widehat{\mathbf{P}^{\natural}}(e^{\theta\xi_1}) = 1,$$

and

$${}^{\beta}\mathbf{Q}(\xi_1^+ e^{\theta_\beta \xi_1}) = \mathbf{E}(\xi_1^-) < \infty.$$

Let ${}^{\beta}\mathbb{Q}_x$ be the law of the α–self–similar Markov process $Y_\beta = (Y_{\beta,t}, t \geq 0)$ associated to the Lévy process with law ${}^{\beta}\mathbf{Q}$ via Lamperti's transformation. By Theorem 1 the process (Y_β, T_0) admits a unique self–similar recurrent extension $\widetilde{Y}_\beta = (\widetilde{Y}_{\beta,t}, \geq 0)$ that leaves 0 continuously. We denote ${}^{\beta}\overline{\mathbf{n}}$ the associated excursion measure.

By the results in Section III.3 we know that there exists a unique self–similar recurrent extension $X_\beta = (X_{\beta,t}, t \geq 0)$ of (X, T_0) that leaves 0 by a jump according to the jumping-in measure

$$\nu_{\theta-\beta}(dx) = d_{\alpha,\theta-\beta} x^{-(1+\theta-\beta)} dx, x > 0,$$

with $d_{\alpha,\theta-\beta} = (\theta-\beta)/\mathbf{E}(I^{\alpha(\theta-\beta)})\Gamma(1-\alpha(\theta-\beta))$.

We now have all the elements required to establish the main result of this section, which is a corollary to Theorems 1 & 2.

Proposition 3. *Let* $\beta \in]0,\theta[$.

(i) For any $x > 0$ *and* T *stopping time for the filtration* $(\mathcal{G}_t, t \geq 0)$ *we have that*

$${}^{\beta}\mathbb{Q}_x(F_T, T < T_0) = x^{-\beta}\widehat{\mathbb{P}^{\natural}}_x(F_T X_T^{\beta}, T < T_0), \qquad F_T \in \mathcal{G}_T.$$

(ii) The process X_β *is in weak duality with the process* $\widetilde{Y}_\beta$ *w.r.t.* $x^{1/\alpha-1-\theta+\beta}dx, x > 0$.

(iii) The image under time reversal of the excursion measure ${}^{\beta}\overline{\mathbf{n}}$ *is given by*

$$\rho({}^{\beta}\overline{\mathbf{n}}(\cdot)) = d_{\alpha,\theta-\beta}\int_0^\infty x^{-1-\theta+\beta}\mathbb{E}_x(\cdot)dx.$$

(iv) The excursion measure ${}^{\beta}\overline{\mathbf{n}}$ *is such that for every* $t > 0$

$${}^{\beta}\overline{\mathbf{n}}(F_t, t < T_0) = (c_\beta)\widehat{\mathbf{n}}(F_t X_t^{\beta}, t < T_0),$$

with $\widehat{\mathbf{n}}$ *the unique normalized excursion measure compatible with the self–similar Markov process* $(\widehat{X}^{\natural}, T_0)$ *such that* $\widehat{\mathbf{n}}(X_{0+} > 0) = 0$, *and* c_β *a normalizing constant.*

Proof. The proof of (i) is a straightforward consequence of the fact that the measure ${}^{\beta}\mathbf{Q}$ is an h-transform of the measure $\widehat{\mathbf{P}^{\natural}}$. The statements in (ii) and (iii) are consequences of the following claim: the measure $\mathbf{P}$ is equal to the measure $\widehat{{}^{\beta}\mathbf{Q}^{\natural}}$. To see this recall that the former is the dual of the measure ${}^{\beta}\mathbf{Q}^{\natural}$ which is in turn the h–transform of ${}^{\beta}\mathbf{Q}$ via the invariant function $h_{\theta-\beta} = e^{(\theta-\beta)x}, x \in \mathbb{R}$. Since under the measures $\mathbf{P}$ and $\widehat{{}^{\beta}\mathbf{Q}^{\natural}}$ the canonical process is a Lévy process with infinite lifetime, all that we have to do to prove the claimed fact is to verify that both have the same 1-dimensional marginals. This is proved in the following sequence of equalities: for every $t > 0, \lambda \in \mathbb{R}$,

$$\widehat{{}^{\beta}\mathbf{Q}^{\natural}}(e^{i\lambda\xi_t}) = {}^{\beta}\mathbf{Q}^{\natural}(e^{-i\lambda\xi_t}) = {}^{\beta}\mathbf{Q}(e^{i\lambda\xi_t}e^{(\theta-\beta)\xi_t}, t < \zeta) = \widehat{\mathbf{P}^{\natural}}(e^{-i\lambda\xi_t}e^{\theta\xi_t}) = \mathbf{P}(e^{i\lambda\xi_t}).$$

Therefore, the laws $\mathbb{P}_x$ and $\widehat{{}^{\beta}\mathbb{Q}^{\natural}}_x$ are equal for all $x > 0$, and the self–similar recurrent extension X_β is equal to the process $Z_{\theta-\beta}$ in Theorem 2 (iii). The statement in (ii) and (iii) follows from Theorem 2.

To prove (iv) recall from Theorem 1 that for every $t > 0$

$${}^{\beta}\overline{\mathbf{n}}(A_t, t < T_0) = (a_{\alpha,\theta_\beta})^{-1}\, {}^{\beta}\mathbb{Q}^{\natural}{}_{0+}(A_t Y_t^{-\theta+\beta}), \quad A_t \in \mathcal{G}_t,$$

with $a_{\alpha,\theta_\beta} = \alpha\, \mathbf{Q}^{\natural}(J^{-(1-(\theta-\beta))})\Gamma(1-\alpha\theta+\alpha\beta)/m^{\natural}$, $m^{\natural} = \mathbf{Q}^{\natural}(\xi_1)$. On the other hand, since the measure ${}^{\beta}\mathbf{Q}^{\natural}$ is equal to $\widehat{\mathbf{P}}$ we have that ${}^{\beta}\mathbb{Q}^{\natural}{}_x$ is equal to $\widehat{\mathbb{P}}_x$ for all $x > 0$. Which implies ${}^{\beta}\mathbb{Q}^{\natural}{}_{0+} = \widehat{\mathbb{P}}_{0+}$ over $\mathcal{G}$. The result is then obtained using the fact that the excursion measure $\widehat{\mathbf{n}}$ is such that for every $t > 0$

$$\widehat{\mathbf{n}}(A_t, t < T_0) = (\widehat{a}_{\alpha,\theta})^{-1}\widehat{\mathbb{E}}_{0+}(A_t, X_t^{-\theta}), \qquad A_t \in \mathcal{G}_t,$$

with $\widehat{a}_{\alpha,\theta} = \alpha\mathbf{E}(I^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)/m$, $m = -\mathbf{E}(\xi_1)$. The constant c_β is determined by $c_\beta = \widehat{a}_{\alpha,\theta}/a_{\alpha,\theta_\beta}$. □

As a final comment, observe that the measure $\mathbf{Q}^{\downarrow}$ of section 5 satisfies the assumptions (H2) of Chapter III and we can therefore apply the construction and results obtained in that section to study the self–similar Markov process $(X^{\downarrow}, \mathbb{Q}^{\downarrow})$ associated to $\mathbb{Q}^{\downarrow}$. In particular, in the a-stable process setting for $a \in]1, 2[$, if $\mathbb{Q}^{\downarrow}$. is the law of the process (X^0, P^0) conditioned to hit 0 continuously we have that the hypotheses (H2) are satisfied for $\theta = 1$, Then, for $\beta = 1 - a\rho$, we have that the process X_β corresponds to the stable process conditioned to stay positive and reflected at its future infimum under the law P and the process Y_β corresponds to the stable process reflected as its infimum under P^*. The latter is equal to the stable process reflected at its supremum under the law P. We leave the details to the interested reader. The restriction $a \in]1, 2[$ is just used to ensure that $0 < (1/a)\theta = 1/a < 1$ and thus the existence of the excursion measure $\widehat{\mathbf{n}}$ in (iv) in Proposition 3. The same result holds without the condition $a \in]1, 2[$, but in (iv) we will have a pseudo excursion measure.

6.3 The case where the process Y has increasing paths

Assume that the Lévy process ξ' of section 2 has increasing paths, that is ξ' is a subordinator. It is well known that the law of a subordinator has negative exponential moments:

$$]-\infty, 0[\subseteq C := \{\lambda \in \mathbb{R} : \mathbf{Q}'(e^{\lambda\xi_1}) < \infty\}.$$

In this case, the Laplace exponent ψ of ξ' is given by

$$\psi(\lambda) = d\lambda + \int_0^{\infty}(e^{\lambda x} - 1)\Pi(dx).$$

We assume that there is a $\theta \in \widehat{C} \cap]0, \infty[\neq \emptyset$ such that $\mathbf{Q}'(\xi_1 e^{\theta \xi_1}) < \infty$ and let $\mathbf{Q}$ be the law of the subordinator ξ' killed at rate $\boldsymbol{k} = \psi(\theta)$. Observe that instead of taking ∞ as cemetery point for the subordinator as usually, we are taking a point Δ such that $e^{\Delta} = 0$. Therefore, the α–self–similar Markov process Y associated to ξ is a process with a.s. increasing paths that suddenly jumps to 0 at some finite time and then dies. By construction, the law $\mathbf{Q}$ satisfies the hypotheses (HI) for $\widehat{\theta}$ and therefore we can construct a self–similar recurrent extension $\widetilde{Y}$ of Y that leaves 0 continuously a.s. By time reversal the dual process $\widehat{Y}^{\natural}$ is the self–similar Markov process associated to the negative of a subordinator whose Laplace exponent is easily derived from ψ. The recurrent extension of the self–similar Markov process $\widehat{Y}^{\natural}$ is that constructed in Example III.1 and is in weak duality with $\widetilde{Y}$.

Observe that in this case the process $\xi^{\natural}$ is a subordinator but it is not equal to ξ'. In general, even if the Lévy process ξ' drifts to ∞, the process $\xi^{\natural}$ is not equal to ξ'.

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