Coverings of Seifert manifolds branched along fibers

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Chapter 1

Introduction

A Seifert manifold M is a 3-manifold which is a disjoint union of circles (fibers). Seifert manifolds M were defined and classified (up to fiber preserving homeomorphisms) by H. Seifert [**Se**] according to a Seifert symbol associated to M. Because of the fact that Seifert manifolds are classified, they play a useful role in the Theory of $3-$manifolds. Since the invention of Seifert manifolds in the 30's, an interesting problem is to understand the branched coverings $\varphi : \tilde{M} \to M$ when M is a closed Seifert manifold.

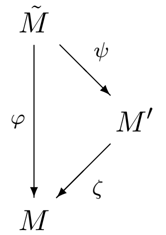
Let M be a closed Seifert manifold and suppose $\varphi : \tilde{M} \to M$ is a covering of M branched along fibers, that is, the branching of φ is a finite union of fibers of M. It is known that $\tilde{M}$ is also a Seifert manifold [**G-H**]. In [**Se**], H. Seifert also found the Seifert symbol for the orientation double covering of M. More recently, V. Núñez and E. Ramírez-Losada [**N-RL**] compute the Seifert symbol for $\tilde{M}$ when M is orientable and $\varphi : \tilde{M} \to M$ satisfies some properties. But in general, if $\varphi : \tilde{M} \to M$ is a covering of a Seifert manifold M branched along fibers, the Seifert Symbol for $\tilde{M}$ is unknown. Therefore a basic problem is to determine the Seifert symbol of $\tilde{M}$ in terms of φ and the Seifert symbol of M. In this work we solve the above problem (Theorem (3.3.8) and Theorem (3.3.15)).

On the other hand, Heegaard genera for almost all Seifert manifolds are known. M. Boileau and H. Zieschang [**B-Z**] computed the Heegaard genera for almost all orientable Seifert manifolds and V. Núñez [**Nu**] computed the Heegaard genera for almost all non-orientable Seifert manifolds. In both cases, orientable or non-orientable, the Heegaard genus of M is expressed in terms of the Seifert symbol of M.

Let M be a Seifert manifold with infinite fundamental group. Suppose $\varphi : \tilde{M} \to M$ is a covering of M branched along fibers. If we know the Heegaard genus of M, $h(M)$, and we compute the Seifert symbol of $\tilde{M}$, we can compare the Heegaard genus of $\tilde{M}$, $h(\tilde{M})$, with $h(M)$. What one can "reasonable" expect is that $h(\tilde{M}) \geq h(M)$. But we find a family of manifolds M, with infinite fundamental group, having a covering $\tilde{M}$ such that $h(\tilde{M}) < h(M)$. This implies (translating into fundamental group) that there is an infinite family of infinite groups G that have a subgroup $H < G$ of finite index with an unexpected and surprising property: $rank(H) < rank(G)$.

In Chapter 1, we deal with basic topics to be used along this work. The basic topics to consider are: Topology of manifolds, Heegaard splittings and Branched coverings. In the last section of Chapter 1, we write a list of Theorems that we will be needed later.

Let M be a Seifert manifold and $\varphi : \tilde{M} \to M$ a branched covering space of M. Suppose $\tilde{M}$ is connected. In chapter 2, we prove that there are coverings $\psi : \tilde{M} \to M'$ and $\zeta : M' \to M$ branched along fibers such that the following diagram commutes



and if ω_ψ and ω_ζ are the representations associated to ψ and ζ, respectively, we have that $\omega_\psi(h') = \varepsilon_n$ and $\omega_\zeta(h) = (1)$, where (1) is the identity permutation in S_n and ε_n is the stardad n-cycle $(1, 2, \ldots, n)$, and h and h' are regular fibers of M and M', respectively. Thus we reduce the study of coverings of M to coverings $\varphi : \tilde{M} \to M$, such that ω_φ, the representation associated to φ, sends a regular fiber h of M into the identity permutation or into the n-cycle $(1, \ldots, n)$. In both cases, $\omega(h) = (1)$ or $\omega(h) = \varepsilon_n$, we calcule the Seifert symbol of $\tilde{M}$.

In chapter 3, given a $\varphi : \tilde{M} \to M$ covering of M branched along fibers such that ω_φ, the representation associated to φ, sends a regular fiber h of M into the identity permutation or into the n-cycle $(1, \ldots, n)$, we apply the theory in Chapter 2 to compare the Heegaard genus of $\tilde{M}$, $h(\tilde{M})$, with the Heegaard genus of M, $h(M)$. The genus $h(\tilde{M})$ is computed in terms of ω_φ and the Seifert symbol of M. We show that there are Seifert manifolds of M and coverings $\tilde{M}$ such that $h(\tilde{M}) < h(M)$.

Chapter 2

Preliminaries

This chapter is a brief review about facts in low-dimensional topology.

2.1 3-manifolds and Heegaard genus

Definition 2.1.1 *Let M be a Hausdorff topological space. We say M is an* ***n-manifold*** *if and only if each element x of M has a neighborhood homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \forall i = 1, \ldots, n\}$.*

If M is an n-manifold and there is a point in M having no neighborhood homeomorphic to $\mathbb{R}^n$, we say that M is an n-manifold with boundary and we call this point ***a boundary point***. The set of boundary points is called ***the boundary of M*** and we denote it by ∂M. The space $M - \partial M$ is called ***the interior of*** M and it is denoted by M^o. An n-manifold M is a ***closed manifold*** if it is compact and $\partial M = \emptyset$.

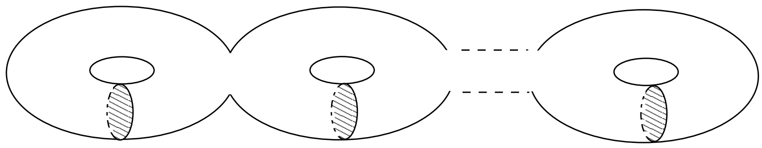
Definition 2.1.2 *A 3-manifold M is* ***irreducible*** *if every $2-$sphere S^2 in M bounds a 3-ball.*

Definition 2.1.3 *A disk D^2 in a 3-manifold with boundary M is said to be* ***properly embedded*** *if $D^2 \cap \partial M = \partial D^2$.*

Definition 2.1.4 *Let V be an orientable irreducible compact and connected 3-manifold with non-empty boundary. If there exist k properly embedded pairwise disjoint 2-disks D_j such that $\cup D_j$ splits V into a 3-ball, we say that V* ***is a handlebody of genus*** *k.*

Note that the boundary of V is a closed, connected and orientable surface of genus k.

Heegaard's theorem 2.1.1 *Let M be a connected closed and orientable $3-$manifold. Then M is union of two handlebodies of genus g, for some $g \geq 0$.*



Handlebody

Proof.
It is well-known that M is triangulable [**Mo**]. Let K be a triangulation for M. Define V_1 to be a regular neighborhood of the 1-skeleton of K and V_2 to be $\overline{M - V_1}$ $\square$

Definition 2.1.5 *Let M be a connected, closed 3-manifold and let $F \subset M$ be a closed, connected and orientable surface. If F splits M into two handlebodies, then (M, F)* **is a Heegaard splitting of** *M.*

Definition 2.1.6 *The genus of a Heegaard splitting is the genus of the surface F, and the* **Heegaard genus of** *M*, *$h(M)$, is the smallest integer h such that M has a Heegaard splitting of genus h.*

Example 2.1.1 $h(S^3) = 0$

2.2 Branched coverings

Definition 2.2.1 *Let X and $\tilde{X}$ be two path-connected topological spaces. A surjective map $f : \tilde{X} \to X$ is a covering space map if and only if for every $x \in X$ there exists a neighborhood V_x of x satisfying the following properties:*

(a) $f^{-1}(V_x) = \cup_{\alpha \in J} \tilde{V}_\alpha$, *with* $\tilde{V}_\alpha \cap \tilde{V}_\beta = \emptyset$ *if* $\alpha \neq \beta$ *and*

(b) $f| : \tilde{V}_\alpha \to V_x$ *is a homeomorphism, for all* $\alpha \in J$.

If $|J| = n$ is a natural number, then f is a ***finite covering space*** and we say that f is a ***covering of*** n***-sheets*** or that f is an $n-$***fold covering***.

Let Ω be a set of n elements; we write $S_n = S(\Omega)$ for the symmetric group on the n elements of Ω. When no confussion arises about the set Ω, we only write S_n.

Let $\tilde{N}$ and N be n-manifolds. Suppose $f : \tilde{N} \to N$ is a map. We say that f is a **proper** map if $f^{-1}(\partial N) = \partial \tilde{N}$. The map f is **finite-to-one** if $f^{-1}(x)$ is finite, for all $x \in N$

Definition 2.2.2 *A proper map $f : \tilde{N} \to N$ between two m-manifolds is called a* **branched covering** *if it is finite-to-one and open.*

Usually one can check if an open map f between manifolds is a branched covering by ***finding a subcomplex*** B ***of*** N ***of codimension two such that*** $f| : \tilde{N} - f^{-1}(B) \to N - B$ ***is a finite covering space***[**Fo**].

The subcomplex B is called ***the branch set of*** f and $f^{-1}(B)$ is called ***the singular set of*** f. In our examples the set B is always a submanifold.

If $f|(\tilde{N} - f^{-1}(B))$ is an n-fold covering, we say that f is a branched covering of n-sheets or that f is an n-fold branched covering.

Note that a finite covering space map (unbranched) between manifolds is a branched covering with $B = \emptyset$.

Remark 2.2.1 *The following facts about coverings and branched coverings are known:*

(a) *An n-fold covering space $\eta : \tilde{X} \to X$ determines and is determined by a homomorphism $\omega_f : \pi_1(X) \to S_n$, where S_n is the symmetric group on n symbols. This homomorphism ω is called a* ***representation of*** *$\pi_1(X)$. Also $\tilde{X}$ is connected if and only if ω is transitive.*

Let $\varphi : \tilde{X} \to X$ be a branched covering and let B be the branch set of φ.

(b) *The covering $\varphi| : \tilde{X} - \varphi^{-1}(B) \to X - B$ determines the branched covering φ through a Fox compactification [**Fo**]. item[(c)] By (a) and (b), a branched covering determines and is determined by a representation $\omega_f : \pi_1(N$ - Branch set of $f) \to S_n$*

(d) *If X is orientable, $\tilde{X}$ is also orientable [**B-E**], for if $w_1(X)$ is the first Stiefel-Whitney class of X then $\varphi^* w_1(X) = w_1(\tilde{X})$.*

2.3 Some preliminary Theorems

If M is 3−manifold, let $w_1(M) : H_1(M) \to \mathbb{Z}_2$ be a homomorphism such that if $\alpha \subset M$ is an orientation preserving curve then $w_1(\alpha) = 1$, and if α is orientation reversing then $w_1(\alpha) = -1$.

The homomorphism $w_1(M)$ is the ***first Stiefel-Whitney class of*** M. If $\varphi : \tilde{M} \to M$ is a branched covering of M, it is proved in [**B-E**] that $w_1(\tilde{M}) = \varphi^*(w_1(M))$ where $\varphi^* : H^1(M, \mathbb{Z}_2) \to H^1(\tilde{M}, \mathbb{Z}_2)$ is the homomorphism induced by φ in the cohomology groups.

We write $PD : H^1(M, \mathbb{Z}_2) \to H_2(M, \mathbb{Z}_2)$ for the Poincaré duality isomorphism associated to the 3-manifold M.

Definition 2.3.1 *Let M be a non-orientable 3-manifold and $F \subset M$ be an orientable surface. We call F a* **Stiefel-Whitney surface for** *M if and only if F is connected and $[F] = PDw_1(M) \in H_2(M; \mathbb{Z}_2)$.*

Assume M is a manifold. Let $\beta : H^i(M, \mathbb{Z}_2) \to H^{i+1}(M, \mathbb{Z})$ denote the Bockstein homomorphism associated to the short exact sequence of coefficients

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$

Lemma 2.3.1 *[**B-E**] Let M be a non-orientable 3-manifold. Then $\beta w_1(M) = 0$ if and only if there exists $S \subset M$ a two-sided Stiefel-Whitney surface for M.*

Let $M = (Xx, g, \beta_1/\alpha_1 \ldots, \beta_r/\alpha_r)$ be a Seifert manifold, where Xx ia symbol in $\{Oo, On, No, NnI, NnII, N$ (See Chapter 3). Write $e_0(M) = \sum \beta_i/\alpha_i$ and, $\lambda(M) = lcm\{\alpha_1, \ldots, \alpha_r\} \cdot e_0(M)$, where $lcm\{\alpha_1, \ldots, \alpha_r\}$ denotes the least common multiple of $\alpha_1, \ldots, \alpha_r$. Notice that $\lambda(M)$ is an integer number.

Theorem 2.3.1 *[**Nu**] If M is a non-orientable Seifert manifold with orbit projection $p : M \to F$, then $\beta w_1(M) \neq 0$ if and only if either $M \in NnII$ or $M \in NnI$, $g(F)$ is odd and $\lambda(M)$ is even.*

Theorem 2.3.2 *[**Nu**] Let M be a non-orientable Seifert manifold. Then there exists a fibered torus $T \subset M$, where fibered means that T is a union of fibers of M, such that T is a Stiefel-Whitney surface for M. In the following cases T is two-sided in M:*

(i) *$M \in (No, g)$.*

(ii) *$M \in (NnI, 2g)$.*

(iii) *$M \in (NnIII, g)$.*

And in the following cases T is one-sided in M:

(iv) *$M \in (NnI, 2g + 1)$.*

(v) *$M \in (NnII, g)$.*

Theorem 2.3.3 *[**Nu**] Let M be a non-orientable Seifert manifold and T be a fibered torus in M.*

- *Suppose $M \in (NnI, 2g+1)$ or $M \in (NnII, g)$. If $T \subset M$ is a two-sided fibered torus, then $M - T$ is non-orientable;*
- *Assume $M \in (No, g)$ or $M \in (NnI, 2g)$ or $M \in (NnIII, g)$. If $T \subset M$ is an one-sided fibered torus, then $M - T$ is non-orientable.*

Chapter 3

Coverings of Seifert manifolds

3.1 Coverings and bundles

Recall that if Ω is a set of n elements, then $S_n = S(\Omega)$ denotes the symmetric group on the n elements of Ω.

The identity permutation of S_n is the permutation that fix all the elements of Ω. We denote the identity permutation of S_n by (1).

Let $\sigma \in S_n$, the order of σ, denoted by $order(\sigma)$, is the smallest natural number n such that $\sigma^n = (1)$.

A cycle $\rho = (a_1, \ldots, a_s)$ in $S_n = S(\Omega)$ is the permutation that fixes the elements in Ω different from a_i, for all $i = 1, \ldots, s$, it sends the element $a_i \in \Omega$ into a_{i+1}, for each $i = 1, \ldots, s-1$, and sends the element a_s into a_1. One can verify easily that if $\rho = (a_1, \ldots, a_s)$ then $order(\rho) = s$. Throughout this work the standard $n-$cycle is the permutation $(1, 2, \ldots, n) \in S_n$ and it will be denoted by ε_n.

Recall that if σ is a permutation in S_n then σ can be represented as a product of disjoint cycles. Throughout this work all permutations in S_n will be represented as a product of disjoint cycles, unless explicitly stated.

Definition 3.1.1 *Suppose $m, n \in \mathbb{N} - \{1\}$ and $H \leq S_{mn} = S(\Omega)$; then we say that H is $m, n-$***imprimitive** *if there are $\Delta_1, \ldots, \Delta_n \subset \Omega$ such that:*

(a) $\Omega = \sqcup_{i=1}^{n} \Delta_i$, *where denotes the disjoint union.*

(b) $\#\Delta_i = m$, *for all $i = 1, \ldots, n$.*

(c) *The elements of H leave the sets Δ_i invariant, that is $\sigma(\Delta_i) = \Delta_j$, for each i and σ*

and for some $j \in \{1, \ldots, n\}$.

The sets $\Delta_1, \ldots, \Delta_n$ *are called* sets of m, n-imprimitivity for H. *Note that if* H *is* $m, n-$*imprimitve then* $H \geq S_{mn}$.

Given $x \in \Omega$, ***the stabilizer of*** x is the subgroup $St(x) = \{\sigma \in S(\Omega) | \sigma(x) = x\} \geq S(\omega)$.

Let H be $m, n-$imprimitive. The quotient $\Delta_1 \sqcup \ldots \sqcup \Delta_n \to \{\Delta_1, \ldots, \Delta_n\}$ which sends all symbols of Δ_i into the symbol Δ_i for each i, induces a "*quotient homomorphism*" $q : H \to S_n = S(\{\Delta_1, \ldots, \Delta_n\})$. If $H_1 = q^{-1}(St(\Delta_1))$, then the "*restriction homomorphism*" $\gamma : H_1 \to S_m = S(\Delta_1)$ such that $\gamma(\sigma) = \sigma|\Delta_1$, is a group homomorphism.

Lemma 3.1.1 *Let* $\varphi : X \to Y$ *be an* $mn-$*fold covering space and let* $\omega : \pi_1(Y) \to S_{mn}$ *be the associated representation; write* $H = Im(\omega)$. *Then* H *is* $m, n-$*imprimitive if and only if* φ *factors through an m-fold covering* $\psi : X \to Z$ *and an* $n-$ *fold covering* $\zeta : Z \to Y$.

Proof.
If H is $m, n-$imprimitive, then there exists sets of $m, n-$imprimitivity, $\Delta_1, \ldots, \Delta_n$, for H. Consider the representation

$$\omega_\zeta : \pi_1(Y) \xrightarrow{\omega} H \xrightarrow{q} S_n = S(\{\Delta_1, \ldots, \Delta_n\}),$$

where q is the quotient homomorphism determined by $\Delta_1, \ldots, \Delta_n$. Let $\zeta : Z \to Y$ be the $n-$fold covering associated to ω_ζ: then Z is a topological space such that $\pi_1(Z) \cong (q \circ \omega)^{-1}(St(\Delta_1))$. Notice that $\omega^{-1}(St(1)) \subset (q \circ \omega)^{-1}(St(\Delta_1))$ by definition of q. Therefore there is an $m-$fold covering $\psi : X \to Z$ such that $\zeta \circ \psi = \varphi$.

Note that the representation associated to ψ is

$$\omega_\psi : \pi_1(Z) \cong (q \circ \omega)^{-1}(St(\Delta_1)) \xrightarrow{\omega} q^{-1}(St(\Delta_1)) \xrightarrow{\gamma} S_\alpha = S(\Delta_1),$$

where ω is the restricition homomorphism determined by $\Delta_1, \ldots, \Delta_n$.

Now suppose there are $\psi : X \to Z$ and $\zeta : Z \to Y$ covering spaces of $m-$sheets and $n-$sheets, respectively, such that $\varphi = \psi \circ \zeta$. Let $y_0 \in Y$. Then $\zeta^{-1}(y_0) = \{z_1, \ldots, z_n\}$ and

$$\varphi^{-1}(y_0) = \{x_{1,1}, \ldots, x_{1,m}, x_{2,1} \ldots, x_{2,m}, \ldots, x_{n,1}, \ldots, x_{n,\alpha}\}.$$

By renumbering the points, if necessary, we can suppose that $\psi(x_{i,j}) = z_i$, for $1 \leq i \leq n$ and for $1 \leq j \leq m$. Define $\Delta_i = \{x_{i,1}, \ldots, x_{i,m}\}$, for each $i \in \{1, \ldots, n\}$. Using the *Path Lifting Theorem* for covering spaces, it is clear that the Δ_i's are sets of $m, n-$imprimitivity.□

Suppose N is an $n-$manifold and $\varphi : \tilde{N} \to N$ is an $m-$fold covering of F. Let $\omega : \pi_1(N) \to S_m$ be the representation determined by φ and $\theta : H_1(N) \to \mathbb{Z}_2$ be an epimorphism, (i.e. θ is a transitive representation).

If $\varphi_\theta : N_\theta \to N$ is the 2-fold covering associated to θ. Define $\tilde{\theta} = \varphi^*(\theta)$, where $\varphi^* : H^1(N, \mathbb{Z}_2) \to H^1(\tilde{N}, \mathbb{Z}_2)$ is the cohomology induced homomorphism. Notice that $\tilde{\theta}$ can be regarded as an element of $H^1(\tilde{N}; \mathbb{Z}_2)$, that is $\tilde{\theta} : H_1(N) \to \mathbb{Z}_2$ is a homomorphism.

Note that if θ is non-trivial, then θ is an epimorphism (i.e. θ is a transitive representation). Consequently $\pi_1(N_\theta) \cong Ker(\theta)$, for φ_θ is regular and thus $Ker(\theta) = \theta^{-1}(St(1))$.

Remark 3.1.1 *If θ is trivial, then $\tilde{\theta}$ is trivial.*

Proof.
In this case $N_\theta = N \sqcup N$, where $\sqcup$ denotes the disjoint union. Suppose $\tilde{\alpha} \in H_1(\tilde{N})$, then $\tilde{\theta}(\tilde{\alpha}) = \theta(\varphi_*(\tilde{\alpha})) = (1)$. □

Remark 3.1.2 *If θ is non-trivial, then $\tilde{\theta}$ is trivial if and only if there exists a $\frac{m}{2}$-fold covering $\psi : \tilde{N} \to N_\theta$ such that $\psi \circ \varphi_\theta = \varphi$.*

Proof.

Let us suppose that $\tilde{\theta}$ is trivial; then $\tilde{\theta}(\tilde{\alpha}) = \theta(\varphi_*(\tilde{\alpha})) = (1)$, for all $\tilde{\alpha} \in H_1(\tilde{N})$. Therefore $\varphi_*(H_1(\tilde{N})) \subset Ker(\theta)$ and there is a $\frac{m}{2}$-fold covering $\psi : \tilde{N} \to N_\theta$ satisfying that $\psi \circ \varphi_\theta = \varphi$.

On the other hand, if there exists a covering $\psi : \tilde{N} \to N_\theta$ such that $\psi \circ \varphi_\theta = \varphi$, then $\varphi_*(H_1(\tilde{N})) \subset Ker(\theta)$ and thus $\tilde{\theta}$ is trivial. □

Theorem 3.1.1 *Assume N is an $n-$manifold and $\varphi : \tilde{N} \to N$ is an $m-$fold covering of F. Let $\omega : \pi_1(N) \to S_m$ be the representation determined by φ and $\theta : H_1(N) \to \mathbb{Z}_2$ be a homomorphism. Let $\tilde{\theta} = \varphi^*(\theta)$. Suppose that θ is non-trivial.*

Then $\tilde{\theta}$ is trivial if and only if $Im(\omega)$ is $\frac{m}{2}, 2-$imprimitive and there are sets of $\frac{m}{2}, 2-$imprimitivity for $Im(\omega)$, Δ_1 and Δ_2, such that the quotient homomorphism $q : Im(\omega) \to S_2$ satisfies that $q \circ \omega = \theta$.

Proof.
If $\tilde{\theta}$ is trivial, by Remark 3.1.2 there exists an $\frac{m}{2}$-fold covering $\psi : \tilde{N} \to N_\theta$ such that $\psi \circ \varphi_\theta = \varphi$. Then, by Lemma 3.1.1, there exist Δ_1 and Δ_2 sets of $\frac{m}{2}, 2-$imprimitivity for $Im(\omega)$ such that the representation induced by φ_θ is $q \circ \omega : \pi_1(N) \to S_2$. Therefore $q \circ \omega = \theta$.

On the other hand, if there are sets of $\frac{m}{2}, 2-$imprimitivity for $Im(\omega)$, Δ_1 and Δ_2, such that $q \circ \omega = \theta$, then by Lemma 3.1.1 there is a covering $\psi : \tilde{N} \to N_\theta$ of $\frac{m}{2}$-sheets such that $\varphi = \psi \circ \varphi_\theta$. Thus, by Remark 3.1.2, $\tilde{\theta}$ is trivial. □

Definition 3.1.2 *Let N be a connected $m-$manifold and let $n \in \mathbb{N}$. Assume $\omega : \pi_1(N) \to S_n$ is a transitive representation and $\theta \in H^1(N, \mathbb{Z}_2)$. We say that ω* **trivializes the bundle of** *θ if and only if $Im(\omega)$ is $\frac{m}{2}, 2-$imprimitive and there are sets of $\frac{m}{2}, 2-$imprimitivity for $Im(\omega)$, Δ_1 and Δ_2, such that the quotient homomorphism $q : Im(\omega) \to S_2$ satisfies that $q \circ \omega = \theta$.*

When a permutation in an imprimitive subgroup contains an odd order cycle, computations are somewhat eased. For example, let us consider the permutations $a = (1,2,3)(4,5,6)$ and $b = (1,4)(2,5)(3,6)$ in S_6. Let $H = \langle a, b\rangle$ be the subgroup in S_6 generated by the permutations a and b. It can be seen that H is $3, 2-$imprimitive. Let us calculate a system of $3, 2-$imprimitivity for H. There exist sets of $3, 2-$imprimitivity, Δ_1 and Δ_2 for H. Note that $a \cdot \Delta_1$ must be equal to Δ_1 or Δ_2 because Δ_1 is a set of $3, 2-$imprimitivity. Assume $1 \in \Delta_1$.

If $a \cdot \Delta_1 = \Delta_1$, then $2, 3 \in \Delta_1$ for $a(1) = 2$ and $a(2) = 3$; thus $\{1, 2, 3\} \subset \Delta_1$ and we get $\Delta_1 = \{1, 2, 3\}$ because $\#\Delta_1 = 3$.

Note that $a \cdot \Delta_1 = \Delta_2$ cannot happen. If $a \cdot \Delta_1 = \Delta_2$, then $2 \in \Delta_2$ for $1 \in \Delta_1$ and $a(1) = 2$. Of course 3 should belong to Δ_2 because $a(3) = 1$; otherwise, if $3 \in \Delta_1$ we have $1 \in \Delta_2$. But $3 \in \Delta_2$ implies that $a \cdot \Delta_2 = \Delta_2$ for $a(2) = 3$ and $2, 3 \in \Delta_2$. Thus $1 \in \Delta_2$ since $a(3) = 1$ and this contradicts our assumption that $1 \in \Delta_1$.

Therefore $\Delta_1 = \{1, 2, 3\}$ and $\Delta_2 = \{4, 5, 6\}$ are the only sets of $3, 2-$imprimitivity for H. One can see easily that if $q : H \to S_2$ is the quotient homomorphism associated to Δ_1 and Δ_2, then $q(a)$ is the identity in $S_2 = S(\{\Delta_1, \Delta_2\})$ and $q(b) = (\Delta_1, \Delta_2) \in S(\{\Delta_1, \Delta_2\})$.

In general, we obtain the following corollary.

Corollary 3.1.1 *Assume N is an $n-$manifold and $\varphi : \tilde{N} \to N$ is an $m-$fold covering of F. Let $\omega : \pi_1(N) \to S_m$ be the representation determined by φ and $\theta : H_1(N) \to \mathbb{Z}_2$ be a homomorphism. Let $\tilde{\theta} = \varphi^*(\theta)$. Suppose that v_j is a generator for $\pi_1(N)$ such that in the disjoint cycle decomposition of $\omega(v_j)$ there is a cycle $(a_{j,1}, \ldots, a_{j,k})$ of odd order and $\theta(v_j) = (1, 2)$.*

Then $\tilde{\theta}$ is non-trivial.

Proof.
Assume that $\tilde{\theta}$ is trivial. Then there are sets Δ_1 and Δ_2 of $\frac{m}{2}, 2-$imprimitive for $Im(\omega)$.

Since $(a_{j,1}\cdots a_{j,k})$ has odd order and $\omega(v_j)$ must leave the sets Δ_1 and Δ_2 invariant, it follows that $\{a_{j,1},\ldots,a_{j,k}\}\subset\Delta_1$ or $\{a_{j,1},\ldots,a_{j,k}\}\subset\Delta_2$. Without loss of generality, we suppose that $\{a_{j,1},\ldots,a_{j,k}\}\subset\Delta_1$, thus $(q\circ\omega(v_j))(\Delta_1)=\Delta_1$ and $q\circ\omega\neq\theta$. Therefore $\tilde{\theta}$ is non-trivial. □

Let N be a manifold and let θ be equal to $w_1(N)$, the first Stiefel-Whitney class of N, and recall that if $\varphi:\tilde{N}\to N$ is a covering space then $w_1(\tilde{N})=\varphi^*(w_1(N))$. Then we can apply the previous theorem to get the following corollary.

Corollary 3.1.2 *Suppose that N is a non-orientable manifold and consider a transitive representation $\omega:\pi_1(N)\to S_m$. Let $\varphi:\tilde{N}\to N$ be the covering space associated to ω and $w_1(N)$ be the first Stiefel-Whitney class of N.*

Then $\tilde{N}$ is orientable if and only if $Im(\omega)$ trivializes the bundle of $w_1(N)$.

Remark 3.1.3 *Let F be a non-orientable surface of genus k and let $\{v_j\}_{j=1}^k$ be a basis for $\pi_1(F)$ such that v_j is an orientation reversing loop, for all $j\in\{1,\ldots,k\}$. Suppose that $n\geq 2$, $\varphi:\tilde{F}\to F$ is a covering space and let $\omega:\pi_1(F)\to S_n$ be the representation associated to φ. By Corollary (3.1.1) and Corollary (3.1.2)*

1. *If the order of a cycle of $\omega(v_m)$ is odd, for some $m\in\{1,\ldots,k\}$, then $\tilde{F}$ is non-orientable.*
2. *If n is an odd number, $\tilde{F}$ is non-orientable.*
3. *Suppose that all the cycles of $w(v_j)$ have even order (therefore n is an even number), for each $j=1,\ldots,k$; then G is orientable if and only if $Im(\omega)$ trivializes the bundle of $w_1(F)$.*

3.2 Seifert manifolds

Let α and β be coprime integers numbers and $\alpha_i\geq 1$; Suppose $r:D^2\to D^2$ is the rotation defined by $r(x)=xe^{2\pi i(\alpha/\beta)}$. Then ***the fibered solid torus*** $T(\beta/\alpha)$ is the quotient space $\dfrac{D^2\times I}{(x,0)\sim(r(x),1)}$, where $I=[0,1]$.

The ***fibers of*** $T(\beta/\alpha)$ are the images of the intervals $\{x\}\times I$ (under the identification). Note that almost all fiber in $T(\beta/\alpha)$ is the union of the images of β intervals; the only exception is the core of $T(\beta/\alpha)$ because this fiber is the image of just the interval from $\{0\}\times I$.

Suppose $T(\beta/\alpha)$ and $T(\beta'/\alpha')$ are fibered solid tori. A ***fiber preserving homeomorphism*** f of $T(\beta/\alpha)$ and $T(\beta'/\alpha')$ is a homeomorphism $f:T(\beta/\alpha)\to T(\beta'/\alpha')$ that sends each fiber of $T(\beta/\alpha)$ onto one fiber of $T(\beta'/\alpha')$.

Definition 3.2.1 *A **Seifert manifold** M is a connected closed 3-manifold that can be decomposed into disjoint circles called fibers of M, such that for every fiber h there exist a neighborhood V_h, and coprime integer numbers $\alpha \geq 1$ and β, and a fiber preserving homeomorphism $f : V_h \to T(\beta/\alpha)$ such that $f(h)$ is the core of $T(\beta/\alpha)$.*

If $\alpha \geq 2$, the core of V_h is called ***an exceptional fiber of multiplicity*** α ***of*** M, otherwise it is ***a regular fiber of*** M.

Note that by collapsing each fiber into a point we get a well-defined ***quotient*** $p : M \to F$, where F is a closed surface of genus g; F is orientable or non-orientable. This quotient is called ***the orbit quotient of*** M or ***the orbit projection of*** M, and F is called ***the orbit surface of*** M. Since each fiber h in M has a neighborhood V_h homeomorphic to a fibered solid torus, one can show that $\{p(V_h)^\circ\}$ is a basis for the topology of F. The image of a regular fiber is a regular point and the image of an exceptional fiber is an exceptional point.

Given a triangulation T of F it is possible to construct a system of neighborhoods of fibers of M, where each neighborhood is homeomorphic to a fibered solid torus and projects onto a triangle of F. Also we can pick T, in such way, that every triangle contains at most one exceptional point. We will consider only triangulations of F with this property.

Assume F is triangulated by T. Let $x_1, y_1 \in F$ and suppose there is a triangle T_1 which misses exceptional points. Let $c_1 \subset T_1$ be a path joining x_1 and y_1. Let us fix an orientation of $p^{-1}(x_1)$. Since $p^{-1}(x)$ and $p^{-1}(y)$ are fibers of the fibered solid torus $p^{-1}(T_1)$, we can induce an orientation on the fiber $p^{-1}(y_1)$ by translating the fiber $p^{-1}(x)$ along the path c_1 and we say that $p^{-1}(y)$ has the orientation induced by $p^{-1}(x)$ along c.

In general, let $x, y \in F$ and suppose there is a path c, connecting x with y, which misses exceptional points, we may assume, refining T, if necessary, that there exist a finite number of s triangles T_i without exceptional points, where $i = 1, \ldots, s$, such that $c \subset \cup_{i=1}^{s} T_i$. Let V_i be the solid torus determined by T_i, for all $i = 1, \ldots, s$. Note that we can also suppose that the set $c_i = c \cap T_i$ does not contain the vertices of T_i. If $p^{-1}(x)$ has an orientation then we can induce an orientation on the fiber $p^{-1}(y)$ by translating the orientation of $p^{-1}(x)$, triangle by triangle, along the curves c_i. Then if $x = y$ and the fiber $p^{-1}(x)$ is oriented we can follow the induced orientation of $p^{-1}(x)$ along loops c based at x. Thus we have a homomorphism $e : \pi_1(F) \to \mathbb{Z}_2$ such that $e(c) = +1$, if c preserves the orientation of the fiber when the fiber is translated along c; otherwise, if c reverses the orientation of the fiber, $e(c) = -1$. This homomorphism is called ***the valuation homomorphism***. Of course, it is enough to define e in a basis for $\pi_1(F)$ or $H_1(F)$.

Since M is compact, the number of exceptional fibers in a Seifert manifold is finite.

Seifert manifolds were classified by H. Seifert [**Se**] according to a ***Seifert symbol*** and six classes, depending on the orientability of F, the valuation homomorphism and the multiplicities of exceptional fibers. In order to state the classification in classes of Seifert manifolds we fix the following facts and notation.

Let $\{h_i\}_{i=1}^{r}$ be a set of fibers of M which contains all the exceptional fibers and some regular fibers. Recall each fiber has a neighborhood V_i fiber preserving homeomorphic to a fibered solid torus. Let $T(\beta_i/\alpha_i)$ be the fibered solid torus homeomorphic to V_i, for all $i = 1, \ldots, r$. Recall that α_i and β_i are coprime numbers and $\alpha_i \geq 1$. ***We always will ask to α_i be greater than or equal to 1 and coprime with β_i.***

We write $M_0 = \overline{M - \cup V_i}$. Note that we have a quotient $p| : M_0 \to F_0$, where F_0 is a surface with boundary. The boundary of F_0 has r components, one for each component of ∂M_0. Let $q_1, \ldots, q_r$ be the components of ∂F_0 and h be a regular fiber. It is very important to note that $e(q_i) = +1$ since q_i bounds a disk in F.

Now the list of classes of Seifert manifolds is the following (we use the notations of the previous paragraphs).

(Oo) M is orientable, the orbit surface F is orientable of genus g and e is the trivial homomorphism.

The Seifert symbol associated to this manifold is

$$M = (Oo, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r).$$

If $\{v_i\}_{i=1}^{2g}$ is a basis for $\pi_1(F)$, presentations for the fundamental groups of M and M_0 are the following:

$$\begin{aligned} \pi_1(M) \cong \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_r, h; [h, v_j] = 1, [h, q_i] = 1, \\ q_1 q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}], q_i^{\alpha_i} h^{\beta_i} = 1 \rangle. \end{aligned}$$

$$\begin{aligned} \pi_1(M_0) \cong \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_r, h; [h, v_j] = 1, [h, q_i] = 1, \\ q_1 q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}] \rangle. \end{aligned}$$

(On) M is orientable, the orbit surface F of M is non-orientable of genus g and if $\{v_1, \ldots, v_g\}$ is a basis for $\pi_1(F)$ such that each v_j is orientation reversing then $e(v_j) = -1$, for $j = 1, \ldots, g$.

The Seifert symbol associated to this manifold is

$$M = (On, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r).$$

Presentations for the fundamental groups of M and M_0 are

$$\begin{aligned}\pi_1(M) \;\cong\; & \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; v_j h v_j^{-1} = h^{-1}, [h, q_i] = 1, \\ & q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2, q_i^{\alpha_i} h^{\beta_i} = 1\rangle.\end{aligned}$$

$$\begin{aligned}\pi_1(M_0) \;\cong\; & \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; v_j h v_j^{-1} = h^{-1}, [h, q_i] = 1, \\ & q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2\rangle.\end{aligned}$$

(No) M is non-orientable, the orbit surface F is orientable of genus g and if $\{v_j\}$ is a basis for $\pi_1(F)$ then $e(v_1) = -1$ and $e(v_j) = +1$, for $j \geq 2$.

The Seifert symbol associated to this manifold is

$$M = (No, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r).$$

Fundamental groups of M and M_0 are isomorphic to the following presentations:

$$\begin{aligned}\pi_1(M) \;\cong\; & \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_s, h; q_1 q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}], \\ & [h, q_i] = 1, q_i^{\alpha_i} h^{\beta_i} = 1, v_1 h v_1^{-1} = h^{-1}, [v_j, h] = 1 \text{ for } j \geq 2\rangle > .\end{aligned}$$

$$\begin{aligned}\pi_1(M_0) \;\cong\; & \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_s, h; q_1 q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}], \\ & [h, q_i] = 1, v_1 h v_1^{-1} = h^{-1}, [v_j, h] = 1 \text{ for } j \geq 2\rangle.\end{aligned}$$

(NnI) M is non-orientable, the orbit surface F is non-orientable of genus g and the valuation is trivial.

The Seifert symbol for this class is

$$M = (NnI, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r).$$

In this case, If $\{v_j\}$ is a basis for $\pi_1(F)$ of orientation reversing curves, then presentations for the fundamental groups of M and M_0 are

$$\begin{aligned}\pi_1(M) \;\cong\; & \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [v_j, h] = 1, [h, q_i] = 1, \\ & q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2, q_i^{\alpha_i} h^{\beta_i} = 1\rangle.\end{aligned}$$

$$\pi_1(M_0) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [v_j, h] = 1, [h, q_i] = 1, \\ q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2 \rangle.$$

(NnII) M is non-orientable, the orbit surface F is non-orientable of genus $g \geq 2$ and if $\{v_j\}$ is a orientation reversing basis for $\pi_1(F)$, then $e(v_1) = +1$ and $e(v_j) = -1$, for all $j \geq 2$.

The Seifert symbol associated to this Seifert manifolds is

$$M = (NnII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r),$$

and, in this case, presentations for the fundamental groups of M and M_0 are

$$\pi_1(M) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [h, q_i] = 1, q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2, \\ q_i^{\alpha_i} h^{\beta_i} = 1, [v_1, h] = 1, v_j h v_j^{-1} = h^{-1}, \text{ for each } j \geq 2 \rangle.$$

$$\pi_1(M_0) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [h, q_i] = 1, q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2, \\ [v_1, h] = 1, v_j h v_j^{-1} = h^{-1}, \text{ for each } j \geq 2 \rangle.$$

(NnIII) M is non-orientable, the orbit surface F is non-orientable of genus $g \geq 3$ and if $\{v_j\}$ is a orientation reversing basis for $\pi_1(F)$, then $e(v_1) = e(v_2) = +1$ and $e(v_j) = -1$, for each $j \geq 2$.

The Seifert symbol associated to these manifolds is

$$M = (NnIII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r).$$

The fundamental groups of M and M_0 have the following presentations:

$$\pi_1(M) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [h, q_i] = 1, q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2, \\ q_i^{\alpha_i} h^{\beta_i} = 1, [v_1, h] = 1, [v_2, h] = 1, v_j h v_j^{-1} = h^{-1}, \text{ for each } j \geq 3 \rangle.$$

$$\pi_1(M_0) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [h, q_i] = 1, q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2, \\ [v_1, h] = 1, [v_2, h] = 1, v_j h v_j^{-1} = h^{-1}, \text{ for each } j \geq 3 \rangle.$$

The set $\{h, q_i, v_j\}$ is called a standard system of generators of $\pi_1(M)$ and of $\pi_1(M_0)$

The Seifert Classification Theorem is:

Theorem 3.2.1 *[**Se**] Two Seifert symbols represent homeomorphic Seifert manifolds by a fiber preserving homeomorphism if and only if one of the symbols can be changed into the other by a finite sequence of the following moves:*

1. *Permute the ratios.*
2. *Add or delete* $0/1$.
3. *Replace the pair* $\beta_i/\alpha_i, \beta_j/\alpha_j$ *by* $(\beta_i + k\alpha_i)/\alpha_i, (\beta_j - k\alpha_j)/\alpha_j$

Definition 3.2.2 *The rational number* $e_0(M) = \sum_{i=1}^{r} \beta_i/\alpha_i$ *is called* **the Euler number of** M.

3.3 Coverings of Seifert manifolds branched along fibers

Definition 3.3.1 *If* M *is a Seifert manifold and* $\varphi : \tilde{M} \to M$ *is a branched covering space of* M*, we say* φ *is* **branched along fibers** *if the branch set of* φ *is a finite union of fibers of* M.

Let $\{h_i\}_{i=1}^{r}$ be a set of fibers of M which contains all the exceptional fibers of M and a finite number of regular fibers of M. Recall each fiber has a fibered neighborhood V_i fiber preserving homeomorphic to a fibered solid torus $T(\beta_i/\alpha_i)$, for $i = 1, \ldots, r$. Recall $M_0 = \overline{M - \cup V_i}$. Note that M_0 is equal to M with all the exceptional fibers and some regular fibers drilled out.

Remember also that $q_i = p(\partial V_i)$, where $p : M \to F$ is the orbit projection.
A covering of M branched along fibers is determined by a representation $\omega : \pi_1(M - \cup_{i=1}^{r} h_i) \to S_n$ and therefore by a representation $\omega : \pi_1(M_0) \to S_n$.

To describe a covering of M branched along fibers our procedure is as follows:

- Let M be a Seifert manifold and consider the subspace M_0.
- Consider a representation $\omega : \pi_1(M_0) \to S_n$. This determines a finite covering space $\varphi_0 : \tilde{M}_0 \to M_0$.
- Let $T_i = q_i \times h$. Let $f_i : \partial V_i \to T_i$ be the glueing homeomorphisms. Using φ_0, lift the homeomorphisms $f_i : \partial V_i \to T_i$ to glueing homeomorphisms $\tilde{f}_i : \tilde{V}_i \to \tilde{T}_i$, where $\tilde{T}_i \subset \varphi^{-1}(T_i)$ is a component.

- In this way we obtain a covering $\varphi : \tilde{M} \to M$ of M branched along fibers.

Lemma 3.3.1 *Suppose M is a Seifert manifold and $\omega : \pi_1(M_0) \to S_n$ is a transitive representation. Assume $\omega(h) \neq (1)$ and $\omega(h) = \sigma_1 \cdots \sigma_k$, is the disjoint cycle decomposition of $\omega(h)$.*
Then $order(\sigma_1) = order(\sigma_2) = \cdots = order(\sigma_k)$.

Proof.
Note that the subgroup generated by h, denoted by $\langle h \rangle$, is a normal subgroup of $\pi_1(M_0)$; thus $\langle \omega(h) \rangle$ is normal in $Im(\omega)$. Let $\sigma_1 = (a_{1,1}, \ldots, a_{1,m})$; then $A = \{a_{1,1}, \ldots, a_{1,m}\}$ is an orbit of $\langle \omega(h) \rangle$.

Let $a_{s,1} \in \{1, \ldots, n\}$. We assume that $a_{s,1}$ appears in the orbit non-trivial of the cycle σ_s. Since ω is transitive there an $\alpha \in \pi_1(M_0)$ such that $\omega(\alpha)(a_{1,1}) = a_{s,1}$. Let us write $\omega(\alpha)(A) = \{a_{s,1}, \ldots, a_{s,m}\}$.

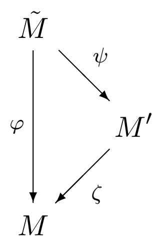
Also

$$\begin{aligned} \langle \omega(h) \rangle \, (\omega(\alpha)(A)) &= (\langle \omega(h) \rangle \omega(\alpha))(A) \\ &= (\omega(\alpha) \langle \omega(h) \rangle)\,(A) \text{ since } \langle \omega(h) \rangle \text{ is normal,} \\ &= \omega(\alpha)\,(\langle \omega(h) \rangle (A)) \\ &= \omega(\alpha)(A) \text{ since } A \text{ is an orbit of } \langle \omega(h) \rangle. \end{aligned}$$

Thus $\{a_{s,1}, \ldots, a_{s,m}\}$ is an orbit of $\langle \omega(h) \rangle$ and $\sigma_s = (a_{s,1} \cdots a_{s,m})$. □

Using Lemma (3.1.1) we can prove the following theorem which is our main tool to study coverings of a Seifert manifold.

Theorem 3.3.1 *Let M be a Seifert manifold and assume that $\varphi : \tilde{M} \to M$ is an n-fold covering branched along fibers of M. Assume $\tilde{M}$ is connected. Then there are coverings $\psi : \tilde{M} \to M'$ and $\zeta : M' \to M$ branched along fibers such that the following diagram is commutative*



Also if ω_ψ and ω_ζ are the representations associated to ψ and ζ, respectively, we have that $\omega_\psi(h') = \varepsilon_t$ and $\omega_\zeta(h) = (1)$, where (1) is the identity permutation of S_n, $\varepsilon_t = (1, 2, \ldots, t)$ is the standard $t-$cycle, and h and h' are regular fibers of M and M', respectively.

Proof.

Since $\tilde{M}$ is connected then ω_φ, the representation determined by φ, is transitive. If $\omega(h) = \sigma_1 \cdots \sigma_k$ is the disjoint cycle decomposition of $\omega(h)$ in the proof of the previous lemma we also proved that each cycle $\sigma_s = (a_{s,1} \cdots a_{s,m})$ of $\omega(h)$ gives us a set of $m, k-$imprimitivity for $Im(\omega)$, namely, $\Delta_s = \{a_{s,1}, \ldots, a_{s,m}\}$.

The quotient homomorphism $q : Im(\omega) \to S(\{\Delta_1, \ldots, \Delta_k\})$ satisfies that $q(\omega(h))(\Delta_i) = \Delta_i$. Therefore $q \circ \omega(h) = (\Delta_1)$, the identity permutation in $S(\{\Delta_1, \ldots, \Delta_k\})$.

Also $\omega(h) \in H_1 = q^{-1}(St(\Delta_1))$ and $\gamma_1 : H_1 \to S_m = S(\Delta_1)$ sends h into an $m-$cycle. □

Therefore in order to understand the connected coverings of a Seifert manifold M branched along fibers, we only need to study representations that send a regular fiber h of M into the identity permutation and representations that send a regular fiber h of M into an standard $n-$cycle.

3.3.1 The case $\omega(h) = (1)$, the identity permutation

If $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$, where Xx is a symbol in $\{Oo, On, No, NnI, NnII, NnIII\}$, we will write M_0 for the manifold obtained from M by drilling out the fibers correponding to the ratios $\beta_1/\alpha_1, \ldots, \beta_r/\alpha_r$.

Along this section $\omega : \pi_1(M_0) \to S_n$ is a transitive representation such that

$$\begin{array}{rcl} \omega(h) & = & (1), \\ \omega(q_i) & = & \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \ \ \textit{and} \\ \omega(v_j) & = & \rho_{j,1} \cdots \rho_{j,s_j}, \end{array}$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively.

Let $\tilde{M}_0 = \varphi^{-1}(M_0)$.

Lemma 3.3.2 *Suppose that M is a Seifert manifold with orbit surface F and $n \in \mathbb{N}$. Let $\omega : \pi_1(M_0) \to S_n$ be a representation defined by*

$$\begin{array}{rcl} \omega(h) & = & (1), \\ \omega(q_i) & = & \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \ \textit{for } i = 1, \ldots, r \ \ \textit{and} \\ \omega(v_j) & = & \rho_{j,1} \cdots \rho_{j,s_j}. \end{array}$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively.

Let $\varphi : \tilde{M} \to M$ be the branched covering associated to ω and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$. Assume $\tilde{g}$ is the genus of G.

i) Suppose F is non-orientable. If G is orientable, then

$$\tilde{g} = 1 - \frac{n(2-g) + \sum_{i=1}^{r} \ell_i - nr}{2};$$

otherwise,

$$\tilde{g} = n(g-2) + 2 + nr - \sum_{i=1}^{r} \ell_i.$$

ii) If F is orientable, then $\tilde{g} = 1 + n(g-1) + \frac{nr - \sum_{i=1}^{r} \ell_i}{2}$.

Proof.

This is essentially the Riemann-Hurwitz formula. Let F_0 be the orbit surface of M_0 and G_0 be the orbit surface of $\tilde{M}_0 = \varphi^{-1}(M_0)$.

Note that $\varphi^{-1}(h)$ has n-components, $\tilde{h}_1, \ldots, \tilde{h}_n$. Thus if $\tilde{x}, \tilde{y} \in \tilde{h}_t$, for some $t \in \{1, \ldots, n\}$, we have $\tilde{p}(\tilde{x}) = \tilde{p}(\tilde{y})$ and $p(\varphi(\tilde{x})) = p(\varphi(\tilde{y}))$; by the Universal Property of Quotients we have a *covering of $n-$sheets* $\overline{\varphi} : G_0 \to F_0$ such that the following diagram is commutative:

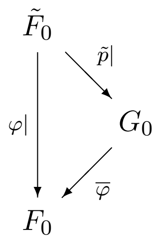
$$\begin{array}{ccc} \tilde{M}_0 & \xrightarrow{\varphi|} & M_0 \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ G_0 & \xrightarrow[\overline{\varphi}]{\cdots\cdots} & F_0 \end{array}$$

The representation $\overline{\omega} : \pi_1(F_0) \to S_n$ associated to $\overline{\varphi}$ is defined as

$$\begin{array}{rcl} \overline{\omega}(q_i) & = & \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and} \\ \overline{\omega}(v_j) & = & \rho_{j,1} \cdots \rho_{j,s_j}, \text{ for } j = 1, \ldots, g. \end{array}$$

That is $\overline{\varphi} = \varphi | G_0$. Since ω is transitive and $\omega(h) = (1)$, $\tilde{F} = \varphi^{-1}(F)$ is connected and let $\tilde{F}_0 = \tilde{F} \cap \tilde{M}_0$. It is easy to see that $\tilde{F}_0$ is a horizontal surface, then $\tilde{p}| : \tilde{F}_0 \to G_0$ is a covering. Also we know that $\varphi| : \tilde{F}_0 \to F_0$ is a covering of n sheets.

Then there exists a commutative diagram



Thus $\tilde{F_0} \cong G_0$ and we conclude $\tilde{F} \cong G$.

Since $\tilde{F_0}$ is a covering of n sheets of F_0, then $\chi(\tilde{F_0}) = n\chi(F_0)$. Since $\omega(q_i) = \sigma_{i,1} \cdots \sigma_{i,s}$, therefore $\varphi^{-1}(q_i)$ has ℓ_i components; thus $\partial\tilde{F_0}$ has $\sum_{i=1}^{r} \ell_i$ components for $\partial F_0 = \sqcup q_i$. Hence

$$\chi(\tilde{F}) = n\chi(F_0) + \sum_{i=1}^{r} \ell_i \tag{3.1}$$

i) Suppose F is non-orientable; then $\chi(F_0) = 2 - g - r$ and Equation (3.1) has the following form

$$\chi(\tilde{F}) = n(2 - g - r) + \sum_{i=1}^{r} \ell_i.$$

If G is orientable, then G has Euler characteristic equal to $2 - 2\tilde{g}$ and

$$\tilde{g} = 1 - \frac{n(2-g) + \sum_{i=1}^{r} \ell_i - nr}{2}.$$

If G is non-orientable, we know that $\chi(G) = 2 - \tilde{g}$. Therefore,

$$\tilde{g} = n(g-2) + 2 + nr - \sum_{i=1}^{r} \ell_i.$$

ii) When F is orientable, G is also orientable. Since $\chi(F_0) = 2 - 2g - r$ and $\chi(G) = 2 - 2\tilde{g}$, by (3.1) we conclude

$$\tilde{g} = 1 + n(g-1) + \frac{nr - \sum_{i=1}^{r} \ell_i}{2}$$

□

Since M_0 is an S^1-bundle over F and $\omega(h) = (1)$, then $\tilde{M_0}$ is the pullback of M_0 by $\overline{\varphi} : G_0 \to F_0$ and the following lemma follows.

Lemma 3.3.3 *If M is a Seifert manifold and $\omega : \pi_1(M_0) \to S_n$ is a representation defined by*

$$\begin{array}{rcl} \omega(h) & = & (1), \\ \omega(q_i) & = & \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and} \\ \omega(v_j) & = & \rho_{j,1} \cdots \rho_{j,s_j}, \end{array}$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively. Let $\varphi : \tilde{M} \to M$ be the covering determined by ω.
Then $\tilde{e} = \varphi^*(e)$, where e and $\tilde{e}$ are the valuations of M and $\tilde{M}$, respectively.

Lemma 3.3.4 *Let M be a non-orientable Seifert manifold. Let F and G be the orbit surfaces of M and $\tilde{M}$, respectively. Consider the orbit projections $\tilde{p} : \tilde{M} \to G$ and $p : M \to F$. Suppose $\overline{\varphi} : G \to F$ is the induced covering of orbit surfaces. Recall that $\overline{\varphi} = \varphi|G$. Let F_0 and G_0 be the orbit surfaces of M_0 and $\tilde{M}_0 = \varphi^{-1}(M_0)$, respectively.*

If v is a simple closed curve in F_0 and if $\tilde{v} \subset G_0$ is the component of $\varphi^{-1}(v)$ corresponding to the cycle $\rho = (a_1, \ldots, a_r)$ of $\omega(v)$, then:

(a) *$\varphi| : \tilde{p}^{-1}(\tilde{v}) \to p^{-1}(v)$ is an r-fold covering space.*

(b) *If $e(v) = +1$, then $\tilde{e}(\tilde{v}) = +1$.*

(c) *Suppose that $e(v) = -1$. Then $\tilde{e}(\tilde{v}) = +1$ if and only if order(ρ) is even.*

Proof.
Note that $p^{-1}(v)$ and $\tilde{p}^{-1}(\tilde{v})$ are S^1-bundles over v and $\tilde{v}$, respectively.

(a) It is easy to see that $\varphi(\tilde{p}^{-1}(\tilde{v})) = p^{-1}(v)$ because $\overline{\varphi}(\tilde{v}) = v$ and the following diagram commutes.

$$\begin{array}{ccc} \tilde{M}_0 & \xrightarrow{\varphi} & M_0 \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ G_0 & \xrightarrow{\varphi|} & F_0 \end{array}$$

Thus $\varphi| : \tilde{p}^{-1}(\tilde{v}) \to p^{-1}(v)$ is a covering space and the representation associated to this covering is $\omega' : \pi_1(p^{-1}(v)) \to S_r = S(\{a_1, \ldots, a_r\})$ defined by

$$\begin{array}{rcl} \omega'(h) & = & (1) \text{ and} \\ \omega'(v) & = & \rho. \end{array}$$

(b) Since $p^{-1}(v)$ and $\tilde{p}^{-1}(\tilde{v})$ are S^1-bundles over v and $\tilde{v}$, respectively, $\varphi| : \tilde{p}^{-1}(\tilde{v}) \to p^{-1}(v)$ is a covering, $\varphi(\tilde{v}) = v$ and $e(v) = +1$ then by Remark (3.1.1) we get $\tilde{e}(\tilde{v}) = +1$.

(c) Note that r odd implies $\tilde{e}(\tilde{v}) = -1$ (Corollary 3.1.1). Thus $\tilde{e}(\tilde{v}) = +1$ only if r is even. On the other hand, suppose r even and let $\rho = (1 \cdots r)$. Define $\Delta_1 = \{a_1, a_3, \ldots, a_{r-1}\}$ and $\Delta_2 = \{a_2, a_4, , \ldots, a_r\}$, then $q : Im(\omega') \to S_2 = S(\{\Delta_1, \Delta_2\})$ sends v into (Δ_1, Δ_2) and we have $q \circ \omega = e$. Therefore $\tilde{e}$ is trivial and $\tilde{e}(\tilde{v}) = +1$ (See Remark 3.1.1) □

Lemma 3.3.5 *Suppose that X and X' are n-manifolds with boundary. Let Y and Y' be connected sub-manifolds of ∂X and $\partial X'$, respectively. If $f : Y \to Y'$ is a homeomorphism, then $Z = X \sqcup X'/f$ is orientable if and only if X and X' are orientable.*

Proof.
Assume O_z is an orientation of Z. Then $O_z|X$ and $O_z|X'$ are orientations for X and X', respectively.

Now, suppose O and O' are orientations of X and X', respectively.

- If f is orientation reversing, it is clear that $O \cup O'$ is an orientation of Z.
- Is f is orientation preserving, then $O \cup (-O')$ is an orientation for Z.

□

Suppose M is a Seifert manifold with orbit projection $p : M \to F$. Let $\omega : \pi_1(M_0) \to S_n$ be a representation such that

$$\begin{aligned} \omega(h) &= (1), \\ \omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \;\; \textit{and} \\ \omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j}, \end{aligned}$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively, and M_0 is the Seifert manifold M with the exceptional fibers drilled out and without some singular fibers that appear in the Seifert symbol, $\sigma_{i,k}$ and $\rho_{j,t}$ are cycles.

Assume $\varphi : \tilde{M} \to M$ is the covering of M branched along fibers associated to ω. Let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$ and recall $\varphi| : G \to F$ is a covering.

Write $F_0 = p(M_0)$ and note that a presentation for $\pi_1(F_0)$ is $\langle v_1, \ldots, v_k, q_1, \ldots, r : -\rangle$: Let $\tilde{M}_0 = \varphi^{-1}(M_0)$ and $G_0 = \varphi^{-1}(F_0)$. Note that $G_0 = G \cap \tilde{M}_0$ and $\varphi| : G_0 \to F_0$ is a covering.

In order to determine what class of Seifert manifold $\tilde{M}$ belong to, we analyze two cases: M orientable and M non-orientable. By Lemma (3.3.5), to see if $\tilde{M}$ and G are orientable we only need to determine the orientability of $\tilde{M}_0 = \varphi^{-1}(M_0)$ and $G_0 = G \cap \tilde{M}_0$.

(a) *The case M orientable.*

Assume $M = (Oo, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is an orientable Seifert manifold with orientable orbit surface F of genus g. Recall also that $\alpha \geq 1$ and β_i are coprime numbers. The numbers β_i/α_i in the Seifert symbol are defined by a fibered torus $T(\beta_i/\alpha_i)$ which is a fibered neighborhood of some fiber h_i of M. All the exceptional fibers are contained in the set $\{h_i\}_{i=1}^r$. Recall that $M_0 = \overline{M - \sqcup T(\beta_i/\alpha_i)}$. Note that $\partial M_0 = \sqcup_{i=1}^r T_i$, where T_i is a torus for $i = 1, \ldots, r$ and $\sqcup_{i=1}^r T_i$ denotes the disjoint union of the tori T_i. Let $q_i = p(T_i)$, where $p : M \to F$ is the orbit projection of M.

If $\{v_i\}_{i=1}^{2g}$ is a basis for $\pi_1(F)$, a presentation for the fundamental groups of M and M_0 are

$$\begin{aligned}\pi_1(M) \;\cong\; & \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_r, h; [h, v_j] = 1, [h, q_i] = 1, \\ & \qquad q_1 q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}], q_i^{\alpha_i} h^{\beta_i} = 1\rangle.\end{aligned}$$

$$\begin{aligned}\pi_1(M_0) \;\cong\; & \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_r, h; [h, v_j] = 1, [h, q_i] = 1, \\ & \qquad q_1 q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}] >\rangle.\end{aligned}$$

Theorem 3.3.2 *Suppose that $M = (Oo, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ and $\omega : \pi_1(M_0) \to S_n$ is a transitive representation defined by*

$$\begin{aligned}\omega(h) &= (1), \\ \omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and} \\ \omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j}, \text{ for } j = 1, \ldots, 2g;\end{aligned}$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively, and $\{h, q_i, v_j\}$ is a standard system of generators of M_0. Assume that $\varphi : \tilde{M} \to M$ is the covering branched along fibers associated to ω and $\tilde{p} : \tilde{M} \to G$ is the orbit projection of $\tilde{M}$.

Then $\tilde{M} \in Oo$, that is, M is orientable and G is orientable.

Proof.

Since M and F are orientable, then M_0 and F_0 are orientable. Thus the first Stiefel-Whitney classes of M_0 and F_0, $w_1(M_0)$ and $w_1(F_0)$, respectively, are trivial. Recall we have coverings $\varphi| : \tilde{M}_0 \to M$ and $\varphi| : G_0 \to F_0$, where $\tilde{M}_0 = \varphi^{-1}(M_0)$ and $G_0 = G \cap \tilde{M}_0 = \varphi^{-1}(F_0)$. Then $\tilde{M}_0$ and G_0 are orientable since $w_1(\tilde{M}_0)$ and $w_1(G_0)$ are (Remark 3.1.1). Therefore $\tilde{M}$ is orientable and G is orientable. □

Let $M = (On, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ be a Seifert manifold: M is orientable and the orbit surface F of M is non-orientable of genus g. Again the numbers β_i/α_i in the Seifert symbol are defined by a fibered torus $T(\beta_i/\alpha_i)$ which is a neighborhood of some fiber h_i of M. All exceptional fibers belong to the set $\{h_i\}_{i=1}^r$. Consider the manifold with boundary $M_0 = \overline{M - \sqcup T(\beta_i/\alpha_i)}$. Note that $\partial M_0 = \sqcup_{i=1}^r T_i$, where T_i is a torus for $i = 1, \ldots, r$. Let $q_i = p(T_i)$, where $p : M \to F$ is the orbit projection of M.

If $\{v_1, \ldots, v_g\}$ is a basis for $\pi_1(F)$ such that each v_j is orientation reversing, then a presentation for the fundamental groups of M and M_0 are

$$\begin{aligned}\pi_1(M) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; v_j h v_j^{-1} = h^{-1}, [h, q_i] = 1, \\ q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2, q_i^{\alpha_i} h^{\beta_i} = 1\rangle.\end{aligned}$$

$$\begin{aligned}\pi_1(M_0) \cong \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; v_j h v_j^{-1} = h^{-1}, [h, q_i] = 1, \\ q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2\rangle.\end{aligned}$$

Theorem 3.3.3 *Let $M = (On, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$. Suppose $\omega : \pi_1(M_0) \to S_n$ is a representation such that*

$$\begin{aligned}\omega(h) &= (1), \\ \omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and} \\ \omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j}, \text{ for } j = 1, \ldots, g;\end{aligned}$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively, and $\{h, q_i, v_j\}$ a standard system of generators of $\pi_1(M_0)$. Assume $\varphi : \tilde{M} \to M$ is the covering of M branched along fibers determined by ω and $\tilde{p} : \tilde{M} \to G$ is the orbit projection of $\tilde{M}$.
Then $\tilde{M} \in Oo$ ($\tilde{M}$ and G are orientable) or $\tilde{M} \in On$ ($\tilde{M}$ is orientable and G is non-orientable).

Also $\tilde{M} \in Oo$ if and only if $\omega|\pi_1(F_0)$ trivializes the bundle of $w_1(F_0)$, where $w_1(F_0)$ is the first Stiefel-Whitney class of F_0.

Proof.
Note that M_0 is orientable since M is orientable. Then the first Stiefel-Whitney class of M_0, $w_1(M_0)$, is trivial. By Lemma 3.1.1, we have that the first Stiefel-Whitney class of $\tilde{M}_0 = \varphi^{-1}(M_0)$, $w_1(\tilde{M}_0)$, is trivial. Thus $\tilde{M}_0$ is orientable and we conclude $\tilde{M}$ is orientable.

We have only two classes of orientable Seifert manifolds, namely, Oo and On. Therefore $\tilde{M} \in Oo$ or $\tilde{M} \in On$. By Corollary 3.1.2, the surface G_0 is orientable (and $\tilde{M} \in Oo$) if and only if $\omega|\pi_1(F_0)$ has sets of $\frac{n}{2}, 2-$imprimitivity, Δ_1 and Δ_2, such that the quotient homomorphism $q : Im(\omega|\pi_1(F_0)) \to S_2$ satisfies that $q \circ \omega = w_1(F_0)$. $\square$

Example 3.3.1

Let $M = (On, 1; 1/2)$. Since $M \in On$, M is orientable and the orbit surface of M, F, is non-orientable. The genus of F is 1, that is, F is a projective plane. Let $T(1/2)$ be the solid fibered torus homeomorphic (under a fiber preserving homeomorphism) to a neighborhood of the only exceptional fiber. The boundary of $M_0 = \overline{M - T(1/2)}$ is a torus T_1. Let $q_1 = p(T_1)$, where $p : M \to F$ is the orbit projection of M. Let v_1 be the generator of $\pi_1(F)$ and let h be a regular fiber of M.

Note that

$$\pi_1(M_0) \cong \langle v_1, q_1, h : [h, q_1] = 1, v_1 h v_1^{-1} = h, q_1 = v_1^2 \rangle$$

and

$$\pi_1(M) \cong \langle v_1, q_1, h : [h, q_1] = 1, v_1 h v_1^{-1} = h^{-1}, q_1 = v_1^2, q_1^2 h = 1 \rangle$$

- Consider the representation $\omega : \pi_1(M_0) \to S_2$ defined by

$$\begin{array}{rcl} \omega(h) & = & (1), \\ \omega(q_1) & = & (1,2) \ \textit{and} \\ \omega(v_1) & = & (1). \end{array}$$

  Assume $\varphi : \tilde{M} \to M$ is the covering determined by ω. Note that the only sets of $1, 2-$imprimitivity for $Im(\omega|\pi_1(F_0))$ are $\Delta_1 = \{1\}$ and $\Delta_2 = \{2\}$. It is clear that $q : Im(\omega|\pi_1(F_0)) \to S_2 = S(\{\Delta_1, \Delta_2\})$ holds the relation: $q(v_1) = (\Delta_1)$, the identity permutation in S_2. Thus $\tilde{M} \in On$ (*Cf.* Theorem 3.3.3).

- If we consider $\omega : \pi_1(M_0) \to S_2$ defined by

$$\begin{array}{rcl} \omega(h) & = & (1), \\ \omega(q_1) & = & (1,2) \ \textit{and} \\ \omega(v_1) & = & (1,2), \end{array}$$

  then $\tilde{M}$ is the 2-fold covering space of orientation and $\tilde{M} \in Oo$ (*Cf.* Theorem 3.3.2).

(b) *The case M non-orientable.*

(i) The case $M \in No$.

Assume $M = (No, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$. Recall that in this kind of Seifert manifolds M is non-orientable and the orbit surface F is orientable of genus g; The numbers β_i/α_i in the Seifert symbol are defined by a fibered torus $T(\beta/\alpha_i)$ which is a fibered neighborhood of some fiber h_i of M. The set of exceptional fibers is contained in the set $\{h_i\}_{i=1}^r$. Recall $M_0 = \overline{M - \sqcup T(\beta_i/\alpha_i)}$. Note that $\partial M_0 = \sqcup_{i=1}^r T_i$, where T_i is a torus for $i = 1, \ldots, r$. Let $q_i = p(T_i)$, where $p : M \to F$ is the orbit projection of M.

If h is a regular fiber and $\{v_j\}_{i=1}^{2g}$ is a basis for $\pi_1(F)$ then the valuation homomorphism $e : \pi_1(M) \to S_n$ satisfies $e(v_1) = -1$ and $e(v_j) = +1$, for $j \geq 2$.

Fundamental groups of M and M_0 have the following presentations:

$$\begin{aligned}\pi_1(M) \quad &\cong \quad \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_s, h; q_1 q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}],\\ &\qquad [h, q_i] = 1, q_i^{\alpha_i} h^{\beta_i} = 1, v_1 h v_1^{-1} = h^{-1}, [v_j, h] = 1 \text{ for } j \geq 2\rangle.\\ \pi_1(M_0) \quad &\cong \quad \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_s, h; q_1 q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}],\\ &\qquad [h, q_i] = 1, v_1 h v_1^{-1} = h^{-1}, [v_j, h] = 1 \text{ for } j \geq 2\rangle.\end{aligned}$$

The orbit projection of M_0 is $p| : M_0 \to F_0$, where $F_0 \subset F$ is a surface. If $e' : \pi_1(F_0) \to S_n$ is the valuation homomorphism in M_0 then $e' = i_{\#} \circ e$, where e is the valuation homomorphism of M and $i : M_0 \to M$ is the natural inclusion map.

Theorem 3.3.4 *Consider $M = (No, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ and suppose $\{v_1, \ldots, v_{2g}\}$ is a basis for the orbit surface F of M. Assume that $\omega : \pi_1(M_0) \to S_n$ is a representation defined by*

$$\begin{aligned}\omega(h) &= (1),\\ \omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and}\\ \omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j}, \text{ for } j = 1, \ldots, 2g,\end{aligned}$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively. Assume $\varphi : \tilde{M} \to M$ is the covering of M branched along fibers determined by ω and $\tilde{p} : \tilde{M} \to G$ is the orbit projection of $\tilde{M}$. Let $e' : \pi_1(F_0) \to S_2$ be the valuation homomorphism of M_0.

Then $\tilde{M} \in Oo$ ($\tilde{M}$ and G are orientable) or $\tilde{M} \in No$ ($\tilde{M}$ is non-orientable and G is orientable). Furthermore $\tilde{M} \in Oo$ if and only if $\omega|\pi_1(F_0)$ trivializes the bundle of e'.

Proof.

Recall $\tilde{M}_0 = \varphi^{-1}(M_0)$, $G_0 = G \cap \tilde{M}_0 = \varphi^{-1}(F_0)$. We have coverings $\varphi| : \tilde{M}_0 \to M_0$ and $\varphi| : G_0 \to F_0$. Since the first Stiefel-Whitney class of F_0, $w_1(F_0)$, is trivial then $w_1(G_0)$ is trivial (Remark 3.1.1). Therefore $\tilde{M} \in No$ or $\tilde{M} \in Oo$.

By Remark 2.2.1.(b), the valuation homomorphism $e : \pi_1(F) \to \mathbb{Z}_2 \cong S_2$ gives us a covering $\varphi_e : (F_e)_0 \to F_0$ of 2-sheets.

Let $e' : \pi_1(F_0 \to \mathbb{Z}_2 \cong S_2$ be the valuation homomorphism of M_0. According to Lemma 3.3.3 and Theorem 3.1.1, e' is trivial if and only if $\omega|\pi_1(F_0)$ trivializes the bundle of e'. In the class No the valuation homomorphism is non-trivial. Therefore $\tilde{M} \in Oo$ if and only if $\omega|\pi_1(F_0)$ trivializes the bundle of e'. □

Remark 3.3.1 *Let $M = (No, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ with orbit projection $p : M \to F$. Suppose $\{v_j\}_{j=1}^{2g}$ is a basis for $\pi_1(F)$ and $M_0 = \overline{M - \sqcup T(\beta_i/\alpha_i)}$, where $T(\beta_i/\alpha_i)$ is a fibered neighborhood of either a exceptional fiber or a regular fiber. Recall $F_0 = F \cap M_0$. Assume $\varphi : \tilde{M} \to M$ is an $n-$fold covering of M branched along fibers, where $\tilde{M}$ is connected. Let $\omega : \pi_1(M_0) \to S_n$ be the transitive representation determined by φ, and let h be a regular fiber of M.*

If $\omega(h) = (1)$, the identity permutation in S_n, a useful criterion to determine if $\tilde{M} \in No$ or $\tilde{M} \in Oo$ is the following:

1. *If n is odd, then $\tilde{M} \in No$*

2. *If $\omega(v_1)$ has a cycle of odd order then $\tilde{M} \in No$*

3. *If $Im(\omega|\pi_1(F_0))$ is not $\frac{n}{2}, 2-$imprimitive then $\tilde{M} \in No$.*

4. *If $Im(\omega|\pi_1(F_0))$ is $\frac{n}{2}, 2-$imprimitive, then $\tilde{M} \in Oo$ if and only if $\omega|\pi_1(F_0)$ trivializes the bundle of e', where $e' : \pi_1(F_0) \to \mathbb{Z}_2 \cong S_2$ is the valuation homomorphism of M_0.*

Example 3.3.2

Let $M = (No, 1; 1/2)$. The manifold M is non-orientable and F, the orbit surface of M, is an orientable surface of genus 1. Note that M has exactly one exceptional fiber h'. Then there exists a fibered neighborhood of h' homeomorphic to the solid fibered torus $T(1/2)$. Consider $M_0 = \overline{M - T(1/2)}$ and $\{v_1, v_2\}$ a basis for $\pi_1(F)$. Note that ∂M_0 is a torus T_1. Let $q_1 = p(T_1)$, where $p : M \to F$ is the orbit projection of M and let h be a regular fiber of M.

Presentations for the fundamental groups of M_0 and M are

$$\pi_1(M_0) \cong \langle v_1, v_2, q_1, h : v_1 h v^{-1} = h^{-1}, [v_2, h] = 1, [h, q_1] = 1, q_1 = [v_1, v_2] \rangle$$

and

$$\pi_1(M_0) \cong \langle v_1, v_2, q_1, h : v_1 h v^{-1} = h^{-1}, [v_2, h] = 1, [h, q_1] = 1, q_1 = [v_1, v_2], q_1^2 h = 1 \rangle.$$

- Let $\omega : \pi_1(M_0) \to S_4$ be the representation defined by

$$\begin{array}{rcl} \omega(h) & = & (1), \\ \omega(v_1) & = & (1,2)(3,4), \\ \omega(v_2) & = & (1,3)(2,4), \text{ and} \\ \omega(q_1) & = & (1). \end{array}$$

  Suppose $\varphi : \tilde{M} \to M$ is the covering of M determined by ω.

  Observe that $\Delta_1 = \{1,3\}$ and $\Delta_2 = \{2,4\}$ are sets of 2, 2-imprimitivity for $Im(\omega|\pi_1(F_0))$ such that $q : Im(\omega|\pi_1(F_0)) \to S(\{\Delta_1, \Delta_2\})$ satisfies

$$\begin{array}{rcl} q(v_1) & = & (\Delta_1, \Delta_2) \\ q(v_2) & = & (\Delta_1), \text{ the identity permutation in } S(\{\Delta_1, \Delta_2\}), \text{ and} \\ q(q_1) & = & (\Delta_1). \end{array}$$

  On the other hand,

$$\begin{array}{rcl} e(v_1) & = & (1,2) = -1 \\ e(v_2) & = & (1) = +1, \text{ and} \\ e(q_1) & = & (1) = +1. \end{array}$$

  Therefore $\tilde{M} \in Oo$ (*Cf* Theorem 3.3.4).
- Suppose $\omega : \pi_1(M_0) \to S_3$ is the representation such that

$$\begin{array}{rcl} \omega(h) & = & (1), \\ \omega(v_1) & = & (1,2,3) \\ \omega(v_2) & = & (1,2,3) \textit{ and} \\ \omega(q_1) & = & (1). \end{array}$$

  Let $\varphi : \tilde{M} \to M$ be the covering of M determined by ω. In this case $\tilde{M} \in No$ because 3 is odd (*Cf.* Theorem 3.3.4).

(ii) The case $M \in NnI$.
Suppose $M = (NnI, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$. That is M is non-orientable, the orbit surface F is non-orientable of genus g and the valuation is trivial. Consider $M_0 = \overline{M - T(\beta_i/\alpha_i)}$, where $T(\beta_i/\alpha_i)$ is the solid fibered torus corresponding to the ratio β_i/α_i. Note that $\partial M_0 = \sqcup_{i=1}^{r} T_i$, where T_i is a torus for $i = 1, \ldots, r$. Let $F_0 = p(M_0)$ and $q_i = p(T_i)$, where $p : M \to F$ is the orbit projection of M.

If h is a regular fiber of M and $\{v_j\}$ is a basis for $\pi_1(F)$ of orientation reversing curves, then presentations for the fundamental groups of M and M_0 are:

$$\begin{aligned}\pi_1(M) \;\cong\; & \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [v_j, h] = 1, [h, q_i] = 1, \\ & q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2, q_i^{\alpha_i} h^{\beta_i} = 1\rangle.\end{aligned}$$

$$\begin{aligned}\pi_1(M_0) \;\cong\; & \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [v_j, h] = 1, [h, q_i] = 1, \\ & q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2\rangle.\end{aligned}$$

The valuation homomorphism of M_0, $e' : \pi_1(F_0) \to S_n$, also is trivial.

Theorem 3.3.5 *Let $M = (NnI, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ be a non-orientable Seifert manifold. Consider a representation $\omega : \pi_1(M_0) \to S_n$ defined by*

$$\begin{aligned}\omega(h) &= (1), \\ \omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and} \\ \omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j},\end{aligned}$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively. Suppose $\varphi : \tilde{M} \to M$ is the covering associated to ω. Let $\tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

Then $\tilde{M} \in Oo$ or $\tilde{M} \in NnI$. Moreover, $\tilde{M} \in Oo$ if and only if $\omega|\pi_1(F_0)$ trivializes the bundle of $w_1(F_0)$, where $w_1(F_0)$ is the first Stiefel-Whitney class of F_0.

Proof.

Recall $\tilde{M}_0 = \varphi^{-1}(M_0)$ and $G_0 = \varphi^{-1}(F_0)$. Let $\tilde{e} : \pi_1(G_0) \to S_2$ be the valuation homomorphism of M_0. Since e is trivial we have $\tilde{e}$ trivial by Lemma 3.3.3 and Remark 3.1.1. There are only two classes of Seifert manifolds having trivial valuation homomorphism, namely, $\tilde{M} \in Oo$ or $\tilde{M} \in NnI$. Therefore $\tilde{M} \in Oo$ or $\tilde{M} \in NnI$.

Since $\varphi| : G \to F$ is a covering, by Corollary (3.1.2), G_0 is orientable if and only if there are sets of $\frac{n}{2}, 2-$imprimitivity, Δ_1 and Δ_2, such that $q \circ (\omega|\pi_1(F_0)) = w_1(F_0)$. Therefore $\tilde{M} \in Oo$ if and only if there are sets of $\frac{n}{2}, 2-$imprimitivity, Δ_1 and Δ_2, such that $q \circ (\omega|\pi_1(F_0)) = w_1(F_0)$. □

Example 3.3.3

Consider $M = (NnI, 1; 1/2)$. Suppose $p : M \to F$ is the orbit projection of M. In this case, F is a non-orientable surface of genus 1. Note that M has exactly one exceptional fiber h'. Then there exists a fibered neighborhood of h' homeomorphic to the solid fibered torus $T(1/2)$. Consider $M_0 = \overline{M - T(1/2)}$ and let $\{v_1\}$ be a basis for $\pi_1(F)$. Note that ∂M_0 is a torus T_1. Let $F_0 = p(M_0)$ and $q_1 = p(T_1)$, where $p : M \to F$ is the orbit projection of M and let h be a regular fiber of M.

Presentations for the fundamental groups of M_0 and M are the following:

$$\pi_1(M_0) \cong \langle v_1, q_1, h : [v_1, h] = 1, [q_1, h] = 1, q_1 = v_1^2 \rangle$$

and

$$\pi_1(M_0) \cong \langle v_1, q_1, h : [v_1, h] = 1, [q_1, h] = 1, q_1 = v_1^2, q_1^2 h = 1 \rangle.$$

- Assume that $\omega : \pi_1(M_0) \to S_3$ is the representation such that

$$\begin{aligned} \omega(h) &= (1), \\ \omega(q_1) &= (1,3,2) \ \textit{and} \\ \omega(v_1) &= (1,2,3). \end{aligned}$$

  Let $\varphi : \tilde{M} \to M$ be the covering determined by ω. Suppose G is the orbit surface of $\tilde{M}$. Then G is non-orientable because n is odd. Therefore $\tilde{M} \in NnI$ (*Cf.* Theorem 3.3.5)
- If $\omega : \pi_1(M_0) \to S_4$ is a representation defined by

$$\begin{aligned} \omega(h) &= (1), \\ \omega(q_1) &= (1,3)(2,4) \ \textit{and} \\ \omega(v_1) &= (1,2,3,4). \end{aligned}$$

  Suppose $\varphi : \tilde{M} \to M$ be the covering associated to ω and G is the orbit surface of $\tilde{M}$.

  Then $\Delta_1 = \{1,3\}$ and $\Delta_2 = \{2,4\}$ are sets of $2, 2-$imprimitivity for $Im(\omega|\pi_1(F_0))$, such that $q(v_1) = (\Delta_1, \Delta_2)$ and $q(q_1) = (\Delta_1)$, the identity permutation in $S(\{\Delta_1, \Delta_2\})$. Of course, $w_1(F_0)(v_1) = (1,2)$ and $w_1(F_0)(q_1) = (1)$. Therefore $\tilde{M} \in Oo$ (*Cf.* Theorem 3.3.5).

(iii) The case $M \in NnII$.

Suppose $M = (NnII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ and $p : M \to F$ is the orbit projection. Since $M \in NnII$ then F is non-orientable. Assume that the genus of F is g. Write $M_0 = \overline{M - T(\beta_i/\alpha_i)}$, where $T(\beta_i/\alpha_i)$ is the solid fibered torus homeomorphic to a neighborhood of either a exceptional fiber or a singular fiber. Then $\partial M_0 = \sqcup_{i=1}^{r} T_i$, where T_i is a torus for $i = 1, \ldots, r$. Let $F_0 = p(M_0)$ and $q_i = p(T_i)$. If h is a regular fiber of M and $\{v_j\}_{j=1}^{g}$ is a basis for $\pi_1(F)$ of orientation reversing curves, then presentations for the fundamental groups of M and M_0 are:

$$\begin{aligned}\pi_1(M) &\cong \langle v_1,\dots,v_g,q_1,\dots,q_r,h;[h,q_i]=1, q_1q_2\cdots q_r=\prod_{j=1}^{g} v_j^2,\\ &\qquad q_i^{\alpha_i}h^{\beta_i}=1,[v_1,h]=1, v_jhv_j^{-1}=h^{-1}, \text{ for each } j\geq 2\rangle.\end{aligned}$$

$$\begin{aligned}\pi_1(M_0) &\cong \langle v_1,\dots,v_g,q_1,\dots,q_r,h;[h,q_i]=1, q_1q_2\cdots q_r=\prod_{j=1}^{g} v_j^2,\\ &\qquad [v_1,h]=1, v_jhv_j^{-1}=h^{-1}, \text{ for each } j\geq 2\rangle.\end{aligned}$$

Lemma 3.3.6 *Suppose that* $M=(NnII,g;\beta_1/\alpha_1,\dots,\beta_r/\alpha_r)$ *and* $\omega:\pi_1(M_0)\to S_n$ *is a representation such that*

$$\begin{aligned}\omega(h) &= (1),\\ \omega(q_i) &= \sigma_{i,1}\cdots\sigma_{i,\ell_i}, \text{ for } i=1,\dots,r \text{ and}\\ \omega(v_j) &= \rho_{j,1}\cdots\rho_{j,s_j}, \text{ for } j=1,\dots,g,\end{aligned}$$

where $\sigma_{i,1}\cdots\sigma_{i,\ell_i}$ *and* $\rho_{j,1}\cdots\rho_{j,s_j}$ *are the disjoint cycle decompositions of* $\omega(q_i)$ *and* $\omega(v_j)$*, respectively. Let* $\varphi:\tilde{M}\to M$ *be the covering associated to* ω *and let* $\tilde{p}:\tilde{M}\to G$ *be the orbit projection of* $\tilde{M}$*. Assume the valuation homomorphism* $e:\pi_1(F)\to\mathbb{Z}_2\cong S_2$ *is non-trivial and* $\tilde{M}$ *is non-orientable (i.e.* $M\in NnII$ *or* $M\in NnIII$*).*

1. *If the number of cycles of* $\omega(v_1)$ *having odd order is odd, then* $M\in NnII$*.*
2. *If the number of cycles of* $\omega(v_1)$ *having odd order is even, then* $M\in NnIII$*.*

Proof.

Note that v_1 is an orientation reversing curve in M_0 because v_1 is orientation reversing in F_0 and $e(v_1)=+1$. Then $p^{-1}(v_1)$ is a 2-sided vertical torus T^2. Let $\mathcal{N}(p^{-1}(v_1))$ be an open regular neighborhood of $p^{-1}(v_1)$. Then $M-\mathcal{N}(p^{-1}(v_1))$ is orientable for $v_2,\dots,v_g,q_1,\dots,q_r$ and h are orientation preserving curves in M_0.

Let $\tilde{v}_{1,j}$ be the components of $\varphi^{-1}(v_1)$ corresponding to $\rho_{1,j}$. Then $\varphi^{-1}(T^2)=\sqcup_{j=1}^{s_1}(\tilde{v}_{i,j}\times S^1)$.

Suppose $\mathcal{N}(\sqcup(\tilde{v}_{1,j}\times S^1))$ is an open regular neighborhood of $\sqcup(\tilde{v}_{1,j}\times S^1)$. It is clear that $\tilde{M}-\mathcal{N}(\sqcup(\tilde{v}_{1,j}\times S^1))$ is orientable because T^2 is a Stiefel-Whitney surface for M_0 (Theorem 2.3.2).

Let $PD: H^1(M,\mathbb{Z}_2)\to H_2(M,\mathbb{Z}_2)$ denote the Poincaré duality isomorphism associated to M.

Since $\varphi^*(w_1(M_0)) = w_1(\tilde{M}_0)$ then

$$\begin{array}{rcl} PDw_1(\tilde{M}_0) & = & [\varphi^{-1}(T^2)] \\ & = & [\sqcup_{j=1}^{s_1}(\tilde{v}_{1,j} \times S^1)] \\ & = & [\sqcup_{j=1}^{s_1}(\tilde{v}_{1,j} \times S^1)] \\ & = & [\tilde{v}_{1,1} \times S^1] + [\tilde{v}_{1,2} \times S^1] + \cdots + [\tilde{v}_{1,s_1} \times S^1], \end{array}$$

where possibly some classes $[\tilde{v}_j \times S^1]$ are trivial. Since the cycles $\rho_{1,j}$ are disjoint and the homology groups are abelian, without loss of generality, we may assume that there is a $k \in \{1, \ldots, s_1\}$, such that $[T_j]$ is trivial for all $k < j \leq s_1$. Thus $PDw_1(\tilde{M}) = [\tilde{v}_{1,1} \times S^1] + [\tilde{v}_{1,2} \times S^1] + \cdots + [\tilde{v}_{1,k} \times S^1]$. Of course, if $\rho_{1,j}$ has odd order then $1 \leq j \leq k$ since $\tilde{v}_{1,j}$ is the core of a Moebius strip contained in G_0 and this is a non-separating curve in G_0; consequently $\tilde{p}^{-1}(\tilde{v}_{1,j}) = \tilde{v}_{1,j} \times S^1$ is a non-separating surface in $\tilde{M}_0$ and the class $[\tilde{p}^{-1}(\tilde{v}_j)]$ is non-trivial in $H_2(\tilde{M}_0)$. Let $\tilde{v}$ be a simple closed curve in G_0 homologous to $\tilde{v}_{1,1} + \cdots + \tilde{v}_{1,k}$ and note that $PDw_1(\tilde{M}_0) = [\tilde{v} \times S^1]$; it means $\tilde{v} \times S^1$ is a Stiefel-Whitney surface for $\tilde{M}_0$ and for $\tilde{M}$. Thus $\tilde{v} \times S^1$ is a vertical torus which is a Stiefel-Whitney surface. Of course, $\tilde{v} \times S^1$ is one-sided in M_0 and M if and only if $\tilde{v}$ is one sided in F_0. By Theorem (2.3.3), if the number of cycles of $\omega(v_1)$ having odd order is odd then $\tilde{M} \in NnII$; Otherwise, $\tilde{M} \in NnIII$. □

Theorem 3.3.6 *Assume that $M = (NnII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ and $n \in \mathbb{N}$. Consider a representation $\omega : \pi_1(M_0) \to S_n$ such that*

$$\begin{array}{rcl} \omega(h) & = & (1), \\ \omega(q_i) & = & \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \ \text{for } i = 1, \ldots, r \ \text{ and} \\ \omega(v_j) & = & \rho_{j,1} \cdots \rho_{j,s_j} \ \text{for } j = 1, \ldots, g, \end{array}$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively. Let $\varphi : \tilde{M} \to M$ be the covering associated to ω and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$. Let $e' : \pi_1(F_0) \to S_n$ be the valuation homomorphism of M_0.

(a) *Suppose that n is an odd number.*

- **(1)** *If $\omega(v_1)$ has an odd number of cycles of odd order, then $\tilde{M} \in NnII$.*
- **(2)** *If $\omega(v_1)$ has an even number of cycles of odd order, then $\tilde{M} \in NnIII$.*

(b) *Assume that n is an even number and that there exists v_j, such that $\omega(v_j)$ has at least a cycle of odd order.*

- **(1)** *Suppose that the number of cycles of $\omega(v_1)$ having odd order is a non-zero even number.*

If there exists $k \neq 1$ such that $\omega(v_k)$ has a cycle of odd order then $\tilde{M} \in NnIII$.
Otherwise, if for $k \neq 1$ each cycle of $\omega(v_k)$ has even order, then $\tilde{M} \in NnI$ or $\tilde{M} \in NnIII$.

Moreover $\tilde{M} \in NnI$ if and only if $\omega|\pi_1(F_0)$ trivializes the bundle of e'.

(2) *If every cycle of $\omega(v_1)$ has even order, then $\tilde{M} \in On$ or $\tilde{M} \in NnIII$. Furthermore, $\tilde{M} \in On$ if and only if ω trivializes the bundle of $w_1(M_0)$, where $w_1(M_0)$ is the first Stiefel-Whitney class of M_0.*

(c) *If n is an even number and every cycle of $\omega(v_j)$ has even order, for $j = 1, \ldots, g$, then $\tilde{M} \notin NnII$. In this case it is possible $\tilde{M} \in Oo$, or $\tilde{M} \in On$, or $\tilde{M} \in No$, or $\tilde{M} \in NnI$ or $\tilde{M} \in NnIII$.*

Proof.
Suppose $\{v_j\}$ is a basis of orientation reversing curves for $\pi_1(F)$. The valuation homomorphism $e : \pi_1(F) \to \mathbb{Z}_2 \cong S_2$ is such that $e(v_1) = +1$ and $e(v_j) = -1$, for $j \geq 2$.

Recall we have $e' : \pi_1(F_0) \to S_2$, the valuation homomorphism of M_0, and $w_1(F_0) : \pi_1(F_0) \to S_2$, the first Stiefel-Whitney class of F_0, and $w_1(M_0) : \pi_1(M_0) \to S_2$, the first Stiefel-Whitney class of M_0. Let $\tilde{e}$ be the valuation homomorphism of $\tilde{M}$.

(a) If n is an odd number. Corollary 3.1.1 applied to $w_1(M_0)$ and to $w_1(F_0)$ give us that $w_1(\tilde{M}_0)$ and $w_1(G_0)$ are non-trivial, where $\tilde{M}_0 = \varphi^{-1}(M_0)$ and $G_0 = G \cap \tilde{M}_0 = \varphi^{-1}(F_0)$. Therefore $\tilde{M}_0$ and G_0 are non-orientable Then $\tilde{M}$ and G are non-orientable. Applying Theorem 3.1.1 to the valuation homomorphism e, we obtain that $\tilde{e}$, the valuation homomorphism of $\tilde{M}$, is non-trivial. Therefore $\tilde{M} \in NnII$ or $\tilde{M} \in NnIII$; The result follows from Lemma 3.3.6.

(b) Recall $\{v_j\}$ is a basis of reversing orientation curves for $\pi_1(F)$.

Since n is an even number and there exists v_j such that $\omega(v_j)$ has at least one cycle of odd order, then the orbit surface G of $\tilde{M}$ is non-orientable (Corollary 3.1.1).

(1) Note that $\tilde{M}$ is non-orientable since Corollary (3.1.1) applied to $\theta = w_1(M_0)$ gives us $w_1(\tilde{M}_0)$ is non-trivial.
If there exists $k \neq 1$ such that v_k has a cycle of odd order, then the valuation homomorphism of $\tilde{M}$, $\tilde{e}$, is non-trivial by Corollary 3.1.1 applied to e. Since the number of cycles of $\omega(v_1)$ having odd order is even, by Lemma 3.3.6 we obtain $\tilde{M} \in NnIII$.

If each cycle of $\omega(v_k)$ has even order, for all $k \neq 1$, then $\tilde{M} \in NnI$ or $\tilde{M} \in NnIII$ and the result follows from Theorem (3.1.1).

(2) First note that G_0 is non-orientable and the valuation homomorphism of $\tilde{M}$, $\tilde{e}$, is non-trivial, by Corollary 3.1.2. Also, by Lemma 3.3.6, we conclude $\tilde{M} \notin NnII$. Thus $\tilde{M} \in On$ or $\tilde{M} \in NnIII$. We can decide if $\tilde{M} \in On$ applying Theorem (3.1.1) to $\theta = w_1(M_0)$ as required.

(c) If n is an even number and every cycle of $\omega(v_j)$ has even order, for all $j = 1, \ldots, g$, then we have the following cases:

If $Im(\omega|\pi_1(M_0))$ and $Im(\omega|\pi_1(F_0))$ are not $\frac{n}{2}, 2-$imprimitive, then $w_1(\tilde{F_0})$, $w_1(\tilde{M_0})$ and $\tilde{e}$ are non-trivial by Theorem (3.1.1) applied to e, to $w_1(M_0)$ and to $w_1(F_0)$. Therefore $\tilde{M}$ and G are non-trivial. Since every cycle of $\omega(v_1)$ has even order and $\tilde{e}$ is non-trivial then $\tilde{M} \in NnIII$ by Lemma 3.3.6.

Assume $Im(\omega|\pi_1(M_0))$ is $\frac{n}{2}, 2-$imprimitive. If $w_1(\tilde{M_0})$ is trivial we have that $\tilde{M} \in Oo$ or $\tilde{M} \in On$. If $w_1(\tilde{M_0})$ is non-trivial, then $\tilde{M} \in No$, or $\tilde{M} \in NnI$, or $\tilde{M} \in NnIII$. Note that $\tilde{M} \notin NnII$ due to Lemma 3.3.6. □

(iv) The case $M \in NnIII$

Let $M = (NnIII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ and let F be the non-orientable orbit surface of M. Assume that the genus of F is g. Consider $M_0 = \overline{M - T(\beta_i/\alpha_i)}$, where $T(\beta_i/\alpha_i)$ is the solid fibered torus homeomorphic to a neighborhood of either a exceptional fiber or a singular fiber. Notice that $\partial M_0 = \sqcup_{i=1}^{r} T_i$, where T_i is a torus for $i = 1, \ldots, r$. Let $F_0 = p(M_0)$ and $q_i = p(T_i)$. Let h be a regular fiber of M and $\{v_j\}_{j=1}^{g}$ be a basis for $\pi_1(F)$ of orientation reversing curves.
The fundamental groups of M and M_0 have the following presentations:

$$\begin{aligned}\pi_1(M) \quad \cong \quad & \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [h, q_i] = 1, q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2, \\ & q_i^{\alpha_i} h^{\beta_i} = 1, [v_1, h] = 1, [v_2, h] = 1, v_j h v_j^{-1} = h^{-1}, \text{ for each } j \geq 3\rangle.\end{aligned}$$

$$\begin{aligned}\pi_1(M_0) \quad \cong \quad & \langle v_1, \ldots, v_g, q_1, \ldots, q_r, h; [h, q_i] = 1, q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2, \\ & [v_1, h] = 1, [v_2, h] = 1, v_j h v_j^{-1} = h^{-1}, \text{ for each } j \geq 3\rangle.\end{aligned}$$

If $e : \pi_1(M) \to \mathbb{Z}_2$ is the valuation homomorphism of M, then $e(v_1) = e(v_2) = +1$ and $e(v_j) = -1$ for $j \geq 3$.

Recall $\beta : H^i(M, \mathbb{Z}_2) \to H^{i+1}(M, \mathbb{Z})$ is the Bockstein homomorphism associated to the short exact sequence of coefficients

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$

Suppose that $M \in NnIII$ and consider a branched covering $\varphi : \tilde{M} \to M$, then $\beta w_1(\tilde{M}) = 0$ for $\beta w_1(M) = 0$ and β is natural with respect to continuous functions ($\varphi_* \beta = \beta \varphi_*$). Thus $\tilde{M} \in Oo$ or $\tilde{M} \in On$ or $\tilde{M} \in No$ or $\tilde{M} \in NnI$ or $\tilde{M} \in NnIII$ by Theorem 2.3.1 (and $\tilde{M} \in NnII$).

Theorem 3.3.7 *Suppose $M \in NnIII$ with $p : M \to F$, the orbit projection of M. Let $n \in \mathbb{N}$. Assume $\{v_j\}$ is a basis of reversing orientation curves for $\pi_1(F)$ Let $\omega : \pi_1(M_0) \to S_n$ be a representation defined by*

$$\begin{aligned} \omega(h) &= (1), \\ \omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and} \\ \omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j}, \text{ for } j = 1, \ldots, g, \end{aligned}$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively. Suppose $\varphi : \tilde{M} \to M$ is the covering determined by ω and $\tilde{p} : \tilde{M} \to G$ is the orbit projection of $\tilde{M}$. Let $e' : \pi_1(F_0) \to S_2$ be the evaluation of M_0.

(a) *If n is an odd number, then $\tilde{M} \in NnIII$.*

(b) *Suppose that n is an even number and there exists v_j such that $\omega(v_j)$ has at least one cycle of odd order.*

- **(i)** *If each cycle of $\omega(v_1)$ and $\omega(v_2)$ has even order, then $\tilde{M} \in On$ or $\tilde{M} \in NnIII$. Also, $\tilde{M} \in On$ if and only if ω trivializes the bundle of $w_1(M_0)$, where $w_1(M_0)$ is the first Stiefel-Whitney class of M_0.*
- **(ii)** *If $\omega(v_1)$ or $\omega(v_2)$ have a cycle of odd order, then $\tilde{M} \in NnI$ or $\tilde{M} \in NnIII$.*

(c) *If n is an even number and each cycle of $\omega(v_j)$ has even order, for all $j = 1, \ldots, g$, then $\tilde{M} \in Oo$ or $\tilde{M} \in No$ or $\tilde{M} \in NnI$ or $\tilde{M} \in NnIII$.*

Proof.

Let $\tilde{e}$ be the valuation homomorphism of $\tilde{M}$.

(a) If n is an odd number, then $w_1(G_0)$ and $w_1(\tilde{M}_0)$ are non-trivial by Corollary 3.1.2; the homomorphism $\tilde{e}$ is also non-trivial by Theorem 3.1.1. Thus $\tilde{M}$ and G are non-orientable. Thus $\tilde{M} \in NnIII$ for $\tilde{e}$ is non-trivial and $\beta(w_1(\tilde{M})) = 0$.

(b) Since there is one $\omega(v_j)$ having a cycle of odd order, then $w_1(G_0)$ is non-trivial because of Corollary (3.1.2). Thus G is non-orientable.

Recall $e(v_1) = e(v_2) = +1$ and $e(v_k) = -1$, for $k \geq 3$.

(i) Since $v_j \neq v_1$ and $v_j \neq v_2$, then $\tilde{e}$ is non-trivial due to Corollary 3.1.1. Therefore $\tilde{M} \in On$ or $\tilde{M} \in NnIII$. By Theorem 3.1.1 applied to $w_1(M_0)$ we can decide when $\tilde{M} \in On$ as stated.

(ii) Suppose that $\omega(v_1)$ or $\omega(v_2)$ have a cycle of odd order. Note that v_1 and v_2 are orientation reversing curves in M_0 since they are 1-sided in F_0 and $e(v_1) = e(v_2) = +1$. By Corollary 3.1.1, $w_1(\tilde{M}_0)$ is non-trivial and we conclude $\tilde{M}$ is non-orientable. Recall G is non-orientable. Therefore $\tilde{M} \in NnI$ or $\tilde{M} \in NnIII$. Furthermore, $\tilde{M} \in NnI$ if and only if $\omega|\pi_1(F_0)$ trivializes the bundle of e'.

(c) Assume n is an even number and every cycle of $\omega(v_j)$ has even order for all $j = 1, \ldots, g$. Then we have the following cases:

- If $Im(\omega|\pi_1(F_0))$ is $\frac{n}{2}, 2-$imprimitive. Then
  1. Suppose $\omega|\pi_1(F_0))$ trivializes the bundle of e'. Then $\tilde{e}$ is trivial (Theorem 3.1.1). Thus, if $\omega|\pi_1(F_0)$ trivializes the bundle of $w_1(M_0)$ then $\tilde{M} \in OO$. Otherwise, $\tilde{M} \in NnI$.
  2. Suppose $\omega|\pi_1(F_0))$ does not trivialize the bundle of e'. Then $\tilde{e}$ is non-trivial (Theorem 3.1.1). Therefore, if $\omega|\pi_1(F_0)$ trivializes the bundle of $w_1(F_0)$, then $w_1(G_0)$ and $w_1(G)$ are trivial (Theorem 3.1.1). Thus G is orientable and we conclude $\tilde{M} \in No$; Otherwise, if ω does not trivialize the bundle $w_1(F_0)$, then $\tilde{M} \in NnIII$ or $\tilde{M} \in On$. Again we can decide if $\tilde{M} \in On$ by means of Theorem 3.1.1 applied to $w_1(M_0)$.
- If $Im(\omega|\pi_1(F_0))$ is not $\frac{n}{2}, 2-$imprimitive, we proceed as before in (2).

To finish our study about representations of Seifert manifolds that send a regular fiber into the identity we prove the following Theorem which let us to compute the Seifert symbol for $\tilde{M}$.

Theorem 3.3.8 *Let $M = (Xx, g; \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_r}{\alpha_r})$ be a Seifert manifold with orbit projection $p : M \to F$, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$. Suppose that F is the orbit surface of M and let g be the genus of F. Consider $\{v_j\}$ a basis for $\pi_1(F)$ such that every curve v_j is orientation reversing in F, if F is non-orientable. Let h be a regular fiber of M. Write $M_0 = \overline{M - \sqcup_{i=1}^r V_i}$, where each V_i is a fibered neighborhood of the fiber corresponding to β_i/α_i, for $i = 1, \ldots, r$. Note that ∂M_0 is the union of r tori, $T_1 \sqcup \cdots \sqcup T_r$. Let $q_i = p(T_i)$, for $i = 1, \ldots, r$. Let $n \in \mathbb{N}$ and $\omega : \pi_1(M_0) \to S_n$ be a transitive representation defined by*

$$\begin{array}{rcl} \omega(h) & = & (1), \\ \omega(q_i) & = & \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \ \textit{for } i = 1, \ldots, r \ \textit{ and} \\ \omega(v_j) & = & \rho_{j,1} \cdots \rho_{j,s_j}, \end{array}$$

where $\sigma_{i,1}\cdots\sigma_{i,\ell_i}$ *and* $\rho_{j,1}\cdots\rho_{j,s_j}$ *are the disjoint cycle decompositions of* $\omega(q_i)$ *and* $\omega(v_j)$*, respectively. Let* $\varphi : \tilde{M} \to M$ *be the covering associated to* ω*. Let* $\tilde{p} : \tilde{M} \to G$ *is the orbit projection of* $\tilde{M}$ *and* G *has genus* $\tilde{g}$*.*

a) *Suppose* F *is non-orientable, then* $\tilde{M}$ *is the manifold*

$$(Yy, \tilde{g}; \frac{B_{1,1}}{A_{1,1}}, \ldots, \frac{B_{1,\ell_1}}{A_{1,\ell_1}}, \ldots, \frac{B_{r,1}}{A_{r,1}}, \ldots, \frac{B_{r,\ell_r}}{A_{r,\ell_r}}),$$

where $Yy \in \{Oo, On, No, NnI, NnII, NnIII\}$ *is determined by Theorems 3.3.3, 3.3.5, 3.3.6 and 3.3.7. If* G *is orientable, then*

$$\tilde{g} = 1 - \frac{n(2-g) + \sum_{i=1}^{r} \ell_i - nr}{2};$$

otherwise,

$$\tilde{g} = n(g-2) + 2 + nr - \sum_{i=1}^{r} \ell_i.$$

b) *If* F *is orientable, then* $\tilde{M}$ *is the manifold*

$$(Yy, \tilde{g}; \frac{B_{1,1}}{A_{1,1}}, \ldots, \frac{B_{1,\ell_1}}{A_{1,\ell_1}}, \ldots, \frac{B_{r,1}}{A_{r,1}}, \ldots, \frac{B_{r,\ell_r}}{A_{r,\ell_r}}),$$

where $Yy \in \{Oo, No\}$ *is determined by Theorems 3.3.2 and 3.3.4; and*

$$\tilde{g} = 1 + n(g-1) + \frac{nr - \sum_{i=1}^{r} \ell_i}{2}.$$

The numbers $B_{i,k}$ *and* $A_{i,k}$ *in the Seifert symbol for* $\tilde{M}$ *in both (a) and (b) are given by:*

$$B_{i,k} = \frac{order(\sigma_{i,k}) \cdot \beta_i}{gcd\{\alpha_i, order(\sigma_{i,k})\}}, \text{ and}$$

$$A_{i,k} = \frac{\alpha_i}{gcd\{\alpha_i, order(\sigma_{i,k})\}},$$

where $gcd\{\alpha_i, order(\sigma_{i,k}\}$ *denotes the greatest common divisor of* α_i *and* $order(\sigma_{i,k})$*.*

Proof.

The genus of G, $\tilde{g}$, is determined by Lemma 3.3.2 and the class Yy is determined by Theorems 3.3.2, 3.3.3, 3.3.4, 3.3.5, 3.3.6 and 3.3.7.

We compute the numbers $B_{i,k}$ and $A_{i,k}$.

Recall $G_0 = \varphi^{-1}(F_0) = G \cap \tilde{M}_0$, where $\tilde{M}_0 = \varphi^{-1}(M_0)$. Then $\varphi| : G \to F$ is a covering. The representation associated to $\varphi| : G \to F$ is $\omega| : \pi_1(F_0) \to S_n$.

The manifold M is obtained from M_0 by glueing a solid tori U_i to $T_i \partial M_0$ with homeomorphisms $f_i : \partial U_i \to T_i$ such that $f_i(m_i) = q_i^{\alpha_i} h^{\beta_i}$, where m_i is a meridian of ∂U_i.

If $i \in \{1, \ldots, r\}$ and we consider the torus $T_i = q_i \times h$, then $\varphi^{-1}(T_i)$ has ℓ_i components for $\varphi : G_0 \to F_0$ is a covering and $\omega(q_i)$ is a product of ℓ_i cycles, in particular, $\varphi^{-1}(q_i)$ has ℓ_i components.

Let $T_{i,k}$ be a component of $\varphi^{-1}(T_i)$, for $k \in \{1, \ldots \ell_i\}$. Note that $T_{i,k}$ is a torus and that φ induces a covering $\varphi_{i,k} : T_{i,k} \to T_i$ with $order(\sigma_{i,k})$ sheets such that, if $\tilde{h}$ is a component of $\varphi^{-1}(h)$ and $\tilde{q}_{i,k}$ is the pre-image of q_i in the torus $T_{i,k}$, then $\{\tilde{h}, \tilde{q}_{i,k}\}$ is a basis for $\pi_1(T_{i,k})$ for $\varphi| : G \to F$ is a covering. Note that $\tilde{q}_{i,k}$ is the union of $order(o\sigma_{i,k})$ liftings of q_i. Then $\varphi_{i,k}(\tilde{h}) = h$ and $\varphi_{i,k}(\tilde{q}_{i,k}) = q_i^{order(\sigma_{i,k})}$. Since $\{\tilde{h}, \tilde{q}_{i,k}\}$ is a basis for $\pi_1(T_{i,k})$, if $\tilde{m}_{i,k} \subset \varphi_{i,k}^{-1}(m_i)$ then there are $A_{i,k}$ and $B_{i,k}$ integer numbers such that $\tilde{m}_{i,k} = \tilde{q}_{i,k}^{A_{i,k}} \tilde{h}^{B_{i,k}}$, and

$$\varphi_{i,k}(\tilde{m}_{i,k}) = \varphi_{i,k}(\tilde{q}_{i,k}^{A_{i,k}} \tilde{h}^{B_{i,k}}) = q_i^{order(\sigma_{i,k})A_{i,k}} h^{B_{i,k}}. \tag{3.2}$$

On the other hand, associated to $\varphi_{i,k}$ we have a representation $\omega_{i,k} : T_i \to S_{order(\sigma_{i,k})}$ such that $\omega(h) = (1)$, the identity permutation in $S_{order(\sigma_{i,k})}$, and $\omega(q_i) = \varepsilon_{order(\sigma_{i,k})}$, the standard $order(\sigma_{i,k})-$cycle in $S_{order(\sigma_{i,k})}$. Note that $\omega_{i,k}$ satisfies that $\omega_{i,k}(m_i) = \omega_{i,k}(q^{\alpha_i} h^{\beta_i}) = (\sigma_{i,k})^{\alpha_i}$. This implies

$$\varphi_{i,k}(\tilde{m}_{i,k}) = m_i^{order((\sigma_{i,k})^{\alpha_i})} = (q_i^{\alpha_i \cdot order((\sigma_{i,k})^{\alpha_i})})(h^{\beta_i \cdot order((\sigma_{i,k})^{\alpha_i})}). \tag{3.3}$$

But in fact $order(\sigma_{i,k})^{\alpha_i}) = \frac{order(\sigma_{i,k})}{gcd\{\alpha_i, order(\sigma_{i,k})\}}$, hence by recalling Equations 3.2 and 3.3, we obtain

$$B_{i,k} = \frac{order(\sigma_{i,k}) \cdot \beta_i}{gcd\{\alpha_i, order(\sigma_{i,k})\}},$$

and

$$A_{i,k} = \frac{\alpha_i}{gcd\{\alpha_i, order(\sigma_{i,k})\}}$$

for $k = 1, \ldots, l_i$ and either $i = 1, \ldots, g$, if F is non-orientable or $i = 1, \ldots, 2g$, if F is orientable. □

3.3.2 The case $\omega(h) = \varepsilon_n$, the stardad n-cycle

Suppose M is a Seifert manifold and h is a regular fiber of M, in this section we focus in representations $\omega : \pi_1(M_0) \to S_n$ such that $\omega(h) = \varepsilon_n$, where ε_n is the standard $n-$cycle of S_n.

Definition 3.3.2 *Let P be an $n-$sided regular polygon with vertices labeled with the numbers from 1 to n. A* **reflection** *ρ in S_n is a permutation determined by a* ***reflection*** *of P restricted to the vertices of P.*

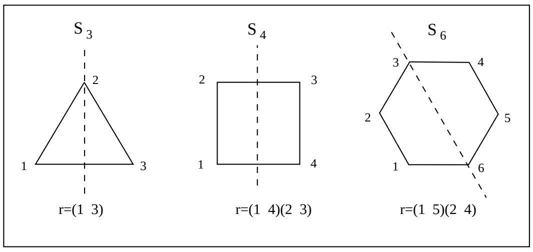


Figure 3.1: Reflections

Note that by definition a reflection ρ has order 2.

We say that $\sigma \in S_n$ **anticommutes** with ε_n if $\sigma\varepsilon_n\sigma^{-1} = \varepsilon_n^{-1}$.

Lemma 3.3.7 *Let $\sigma \in S_n$. Then σ anticommutes with ε_n if and only if σ is a reflection.*

Proof.
Let P be a $n-$sided regular polygon and $\sigma \in S_n$ be a reflection. Note that ε_n is induced by a rotation of P through an angle $2\pi/n$; by inspections it is easy to see that σ anticommutes with ε_n.

In a $n-$sided regular polygon P we have n reflections, then if $A = \{h \in S_n : h\varepsilon_n h^{-1} = \varepsilon_n^{-1}\}$ we have that $|A| \geq n$.

Now we prove $|A| = n$.

Suppose $\rho \in A$, then $\rho\varepsilon_n\rho^{-1} = \varepsilon_n^{-1}$. Let $\cdot : S_n \times S_n \to S_n$ be the group action defined by $g \cdot h = ghg^{-1}$. With this action the stabilizer of ε_n is the subgroup $Stabilizer(\varepsilon_n) = \{g \in S_n : g \cdot \varepsilon_n = \varepsilon_n\} = \{g \in S_n : g\varepsilon_n g^{-1} = \varepsilon_n\}$. Consider $S_n/Stabilizer(\varepsilon_n) = \{g(Stabilizer(\varepsilon_n)) : g \in S_n\}$ and note that $r \in \rho(Stabilizer(\varepsilon_n))$ if and only if $r\varepsilon_n r^{-1} = \rho\varepsilon_n\rho^{-1}$. Thus $\sigma(Stabilizer(\varepsilon_n)) = \{r \in S_n | r\varepsilon_n r^{-1} = \varepsilon_n^{-1}\} = A$.
On the other hand, the orbit of ε_n under this action is the set $O_{\varepsilon_n} = \{h \in S_n | h = g\varepsilon_n g^{-1}$for some $g \in S_n\}$. Note that O_{ε_n} is the set of $n-$cycles for the conjugates of an n-cycle have also order n.

We have a bijection $S_n/Stabilizer(\varepsilon_n) \to O_{\varepsilon_n}$. Then $n! = |S_n| = (|Stabilizer(\varepsilon_n)|)(|O_{\varepsilon_n}|)$. Since $|O_{\varepsilon_n}| = (n-1)!$, we obtain $|Stabilizer(\varepsilon)| = n$.

Therefore $|A| = n$ because $|A| = |\rho(Stabilizer(\varepsilon_n))| = |Stabilizer(\varepsilon_n)| = n$. □

Lemma 3.3.8 *Let $\sigma \in S_n$. Then σ commutes with ε_n if and only if there is $k \in \mathbb{Z}$ such that $\sigma = \varepsilon_n^k$.*

Proof.

Consider again the group action $\cdot : S_n \times S_n \to S_n$ given by $g \cdot h = ghg^{-1}$. Recall from the proof of the previous lemma that $|Stabilizer(\varepsilon)| = n$. Since $\{(1), \varepsilon_n, \ldots, \varepsilon_n^{n-1}\} \subset Stabilizer(\varepsilon_n)$ we obtain $Stabilizer(\varepsilon) = \{(1), \varepsilon_n, \ldots, \varepsilon_n^{n-1}\}$. Therefore, $\sigma = \varepsilon_n^k$, for some $k \in \mathbb{Z}$. □

Lemma 3.3.9 (Torus Lemma)*[**N-RL**] Let T be a torus and let $h, q \subset T$ be a basis for $\pi_1(T)$. Let $n \in \mathbb{Z}$ and assume that $\omega : \pi_1(T) \to S_n$ is the representation such that*

$$\begin{aligned} \omega(h) &= \varepsilon_n, \\ \omega(q) &= \varepsilon_n^k, \end{aligned}$$

where $\varepsilon_n = (1, 2, \ldots, n)$ is the standard $n-$cycle. Suppose that $\varphi : \tilde{T} \to T$ is the covering space defined by ω. Then there exist a basis $\tilde{h}, \tilde{q} \subset \tilde{T}$ for $\pi_1(\tilde{T})$ such that $\varphi(\tilde{h}) = h^n$ and $\varphi(\tilde{q}) = qh^{-k}$.

Proof.
Cut T along h and q to get the identification square S shown in Figure 3.2.

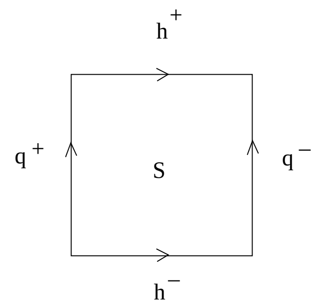


Figure 3.2: Square S

The boundary of S is the union of h^+, h_-, q^+ and q_-. If $S(1), \ldots, S(n)$ are n copies of S and the boundary of $S(i)$ is the union of $h(i)^+, h(i)^-, q(i)^+, q(i)^-$, we can construct $\tilde{T}$ by glueing $q(i)^+ \subset S(i)$ with $q(\varepsilon_n(i))^- \subset S(\varepsilon_n(i))$ and $h(i)^+$ with $h(\varepsilon_n(i))^-$.

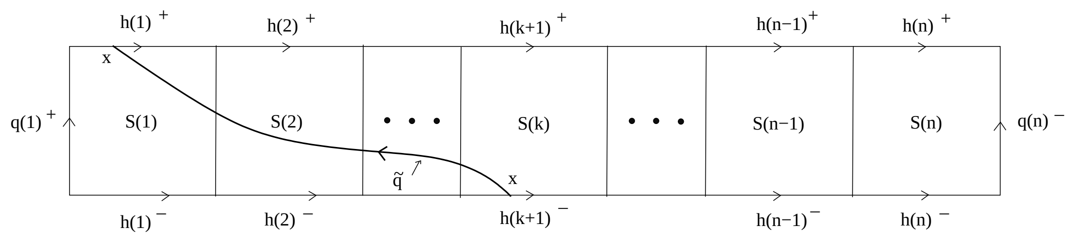


Figure 3.3: $\tilde{T}$

Suppose $x \in h(1)^+$ and let $y \in h(k+1)^+$ be the image of x under the identification. Let $\tilde{h} = \varphi^{-1}(h)$ and $\tilde{q}$ a shortest curve in $S(1) \cup \cdot \cup S(n)$ connecting x and y, as shown in Figure 3.3. Observe that $\tilde{h} \cap \tilde{q} = \{x\}$, then it is clear that $\tilde{h}, \tilde{q} \subset \tilde{T}$ is a basis for $\pi_1(T)$. By construction $\varphi(\tilde{h}) = h^n$ and $\varphi(\tilde{q}) = qh^{-k}$. □

Lemma 3.3.10 (Klein Bottle Lemma) *Let K be a Klein bottle with $\pi_1(K) = \langle h, v : vhv^{-1} = h^{-1}\rangle$. Consider a representation $\omega : \pi_1(K) \to S_n$ such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2 \ldots, n)$. Assume $\varphi : \tilde{K} \to K$ is the covering associated to ω. Then $\omega(v)$ is a reflection ρ, the covering space $\tilde{K}$ is also a Klein bottle and, if $\rho(1) = t$, then there exists a basis $\{\tilde{h}, \tilde{v}\}$ for $\tilde{K}$ such that $\varphi(\tilde{h}) = h^n$ and $\varphi(\tilde{v}) = vh^{-(t-1)}$.*

Proof.
Note that $\omega(v)\varepsilon_n\omega(v)^{-1} = \varepsilon^{-1}$, for $\omega(h) = \varepsilon_n$ and $vhv^{-1} = h^{-1}$. By Lemma (3.3.7), $\omega(v)$ is a reflection ρ. The surface $\tilde{K}$ is a closed surface. Also $\chi(\tilde{K}) = n\chi(K) = 0$ for $\chi(K) = 0$, where $\chi(\tilde{K})$ and $\chi(K)$ are the Euler characteristic of $\tilde{K}$ and K, respectively. Thus $\tilde{K}$ could be either a Klein bottle or a torus.

To construct $\tilde{K}$, cut K along h and v to get the identification square S shown in Figure 3.4.

The boundary of S is the union of h^+, h^-, v^+ and v^-. If $S(1), \ldots, S(n)$ are n copies of S and the boundary of $S(i)$ is the union of $h(i)^+, h(i)^-, v(i)^+, v(i)^-$, then $\tilde{K}$ is constructed by glueing $v(i)^+ \subset S(i)$ along $v(\varepsilon_n(i))^- \subset S(\varepsilon_n(i))$ and $h(i)^+$ with $h(\rho(i))^-$.

Suppose $x \in h(1)^+$ and let $y \in h(t)^-$ be the image of x under the identification. Let $\tilde{h} = \varphi^{-1}(h)$ and $\tilde{v}$ be a shortest curve in $S(1) \cup \cdots \cup S(n)$ connecting x and y, as shown in the Figure 3.5 Then $\varphi_{\#}(\tilde{h}) = h^n$, $\varphi_{\#}(\tilde{v}) = vh^{-(t-1)}$ by construction.

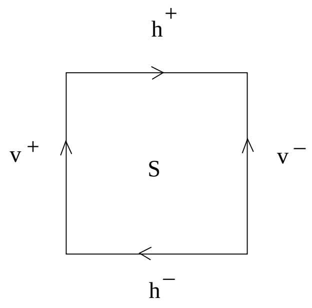


Figure 3.4: Square S

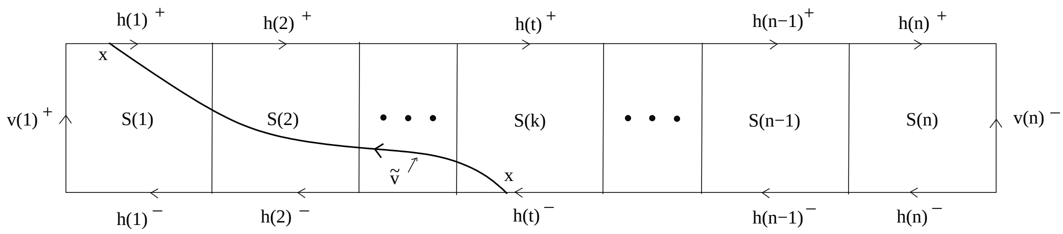


Figure 3.5: $\tilde{T}$

Notice that

$$\begin{aligned}
\varphi_{\#}(\tilde{v}\tilde{h}\tilde{v}^{-1}\tilde{h}) &= \varphi_{\#}(\tilde{v})\varphi_{\#}(\tilde{h})\varphi_{\#}(\tilde{v}^{-1})\varphi_{\#}(\tilde{h}) \\
&= (vh^{-(t-1)})h^n(h^{(t-1)}v^{-1})h^n \\
&= vh^nv^{-1}h^n \\
&= \underbrace{vhv^{-1}vhv^{-1}\cdots vhv^{-1}}_{n-times}h^n \\
&= h^{-n}h^n \text{ (because of the relation } v_jhv - j^{-1} = h^{-1}) \\
&= 1.
\end{aligned}$$

Thus $\tilde{v}\tilde{h}\tilde{v}^{-1} = \tilde{h}^{-1}$ for $\varphi_{\#}$ is injective.

Observe that $\tilde{h}$ intersects transversally $\tilde{v}$ only in one single point, thus $\tilde{K}$ must be a Klein bottle. Otherwise, $\{\tilde{h}, \tilde{v}\}$ would be a non-commuting pair in $\pi_1(K)$, the fundamental group of the torus $\tilde{K}$. Finally, $\{\tilde{h}, \tilde{v}\}$ is a basis for $\pi_1(\tilde{K})$ because the complement of these curves is a 2-disk, by construction. □

Remark 3.3.2 *Suppose M is a Seifert manifold with orbit projection $p : M \to F$. Assume F is of genus g. Let $\{h_i\}_{i=1}^r$ be a set of fibers of M which contains all the exceptional* *fibers and a finite number of regular fibers. Recall each fiber has a neighborhood V_i fiber preserving homeomorphic to a fibered solid torus $T(\beta_i/\alpha_i)$.*

Write $M_0 = \overline{M - \cup V_i}$. Note that we have a quotient $p| : M_0 \to F_0$, where F_0 is a surface with boundary. Recall $F_0 = F \cap M_0$. The boundary of F_0 has r components, one for each component of ∂M_0. Let $q_1, \dots, q_r$ be the components of ∂F_0 and h be a regular fiber in M_0.

Suppose $\{v_j\}$ is a basis for $\pi_1(F)$ such that v_j is orientation reversing in F, if F is non-orientable.

- *Assume $M \in Oo$, a presentation for $\pi_1(M_0)$ is*

$$\begin{aligned}\pi_1(M_0) \;\cong\; \langle v_1, \dots, v_{2g}, q_1, \dots, q_r, h; [h, v_j] = 1, [h, q_i] = 1, \\ q_1 q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}] \rangle.\end{aligned}$$

Let $\omega : \pi_1(M_0) \to S_n$ be a representation such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2, \dots, n)$. Then $\omega(v_j)$ and $\omega(q_i)$ commute with ε_n, for $[h, v_j] = [h, q_i] = 1$, $j = 1, \dots, 2g$ and $i = 1, \dots, r$, By Lemma (3.3.8), there are integer numbers k_i and s_j such that

$$\begin{aligned}\omega(q_i) &= \varepsilon_n^{k_i}, \forall i = 1, \dots, r \quad \text{and} \\ \omega(v_j) &= \varepsilon_n^{s_j}, \forall j = 1, \dots, 2g.\end{aligned}$$

In $\pi_1(M_0)$ we have the relation $q_1 \cdots q_r = \prod [v_{2j-1}, v_{2j}]$. Then

$$\omega(q_1 \cdots q_r (\prod [v_{2j-1}, v_{2j}])^{-1}) = \varepsilon^{\sum k_i} = (1).$$

Since ε_n has order n, there is an integer number p such that $\sum k_i = np$. Define $k'_1 = k_1 - np$ and $k'_j = k_j$, if $j \neq 1$. Then we get a representation $\omega' : \pi_1(M_0) \to S_n$ such that

$$\begin{aligned}\omega'(h) &= \varepsilon_n \\ \omega'(q_i) &= \varepsilon_n^{k'_i}, \forall i = 1, \dots, r \quad \text{and} \\ \omega'(v_j) &= \varepsilon_n^{s_j}, \forall j = 1, \dots, 2g.\end{aligned}$$

Clearly $\sum k'_i = 0$ and $\varepsilon_n^{k_1} = \varepsilon_n^{k'_1}$ because ε_n has order n. Therefore $\omega' = \omega$ and we can always assume $\sum k_i = 0$.

- *If $M \in On$, then a presentation for $\pi_1(M_0)$ is*

$$\begin{aligned}\pi_1(M_0) \;\cong\; \langle v_1, \dots, v_g, q_1, \dots, q_r, h; v_j h v_j^{-1} = h^{-1}, [h, q_i] = 1, \\ q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2 \rangle.\end{aligned}$$

Let $\omega : \pi_1(M_0) \to S_n$ be a representation such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2, \ldots, n)$. Note that $\omega(v_j)$ anticommutes with ε_n, that is, $\omega(v_j)\varepsilon_n\omega(v_j)^{-1} = \varepsilon^{-1}$, and $\omega(q_i)$ commute with ε_n, since we have that relations $v_jhv_j^{-1} = h^{-1}$ and $[h, q_i] = 1$, $j = 1, \ldots, 2g$ and $i = 1, \ldots, r$, By Lemmas 3.3.8 and 3.3.7 there are integer numbers k_i and reflections ρ_j such that $\omega : \pi_1(M_0) \to S_n$ is defined by

$$\begin{array}{rcl} \omega(h) &=& \varepsilon_n \\ \omega(q_i) &=& \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \quad \textit{and} \\ \omega(v_j) &=& \rho_j, \forall j = 1, \ldots, g. \end{array}$$

Since we have the relation $q_1 \cdots q_r = \prod v_j^2$ in $\pi_1(M_0)$ and reflections have order 2, then

$$\omega(q_1 \cdots q_r(\prod v_j^2)^{-1}) = \varepsilon^{\sum k_i} = (1).$$

Therefore there is an integer number p such that $\sum k_i = np$. Let $k_1' = k_1 - np$ and $k_j' = k_j$, if $j \neq 1$. We define a representation $\omega' : \pi_1(M_0) \to S_n$ by

$$\begin{array}{rcl} \omega'(h) &=& \varepsilon_n \\ \omega'(q_i) &=& \varepsilon_n^{k_i'}, \forall i = 1, \ldots, r \quad \textit{and} \\ \omega'(v_j) &=& \rho_j, \forall j = 1, \ldots, g. \end{array}$$

Note that $\omega' = \omega$ and $\sum k_i' = 0$. Therefore we can always assume $\sum k_i = 0$.

- *If $M \in No$, then a presentation for $\pi_1(M_0)$ is*

$$\begin{array}{rcl} \pi_1(M_0) &\cong& \langle v_1, \ldots, v_{2g}, q_1, \ldots, q_s, h; q_1q_2 \cdots q_r = \prod_{j=1}^{g} [v_{2j-1}, v_{2j}], \\ && [h, q_i] = 1, v_1hv_1^{-1} = h^{-1}, [v_j, h] = 1 \textit{ for } j \geq 2\rangle. \end{array}$$

Assume $\omega : \pi_1(M_0) \to S_n$ is a representation such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2, \ldots, n)$. Then $\omega(v_1)$ anticommutes with ε_n for $v_1hv_1^{-1}$; $\omega(v_j)$ and $\omega(q_i)$ commute with ε_n, for $[h, v_j] = [h, q_i] = 1$, $j = 2, \ldots, 2g$ and $i = 1, \ldots, r$, By Lemma 3.3.7, there is a reflection ρ_1 and by Lemma 3.3.8 there are integer numbers $k_1, \ldots, k_r$, $s_2, s_3, \ldots, s_{2g-1}$ and s_{2g} such that $\omega : \pi_1(M_0) \to S_n$ is defined by

$$\begin{array}{rcl} \omega(h) &=& \varepsilon_n \\ \omega(q_i) &=& \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \quad \textit{and} \\ \omega(v_1) &=& \rho_1 \\ \omega(v_j) &=& \varepsilon_n^{s_j}, \forall j = 2, \ldots, 2g. \end{array}$$

. In $\pi_1(M_0)$ we have the relation $q_1 \cdots q_r = \prod [v_{2j-1}, v_{2j}]$. Then

$$\omega(q_1 \cdots q_r(\prod [v_{2j-1}, v_{2j}])^{-1}) = \varepsilon^{\sum k_i + 2s_2} = (1).$$

Thus there is an integer number p *such that* $\sum k_i + 2s_2 = np$. *Define* $k'_1 = k_1 - np$ *and* $k'_j = k_j$, *if* $j \neq 1$. *We get a representation* $\omega' : \pi_1(M_0) \to S_n$ *such that*

$$\begin{aligned} \omega'(h) &= \varepsilon_n \\ \omega'(q_i) &= \varepsilon_n^{k'_i}, \forall i = 1, \dots, r \quad and \\ \omega'(v_1) &= \rho_1 \\ \omega'(v_j) &= \varepsilon_n^{s_j}, \forall j = 2, \dots, 2g. \end{aligned}$$

It is easy to see $\sum k'_i + 2s_2 = 0$ *and* $\varepsilon_n^{k_1} = \varepsilon_n^{k'_1}$ *for* ε_n *has order* n. *Therefore* $\omega' = \omega$ *and we can always assume* $\sum k_i + 2s_2 = 0$.

- *If* $M \in NnI$, *then a presentation for* $\pi_1(M_0)$ *is*

$$\begin{aligned} \pi_1(M_0) \cong \langle & v_1, \dots, v_g, q_1, \dots, q_r, h; [v_j, h] = 1, [h, q_i] = 1, \\ & q_1 q_2 \cdots q_r = \prod_{j=1}^{g} v_j^2 \rangle. \end{aligned}$$

Suppose $\omega : \pi_1(M_0) \to S_n$ *is a representation such that* $\omega(h) = \varepsilon_n$, *where* $\varepsilon_n = (1, 2, \dots, n)$. *Then* $\omega(v_j)$ *and* $\omega(q_i)$ *commute with* ε_n, *for* $[h, v_j] = [h, q_i] = 1$. *By Lemma (3.3.8),* $j = 1, \dots, 2g$ *and* $i = 1, \dots, r$, *there are integer numbers* k_i *and* s_j *such that*

$$\begin{aligned} \omega(q_i) &= \varepsilon_n^{k_i}, \forall i = 1, \dots, r \quad and \\ \omega(v_j) &= \varepsilon_n^{s_j}, \forall j = 1, \dots, g. \end{aligned}$$

Recall in $\pi_1(M_0)$ *we have the relation* $q_1 \cdots q_r = \prod v_j^2$. *Then*

$$\omega(q_1 \cdots q_r (\prod v_j^2)^{-1}) = \varepsilon^{\sum k_i - 2\sum s_j} = (1).$$

Since ε_n *has order* n, *there is an integer number* p *such that* $\sum k_i - 2\sum s_j = np$. *Define* $k'_1 = k_1 - np$ *and* $k'_j = k_j$, *if* $j \neq 1$. *Then we get a representation* $\omega' : \pi_1(M_0) \to S_n$ *such that*

$$\begin{aligned} \omega'(h) &= \varepsilon_n \\ \omega'(q_i) &= \varepsilon_n^{k'_i}, \forall i = 1, \dots, r \quad and \\ \omega'(v_j) &= \varepsilon_n^{s_j}, \forall j = 1, \dots, g. \end{aligned}$$

Clearly $\sum k'_i - 2\sum s_j = 0$ *and* $\varepsilon_n^{k_1} = \varepsilon_n^{k'_1}$ *because* ε_n *has order* n. *Therefore* $\omega' = \omega$ *and we can always assume* $\sum k_i - 2\sum s_j = 0$.

- *If* $M \in NnII$, *then a presentation for* $\pi_1(M_0)$ *is*

$$\begin{aligned}\pi_1(M_0) &\cong \langle v_1,\ldots,v_g,q_1,\ldots,q_r,h;[h,q_i]=1,q_1q_2\cdots q_r=\prod_{j=1}^{g}v_j^2,\\ &\qquad [v_1,h]=1,v_jhv_j^{-1}=h^{-1},\ \textit{for each } j\geq 2\rangle.\end{aligned}$$

Assume $\omega:\pi_1(M_0)\to S_n$ *is a representation such that* $\omega(h)=\varepsilon_n$*, where* $\varepsilon_n=(1,2,\ldots,n)$*. Then* $\omega(v_1)$ *and* $\omega(q_i)$ *commute with* ε_n *for* $[v_1,h]=[h,q_i]=1$*; if* $j\geq 2$*, then* $\omega(v_j)$ *anticommutes with* ε_n *because* $[h,v_j]=[h,q_i]=1$*, for* $j\geq 2$*. By Lemma 3.3.7 and 3.3.8, there are reflections* ρ_j*,* $j\geq 2$*, and there are integer numbers* k_i *and* s_1 *such that* $\omega:\pi_1(M_0)\to S_n$ *is defined by*

$$\begin{aligned}\omega(h) &= \varepsilon_n\\ \omega(q_i) &= \varepsilon_n^{k_i},\forall i=1,\ldots,r\\ \omega(v_1) &= \varepsilon_n^{s_1},\quad \textit{and}\\ \omega(v_j) &= \rho_j,\forall j=2,\ldots,g.\end{aligned}$$

Note that

$$\omega(q_1\cdots q_r(\prod v_j^2)^{-1})=\varepsilon^{\sum k_i-2s_1}=(1)$$

because of relation $q_1\cdots q_r=\prod v_j^2$ *and because reflections have order 2.*

Thus there is an integer number p *such that* $\sum k_i-2s_1=np$*. Define* $k_1'=k_1-np$ *and* $k_j'=k_j$*, if* $j\neq 1$*. We get a representation* $\omega':\pi_1(M_0)\to S_n$ *such that*

$$\begin{aligned}\omega'(h) &= \varepsilon_n\\ \omega'(q_i) &= \varepsilon_n^{k_i'},\forall i=1,\ldots,r;\\ \omega'(v_1) &= \varepsilon_n^{s_1},\quad \textit{and}\\ \omega'(v_j) &= \rho_j,\ \textit{for } j=2,\ldots,g.\end{aligned}$$

It is easy to see $\sum k_i'-2s_1=0$ *and* $\varepsilon_n^{k_1}=\varepsilon_n^{k_1'}$ *since* ε_n *has order* n*. Therefore* $\omega'=\omega$ *and we can always assume* $\sum k_i-2s_1=0$*.*

- *If* $M\in NnIII$*, then a presentation for* $\pi_1(M_0)$ *is*

$$\begin{aligned}\pi_1(M_0) &\cong \langle v_1,\ldots,v_g,q_1,\ldots,q_r,h;[h,q_i]=1,q_1q_2\cdots q_r=\prod_{j=1}^{g}v_j^2,\\ &\qquad [v_1,h]=1,[v_2,h]=1,v_jhv_j^{-1}=h^{-1},\ \textit{for each } j\geq 3\rangle.\end{aligned}$$

Suppose $\omega:\pi_1(M_0)\to S_n$ *is a representation such that* $\omega(h)=\varepsilon_n$*, where* $\varepsilon_n=(1,2,\ldots,n)$*. Then* $\omega(v_1)$*,* $\omega(v_2)$ *and* $\omega(q_i)$ *commute with* ε_n *for* $[v_1,h]=[v_2,h]=[h,q_i]=1$*; if* $j\geq 3$*, then* $\omega(v_j)$ *anticommutes with* ε_n *for if* $j\geq 3$ *then* $[h,v_j]=$

$[h, q_i] = 1$. By Lemma (3.3.7) and (3.3.8), there are reflections ρ_j, $j \geq 3$, and there are integer numbers k_i, s_1 and s_2 such that $\omega : \pi_1(M_0) \to S_n$ is defined by

$$\begin{array}{rcl} \omega(h) & = & \varepsilon_n \\ \omega(q_i) & = & \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \\ \omega(v_1) & = & \varepsilon_n^{s_1}, \\ \omega(v_2) & = & \varepsilon_n^{s_2}, \quad and \\ \omega(v_j) & = & \rho_j, \forall j = 3, \ldots, g. \end{array}$$

Note that

$$\omega(q_1 \cdots q_r (\prod v_j^2)^{-1}) = \varepsilon^{\sum k_i - 2s_1 - 2s_2} = (1)$$

since $q_1 \cdots q_r = \prod v_j^2$ and because reflections have order 2.

Thus there is an integer number p such that $\sum k_i - 2s_1 - 2s_2 = np$. Let $k_1' = k_1 - np$ and $k_j' = k_j$, if $j \neq 1$. We obtain a representation $\omega' : \pi_1(M_0) \to S_n$ such that

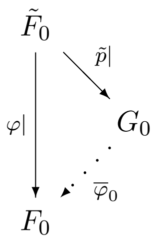
$$\begin{array}{rcl} \omega'(h) & = & \varepsilon_n \\ \omega'(q_i) & = & \varepsilon_n^{k_i'}, \forall i = 1, \ldots, r \\ \omega'(v_1) & = & \varepsilon_n^{s_1}, \\ \omega'(v_2) & = & \varepsilon_n^{s_2}, \quad and \\ \omega'(v_j) & = & \rho_j, \forall j = 3, \ldots, g; \end{array}$$

It is easy to see $\sum k_i' - 2s_1 - 2s_2 = 0$ and $\varepsilon_n^{k_1} = \varepsilon_n^{k_1'}$ for ε_n has order n. Therefore $\omega' = \omega$ and we can always assume $\sum k_i - 2s_1 - 2s_2 = 0$.

Lemma 3.3.11 *Let M be a Seifert manifold. Assume M_0, F and F_0 are as in las remark. Suppose h is a regular fiber of M and $\omega : \pi_1(M_0) \to S_n$ is a representation such that $\omega(h) = \varepsilon_n$. Let $\varphi : \tilde{M} \to M$ be the covering of M branched along fibers of M determined by ω. Assume $\tilde{p} : \tilde{M} \to G$ is the orbit projection of $\tilde{M}$. Then $F \cong G$.*

Proof.
Let $\tilde{M}_0 = \varphi^{-1}(M_0)$, $\tilde{F}_0 = \varphi^{-1}(F_0)$ and $G_0 = \tilde{p}(\tilde{M}_0)$. Then $\varphi| : \tilde{F}_0 \to F_0$ is a covering space of n sheets. Since $\omega(h) = \varepsilon_n$, each fiber of $\tilde{M}_0$ is the preimage of a fiber h' in M_0 under φ. Thus the projection $\tilde{p}| : \tilde{F}_0 \to G_0$ is also an n-fold covering for each fiber of $\tilde{M}_0$ intersects $\tilde{F}_0$ in n points. Suppose that $\tilde{x}, \tilde{y} \in \tilde{F}_0$ and $\tilde{p}(\tilde{x}) = \tilde{p}(\tilde{y})$. Then there is one fiber $\tilde{h}$ in $\tilde{M}_0$ such that $\tilde{x}, \tilde{y} \in \tilde{h} \cap \tilde{F}_0$. Also there is a fiber h' of M_0 such that $\varphi(\tilde{h}) = (h')^n$ for $\omega(h) = \varepsilon_n$. We conclude $\varphi|(\tilde{x}) = \varphi|(\tilde{y})$ for $\varphi|(\tilde{x}), \varphi|(\tilde{y}) \in h' \cap F_0$ and each fiber intersects F_0

in one single point. Thus there exists the following commutative diagram:



The map $\overline{\varphi}_0 : G_0 \to F_0$ is defined as usual: Let $x \in G_0$ and consider $\tilde{x} \in (\tilde{p}|)^{-1}(x)$ then $\overline{\varphi}_0(x) = \varphi|(\tilde{x})$. Of course, $\overline{\varphi}_0(x)$ does not depend on $\tilde{x}$ because $(\varphi|)((\tilde{p}|)^{-1}(x))$ is one point. Note that $\overline{\varphi}_0$ is a covering of 1 sheet for $\tilde{p}| : \tilde{F}_0 \to G_0$ and $\varphi| : \tilde{F}_0 \to F_0$ are $n-$fold coverings and for the diagram above is a commutative diagram. Thus $\overline{\varphi}_0$ is a homeomorphism. Therefore there is a homeomorphism $\overline{\varphi} : G \to F$. □

Note that in this context $\tilde{M}$ is no longer a pullback.

Lemma 3.3.12 *Let M be a Seifert manifold and $\varphi : \tilde{M} \to M$ be a covering of M branched along fibers. Assume $\tilde{p} : \tilde{M} \to G$ and $p : M \to F$ are the orbit projections of $\tilde{M}$ and M, respectively. Let h be a regular fiber of M. Let $\omega : \pi_1(M_0) \to S_n$ be the representation determined by φ. Suppose $\omega(h) = \varepsilon_n$. Let G_0 and F_0 be as the proof of the previous lemma. Let $\overline{\varphi}_0 : G_0 \to F_0$ be the homeomorphism obtained in the previous lemma. Recall $\pi_(F) \to \mathbb{Z}_2$ is the valuation homomorphism. Let $\tilde{v} \subset G_0$ and $v \subset F_0$ be simple closed curves such that $\overline{\varphi}_0(\tilde{v}) = v$.*
Then:

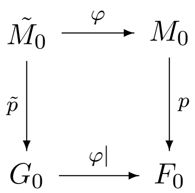
(a) *The map $\varphi| : \tilde{p}^{-1}(\tilde{v}) \to p^{-1}(v)$ is an $n-$fold covering space.*

(b) *If $e(v) = +1$, then $\tilde{e}(\tilde{v}) = +1$.*

(c) *If $e(v) = -1$, Then $\tilde{e}(\tilde{v}) = -1$.*

Proof.

(a) Note that the following diagram commutes.



Thus $\varphi| : \tilde{p}^{-1}(\tilde{v}) \to p^{-1}(v)$ is a covering space and $\omega' : \pi_1(p^{-1}(v)) \to S_r = S(\{a_1, \ldots, a_r\})$, the representation associated to this covering, sends h into ε_n. Note that $\tilde{p}^{-1}(\tilde{v})$ and $p^{-1}(v)$ are S^1-bundles over the simple closed curves $\tilde{v}$ and v, respectively. Then $\tilde{p}^{-1}(\tilde{v})$ and $p^{-1}(v)$ are either tori or Klein bottles depending on the triviality of the S^1-bundles.

(b) Since $e(v) = +1$, then $p^{-1}(v)$ is a torus and $\tilde{p}^{-1}(\tilde{v})$ is a torus. Thus $\tilde{e}(\tilde{v}) = +1$ for $\tilde{p}^{-1}(\tilde{v})$ is an S^1-bundle over $\tilde{v}$.

(c) If $e(v) = -1$, then $p^{-1}(v)$ is a Klein bottle. According to Lemma 3.3.10, we conclude $\tilde{p}^{-1}(\tilde{v})$ is a Klein bottle and therefore $\tilde{e}(\tilde{v}) = -1$. □

Theorem 3.3.9 *Assume $M = (Oo, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is a Seifert manifold. Let v_j and q_i be as in Remark 3.3.2 and $\omega : \pi_1(M_0) \to S_n$ be a representation defined by*

$$\begin{array}{rcl} \omega(h) & = & \varepsilon_n \\ \omega(q_i) & = & \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \ \text{ and} \\ \omega(v_j) & = & \varepsilon_n^{s_j}, \forall j = 1, \ldots, 2g; \end{array}$$

where $\sum k_i = 0$.

Let $\varphi : \tilde{M} \to M$ be the covering defined by ω. Then $\tilde{M} \in Oo$.

Proof.
Let $p : M \to F$ be the orbit projection of M and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$. By Lemma 3.3.11, there exists a homeomorphism $\overline{\varphi} : G \to F$. Then G is orientable. Let $\tilde{M}_0 = \varphi^{-1}(M_0)$. Since $\varphi| : \tilde{M}_0 \to M_0$ is a covering and M_0 is orientable, then $\tilde{M}_0$, and consequently, $\tilde{M}$ are orientable by Lemma 3.3.5 and Corollary 3.1.2. Therefore $\tilde{M} \in Oo$.□

Theorem 3.3.10 *Assume $M = (On, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is a Seifert manifold. Let v_j and q_i be as in Remark 3.3.2 and $\omega : \pi_1(M_0) \to S_n$ be a representation defined by*

$$\begin{array}{rcl} \omega(h) & = & \varepsilon_n \\ \omega(q_i) & = & \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \ \text{ and} \\ \omega(v_j) & = & \rho_j, \forall j = 1, \ldots, g; \end{array}$$

where $\sum k_i = 0$ and ρ_j is a reflection, for $j = 1, \ldots, g$.

Let $\varphi : \tilde{M} \to M$ be the covering defined by ω. Then $\tilde{M} \in On$.

Proof.

Let $p : M \to F$ be the orbit projection of M and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

By Lemma 3.3.11, there exists a homeomorphism $\overline{\varphi} : G \to F$. Then G is non-orientable. Let $\tilde{M}_0 = \varphi^{-1}(M_0)$. Since $\varphi| : \tilde{M}_0 \to M_0$ is a covering and M_0 is orientable, then $\tilde{M}_0$ is orientable; $\tilde{M}$ as also orientable by Lemma 3.3.5 and Corollary 3.1.2. Therefore $\tilde{M} \in On$.□

Theorem 3.3.11 *Assume $M = (No, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is a Seifert manifold. Let v_j and q_i be as in Remark 3.3.2 and $\omega : \pi_1(M_0) \to S_n$ be a representation defined by*

$$\begin{array}{rcl} \omega(h) & = & \varepsilon_n \\ \omega(q_i) & = & \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \quad \textit{and} \\ \omega(v_1) & = & \rho_1 \\ \omega(v_j) & = & \varepsilon_n^{s_j}, \forall j = 2, \ldots, 2g; \end{array}$$

where $\sum k_i + 2s_2 = 0$ and ρ_1 is a reflection. Suppose $\rho_1(1) = t_1\{1, \ldots, n\}$.

Let $\varphi : \tilde{M} \to M$ be the covering defined by ω. Then $\tilde{M} \in No$.

Proof.

Let $p : M \to F$ be the orbit projection of M and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$. Recall $e : \pi_1(F) \to \mathbb{Z}_2$, the valuation homomorphism of M, is defined by $e(v_1) = -1$ and $e(v_2) = +1$, for $i = 2, \ldots, 2g$. By Lemma 3.3.11, there is a homeomorphism $\overline{\varphi} : G \to F$. Thus G is orientable. Let $\{v'_j\}_{j=1}^{2g}$ be a basis for $\pi_1(G)$ such that $\overline{\varphi}(v'_j) = v_j$. By Lemma (3.3.12), the map $\varphi| : \tilde{p}^{-1}(v'_j) \to p^{-1}(v_j)$ is a covering and $\tilde{e}(v'_j) = e(v_j)$, for $j = 1, \ldots, 2g$, where $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ is the valuation homomorphism of $\tilde{M}$. Therefore $\tilde{M} \in No$. □

Theorem 3.3.12 *Assume $M = (NnI, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is a Seifert manifold. Let v_j and q_i be as in Remark 3.3.2 and $\omega : \pi_1(M_0) \to S_n$ be a representation defined by*

$$\begin{array}{rcl} \omega(h) & = & \varepsilon_n \\ \omega(q_i) & = & \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \quad \textit{and} \\ \omega(v_j) & = & \varepsilon_n^{s_j}, \forall j = 1, \ldots, g; \end{array}$$

where $\sum k_i - 2\sum s_j = 0$.

Let $\varphi : \tilde{M} \to M$ be the covering defined by ω. Then $\tilde{M} \in NnI$.

Proof.

Let $p : M \to F$ be the orbit projection of M and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

Recall $\{v_j\}$ is a basis of orientation reversing curves for $\pi_1(F)$ and $e : \pi_1(F) \to \mathbb{Z}_2$, the valuation homomorphism of M, is trivial. By Lemma 3.3.11, there is an homeomorphism $\overline{\varphi} : G \to F$. Thus G is non-orientable. Since $\overline{\varphi}$ is a homeomorphism, there exists a basis $\{v'_j\}_{j=1}^{g}$ of orientation reversing curves for $\pi_1(G)$ such that $\overline{\varphi}(v'_j) = v_j$. By Lemma 3.3.12, the map $\varphi| : \tilde{p}^{-1}(v'_j) \to p^{-1}(v_j)$ is a covering and $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ is trivial, where $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ is the valuation homomorphism of $\tilde{M}$. Therefore $\tilde{M} \in NnI$. □

Theorem 3.3.13 *Assume $M = (NnII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is a Seifert manifold. Let v_j and q_i be as in Remark 3.3.2 and $\omega : \pi_1(M_0) \to S_n$ be a representation defined by*

$$\begin{array}{rcl} \omega(h) & = & \varepsilon_n \\ \omega(q_i) & = & \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \\ \omega(v_1) & = & \varepsilon_n^{s_1}, \quad \textit{and} \\ \omega(v_j) & = & \rho_j, \forall j = 2, \ldots, g; \end{array}$$

where $\sum k_i - 2s_1 = 0$ and ρ_j is a reflection, for all $j = 2, \ldots, g$.

Let $\varphi : \tilde{M} \to M$ be the covering defined by ω. Then $\tilde{M} \in NnII$.

Proof.

Let $p : M \to F$ be the orbit projection of M and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

Recall $\{v_j\}$ is a basis of orientation reversing curves for $\pi_1(F)$ and $e : \pi_1(F) \to \mathbb{Z}_2$, the valuation homomorphism of M, is defined by $e(v_1) = +1$ and $e(v_j) = -1$, for $j = 2, \ldots, g$. By Lemma 3.3.11, there is an homeomorphism $\overline{\varphi} : G \to F$. Then G is non-orientable. Also there exists a basis $\{v'_j\}_{j=1}^{g}$ of orientation reversing curves for $\pi_1(G)$ such that $\overline{\varphi}(v'_j) = v_j$,because $\overline{\varphi}$ is a homeomorphism. By Lemma 3.3.12, the map $\varphi| : \tilde{p}^{-1}(v'_j) \to p^{-1}(v_j)$ is a covering and $\tilde{e}(v'_j) = e(v_j)$, for $j = 1, \ldots, g$, where $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ is the valuation homomorphism of $\tilde{M}$. Therefore $\tilde{M} \in NnII$. □

Theorem 3.3.14 *Assume $M = (NnIII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is a Seifert manifold. Let v_j and q_i be as in Remark 3.3.2 and $\omega : \pi_1(M_0) \to S_n$ be a representation defined by*

$$\begin{array}{rcl} \omega(h) & = & \varepsilon_n \\ \omega(q_i) & = & \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \\ \omega(v_1) & = & \varepsilon_n^{s_1}, \\ \omega(v_2) & = & \varepsilon_n^{s_2}, \quad \textit{and} \\ \omega(v_j) & = & \rho_j, \forall j = 3, \ldots, g; \end{array}$$

where $\sum k_i - 2s_1 - 2s_2 = 0$ and ρ_j is a reflection, for $j = 3, \ldots, g$.

Let $\varphi : \tilde{M} \to M$ be the covering defined by ω. Then $\tilde{M} \in NnIII$.

Proof.
Let $p : M \to F$ be the orbit projection of M and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

Recall $\{v_j\}$ is a basis of orientation reversing curves for $\pi_1(F)$ and $e : \pi_1(F) \to \mathbb{Z}_2$, the valuation homomorphism of M, is defined by $e(v_1) = +1$ and $e(v_j) = -1$, for $j = 2, \dots, g$. By Lemma 3.3.11, there is an homeomorphism $\overline{\varphi} : G \to F$. Then G is non-orientable. Also there exists a basis $\{v'_j\}_{j=1}^{g}$ of orientation reversing curves for $\pi_1(G)$ such that $\overline{\varphi}(v'_j) = v_j$, for $\overline{\varphi}$ is a homeomorphism. By Lemma 3.3.12, the map $\varphi| : \tilde{p}^{-1}(v'_j) \to p^{-1}(v_j)$ is a covering and $\tilde{e}(v'_j) = e(v_j)$, for $j = 1, \dots, g$, where $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ is the valuation homomorphism of $\tilde{M}$. Therefore $\tilde{M} \in NnIII$. □

Corollary 3.3.1 *Let $M = (Xx, g; \beta_1/\alpha_1, \dots, \alpha_r/\beta_r)$ and M_0 as in Remark* **??** *Assume h is a regular fiber of M. Let $\omega : \pi_1(M_0) \to S_n$ be a representation such that $\omega(h) = \varepsilon_n$ and let $\varphi : \tilde{M} \to M$ be covering space determined by ω.*
Then $\tilde{M}$ is in the same class of M.

Lemma 3.3.13 *Suppose $M = (Oo, g; \beta_1/\alpha_1, \dots, \beta_r/\alpha_r)$ is a Seifert manifold. Assume h is a regular fiber of M. Let $\omega : \pi_1(M_0) \to S_n$ such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2, \dots, n)$. By Remark 3.3.2, $\omega : \pi_1(M_0) \to S_n$ is defined by*

$$\begin{aligned} \omega(h) &= \varepsilon_n \\ \omega(q_i) &= \varepsilon_n^{k_i}, \forall i = 1, \dots, r \ \ \textit{and} \\ \omega(v_j) &= \varepsilon_n^{s_j}, \forall j = 1, \dots, 2g; \end{aligned}$$

where v_j and q_i are considered as in Remark 3.3.2 and $\sum k_i = 0$.

Let $\varphi : \tilde{M} \to M$ be the covering defined by ω.

Then there are an orbit surface G'_0 of $\tilde{M}_0$ and a basis $\tilde{v}_1, \dots, \tilde{v}_g$ for $\pi_1(G'_0)$ and curves $\tilde{q}_i$ in the boundary of G'_0 such that $\varphi_\#(\tilde{q}_i) = q_i h^{-k_i}$, $\varphi_\#(\tilde{v}_j) = v_j h^{-s_j}$, for all j.

In particular, we have an orbit surface G' of $\tilde{M}$ such that $\tilde{v}_1, \dots, \tilde{v}_g$ is a basis for $\pi_1(G')$.

Proof.
Let $p : M \to F$ be the orbit projection of M and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

Recall $F_0 = p(M_0)$. By Lemma 3.3.11, there exists a homeomorphism $\overline{\varphi}_0 : G_0 \to F_0$, where $F_0 = p(M_0)$ and $G_0 = \tilde{p}(\varphi^{-1}(M_0))$. Then there exists a basis $\{v'_j, q'_i\}$, where $j = 1, \dots, 2g$ and $i = 1, \dots, r$, for $\pi_1(G_0)$ such that $\overline{\varphi}_0(v'_j) = v_j$ and $\overline{\varphi}_0(q'_i) = q_i$, for all $j = 1, \dots, 2g$ and for $i = 1, \dots, r$.

Recall $e : \pi_1(F) \to \mathbb{Z}_2$, the valuation homomorphism of M, is trivial. By Lemma 3.3.12 $\tilde{e}(v'_j) = \tilde{e}(q'_i) = +1$, where $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ is the valuation homomorphism of $\tilde{M}$.

By Lemma 3.3.12, $\varphi| : \tilde{p}^{-1}(q'_i) \to p^{-1}(q_i)$ is a covering space; using Lemma 3.3.9 we obtain a basis $\{\tilde{h}, \tilde{q}_i\}$ for $\pi_1(\tilde{p}^{-1}(q'_i))$ such that $\varphi_\#(\tilde{h}) = h^n$ and $\varphi_\#(\tilde{q}_i) = q_i h^{-k_i}$.

Analogously, there is a basis $\{\tilde{v}_j, \tilde{h}\}$ for $\pi_1(\tilde{p}^{-1}(v'_j))$ such that $\varphi_\#(\tilde{h}) = h^n$ and $\varphi_\#(\tilde{v}_j) = v_j h^{-s_j}$, for all j. Note that, by construction, $\tilde{v}_j$ and $\tilde{q}_i$ intersect every fiber of $\tilde{p}^{-1}(v'_j)$ and $\tilde{p}^{-1}(q'_i)$, respectively, in exactly one point.

Since h commutes with v_j, for $j = 1, \ldots, 2g$, we obtain

$$\begin{array}{rcl} \varphi_\# \left(\tilde{q}_1 \cdots \tilde{q}_r (\prod[\tilde{v}_{2j-1}, \tilde{v}_{2j}])^{-1}\right) & \simeq & q_1 h^{-k_1} \cdots q_r h^{-k_r} (\prod[v_{2l-1}, v_{2l}])^{-1} \\ & \simeq & h^{-\sum k_i} q_1 \cdots q_r (\prod[v_{2l-1}, v_{2l}])^{-1} \text{ (recall } \sum k_i = 0.) \\ & \simeq & q_1 \cdots q_r (\prod[v_{2l-1}, v_{2l}])^{-1} \\ & \simeq & 1, \end{array}$$

where all homotopies are $rel \partial I$. Thus $\tilde{q}_1 \cdots \tilde{q}_r (\prod[\tilde{v}_{2j-1}, \tilde{v}_{2j}])^{-1} \simeq 1$ for $\varphi_\#$ is injective.

Then the curves $\tilde{q}_1, \ldots, \tilde{q}_r$ span a surface G'_0 in M_0. After some isotopies of G'_0 in $\tilde{M}$ fixing $\partial G'_0$, we obtain G'_0 is an orbit surface. After filling the holes of $\tilde{M}_0$, G'_0 gives rise to G' as required. □

Lemma 3.3.14 *Suppose $M = (On, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is a Seifert manifold. Assume h is a regular fiber of M. Let M_0 be as in Remark 3.3.2 and $\omega : \pi_1(M_0) \to S_n$ such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2, \ldots, n)$. By Remark 3.3.2, $\omega : \pi_1(M_0) \to S_n$ is defined by*

$$\begin{array}{rcl} \omega(h) & = & \varepsilon_n \\ \omega(q_i) & = & \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \ \text{ and} \\ \omega(v_j) & = & \rho_j, \forall j = 1, \ldots, g; \end{array}$$

where $\sum k_i = 0$ and ρ_j is a reflection, for $j = 1, \ldots, g$. Suppose $\rho_j(1) = t_j \in \{1, \ldots, n\}$, for $j = 1, \ldots, g$.

Let $\varphi : \tilde{M} \to M$ be the covering defined by ω.

Then there are an orbit surface G'_0 of $\tilde{M}_0$ and a basis $\tilde{v}_1, \ldots, \tilde{v}_g$ for $\pi_1(G'_0)$ and curves $\tilde{q}_i$ in the boundary of G'_0 such that $\varphi_\#(\tilde{q}_i) = q_i h^{-k_i}$, $\varphi_\#(\tilde{v}_j) = v_j h^{-(t_j - 1)}$, for all j.

In particular, we have an orbit surface G' of $\tilde{M}$ such that $\tilde{v}_1, \ldots, \tilde{v}_g$ is a basis for $\pi_1(G')$.

Proof.

Let $p : M \to F$ be the orbit projection of M and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

Recall $F_0 = p(M_0)$ and $\{v_j\}$ is a basis of orientation reversing curves for $\pi_1(F)$. By Lemma 3.3.11, there exists a homeomorphism $\overline{\varphi}_0 : G_0 \to F_0$, where $F_0 = p(M_0)$ and $G_0 = \tilde{p}(\varphi^{-1}(M_0))$. Then there exists a basis $\{v'_j, q'_i\}$, where $j = 1, \ldots, g$ and $i = 1, \ldots, r$, for $\pi_1(G_0)$ such that $\overline{\varphi}_0(v'_j) = v_j$ and $\overline{\varphi}_0(q'_i) = q_i$, for all $j = 1, \ldots, g$ and for $i = 1, \ldots, r$.

Recall $e : \pi_1(F) \to \mathbb{Z}_2$, the valuation homomorphism of M, is defined by $e(v_j) = -1$, for $j = 1, \ldots, g$, and $e(q_i) = +1$, for $i = 1, \ldots, r$. Let $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ be the valuation homomorphism of $\tilde{M}$; by Lemma 3.3.12 we have that $\varphi| : \tilde{p}^{-1}(q'_i) \to p^{-1}(q_i)$ is a covering, $\tilde{e}(v'_j) = -1$ and $\tilde{e}(q'_i) = +1$.

From Lemma 3.3.9 it follows that we have a basis $\{\tilde{h}, \tilde{q}_i\}$ for $\pi_1(\tilde{p}^{-1}(q'_i))$ such that $\varphi_{\#}(\tilde{h}) = h^n$ and $\varphi_{\#}(\tilde{q}_i) = q_i h^{-k_i}$.

Recall $\rho_j(1) = t_j$. By Lemma 3.3.10 there is a basis $\{\tilde{v}_j, \tilde{h}\}$ for $\pi_1(\tilde{p}^{-1}(v'_j))$ such that $\varphi_{\#}(\tilde{h}) = h^n$ and $\varphi_{\#}(\tilde{v}_j) = v_j h^{-(t_j-1)}$, for $j = 1, \ldots, g$.

Note that, by construction, $\tilde{v}_j$ and $\tilde{q}_i$ intersect each fiber of $\tilde{p}^{-1}(v'_j)$ and $\tilde{p}^{-1}(q'_i)$, respectively, in exactly one point.

Since h anticommutes with v_j, we obtain $v_j h^{-(t_j-1)} = h^{(t_j-1)} v_j$ and $v_j h^(t_j - 1) = h^{-(t_j-1) v_j}$, for $j = 1, \ldots, 2g$. Then $v_j h^{-(t_j-1)} v_j h^{(-(t_j-1))} = h^{(t_j-1)-(t_j-1)} v_j^2 = v_j^2$.

Note that

$$\begin{aligned} \varphi_{\#}\left(\tilde{q}_1 \cdots \tilde{q}_r (\prod \tilde{v}_j^2)^{-1}\right) &\simeq q_1 h^{-k_1} \cdots q_r h^{-k_r} (\prod (v_j h^{-(t_j-1)})^2)^{-1} \\ &\simeq h^{-\sum k_i} q_1 \cdots q_r (\prod v_j h^{-(t_j-1)} v_j h^{-(t_j-1)})^{-1}, \text{ (recall } \textstyle\sum k_i = 0.) \\ &\simeq q_1 \cdots q_r (\prod v_j^2)^{-1}, \\ &\simeq 1. \end{aligned}$$

Thus $\tilde{q}_1 \cdots \tilde{q}_r (\prod \tilde{v}_j^2])^{-1} \simeq 1$ because for $\varphi_{\#}$ is injective.

Then the curves $\tilde{q}_1, \ldots, \tilde{q}_r$ span a surface G'_0 in M_0. After some isotopies of G'_0 in $\tilde{M}$ fixing $\partial G'_0$, we obtain G'_0 is an orbit surface. After filling the holes of $\tilde{M}_0$, G'_0 gives rise to G' as required. □

Lemma 3.3.15 *Suppose $M = (No, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is a Seifert manifold. Assume h is a regular fiber of M. Let M_0 be as in Remark 3.3.2 and $\omega : \pi_1(M_0) \to S_n$ such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2, \ldots, n)$. Let $\omega : \pi_1(M_0) \to S_n$ be a representation is defined by*

$$\begin{aligned} \omega(h) &= \varepsilon_n \\ \omega(q_i) &= \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \quad \textit{and} \\ \omega(v_1) &= \rho_1 \\ \omega(v_j) &= \varepsilon_n^{s_j}, \forall j = 2, \ldots, 2g; \end{aligned}$$

where $\sum k_i + 2s_2 = 0$ and ρ_1 is a reflection. Suppose $\rho_1(1) = t_1 \in \{1, \dots, n\}$.

Let $\varphi : \tilde{M} \to M$ be the covering defined by ω.

Then there are an orbit surface G'_0 of $\tilde{M}_0$ and a basis $\tilde{v}_1, \dots, \tilde{v}_g$ for $\pi_1(G'_0)$ and curves $\tilde{q}_i$ in the boundary of G'_0 such that $\varphi_\#(\tilde{q}_i) = q_i h^{-k_i}$, $\varphi_\#(\tilde{v_1}) = v_1 h^{-(t_1-1)}$ and $\varphi_\#(\tilde{v_j}) = v_j h^{-s_j}$, for $j = 2, \dots, 2g$.

In particular, we have an orbit surface G' of $\tilde{M}$ such that $\tilde{v}_1, \dots, \tilde{v}_g$ is a basis for $\pi_1(G')$.

Proof.
Let $p : M \to F$ be the orbit projection of M and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

Recall $F_0 = p(M_0)$. By Lemma 3.3.11, there exists a homeomorphism $\overline{\varphi}_0 : G_0 \to F_0$, where $F_0 = p(M_0)$ and $G_0 = \tilde{p}(\varphi^{-1}(M_0))$. Then there exists a basis $\{v'_j, q'_i\}$, where $j = 1, \dots, g$ and $i = 1, \dots, r$, for $\pi_1(G_0)$ such that $\overline{\varphi}_0(v'_j) = v_j$ and $\overline{\varphi}_0(q'_i) = q_i$, for $j = 1, \dots, g$ and for $i = 1, \dots, r$.

Recall $e(v_1) = -1$, $e(v_j) = +1$, for $j = 2, \dots, 2g$, and $e(q_i) = +1$, for $i = 1, \dots, r$, where $e : \pi_1(F) \to \mathbb{Z}_2$ is the valuation homomorphism of M. Let $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ be the valuation homomorphism of $\tilde{M}$; by Lemma 3.3.12 we have that $\varphi| : \tilde{p}^{-1}(q'_i) \to p^{-1}(q_i)$ is a covering space, $\tilde{e}(v'_1) = -1$, $\tilde{e}(v'_j) = +1$, for $j = 2, \dots, 2g$ and $\tilde{e}(q'_i) = +1$, for $i = 1, \dots, r$.

From Lemma 3.3.9 it follows we have basis $\{\tilde{h}, \tilde{v_j}\}$ and $\{\tilde{h}, \tilde{q}_i\}$ for $\pi_1(\tilde{p}^{-1}(v'_j))$ and $\pi_1(\tilde{p}^{-1}(q'_i))$, respectively, such that $\varphi_\#(\tilde{h}) = h^n$, $\varphi_\#(\tilde{v_j}) = v_j h^{-s_j}$ and $\varphi_\#(\tilde{q}_i) = q_i h^{-k_i}$, for $j = 2, \dots, 2g$ and for $i = 1 \dots, r$.

Recall $\rho_1(1) = t_1$. By Lemma 3.3.10 there is a basis $\{\tilde{v}_1, \tilde{h}\}$ for $\pi_1(\tilde{p}^{-1}(v'_1))$ such that $\varphi_\#(\tilde{h}) = h^n$ and $\varphi_\#(\tilde{v_1}) = v_1 h^{-(t_1-1)}$. By construction, $\tilde{v_j}$ and $\tilde{q_i}$ intersect each fiber of $\tilde{p}^{-1}(v'_j)$ and $\tilde{p}^{-1}(q'_i)$, respectively, in exactly one point.

Since h anticommutes with v_1 we obtain $v_1^{-1} h^{s_j} = h^{-s_j} v_1^{-1}$. Then $v_1 h^{-(t_1-1)} v_2 h^{-s_2} h^{(t_1-1)} v_1^{-1} h^{s_2} v_2^{-1} = v_1 v_2 v_1^{-1} v_2^{-1} h^{2s_2}$ because h commutes with v_2.

Thus

$$\begin{aligned}
\varphi_\# \left(\tilde{q}_1 \cdots \tilde{q}_r (\textstyle\prod_{j=1}^g [\tilde{v}_{2j-1}, \tilde{v}_{2j}])^{-1} \right) &\simeq q_1 h^{-k_1} \cdots q_r h^{-k_r} (\textstyle\prod_{j=1}^g [\varphi\#(\tilde{v}_{2j-1}), \varphi_\#(\tilde{v}_2 j)])^{-1} \\
&\simeq h^{-\sum k_i} q_1 \cdots q_r (\textstyle\prod_{j=1}^g [v_{2j-1}, v_{2j}] h^{2s_2})^{-1} \\
&\simeq h^{-\sum k_i} q_1 \cdots q_r h^{-2s_2} (\textstyle\prod_{j=1}^g [v_{2j-1}, v_{2j}])^{-1}, \text{ (since } [q_i, h] = 1) \\
&\simeq h^{-\sum k_i - 2s_2} q_1 \cdots q_r (\textstyle\prod_{j=1}^g [v_{2j-1}, v_{2j}])^{-1} \\
&\simeq 1 \text{ (for } \textstyle\sum k_i + 2s_2 = 0).
\end{aligned}$$

Thus $\tilde{q}_1 \cdots \tilde{q}_r (\prod [\tilde{v}_{2j-1}, \tilde{v}_{2j}])^{-1} \simeq 1$ for $\varphi_\#$ is injective. Then the curves $\tilde{q}_1, \ldots, \tilde{q}_r$ span a surface G_0' in M_0. After some isotopies of G_0' in $\tilde{M}$ fixing $\partial G_0'$, we obtain G_0' is an orbit surface. After filling the holes of $\tilde{M}_0$, G_0' gives rise to G' as required. □

Lemma 3.3.16 *Suppose $M = (NnI, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is a Seifert manifold. Assume h is a regular fiber of M. Let $\omega : \pi_1(M_0) \to S_n$ be a representation such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2, \ldots, n)$. By Remark 3.3.2, $\omega : \pi_1(M_0) \to S_n$ is defined by*

$$\begin{array}{rcl} \omega(h) & = & \varepsilon_n \\ \omega(q_i) & = & \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \quad \text{and} \\ \omega(v_j) & = & \varepsilon_n^{s_j}, \forall j = 1, \ldots, g. \end{array}$$

where $\sum k_i - 2 \sum s_j = 0$.

Let $\varphi : \tilde{M} \to M$ be the covering defined by ω.

Then there are an orbit surface G_0' of $\tilde{M}_0$ and a basis $\tilde{v}_1, \ldots, \tilde{v}_g$ for $\pi_1(G_0')$ and curves $\tilde{q}_i$ in the boundary of G_0' such that $\varphi_\#(\tilde{q}_i) = q_i h^{-k_i}$, $\varphi_\#(\tilde{v}_j) = v_j h^{-(s_j)}$, for all $j = 1 \ldots, g$.

In particular, we have an orbit surface G' of $\tilde{M}$ such that $\tilde{v}_1, \ldots, \tilde{v}_g$ is a basis for $\pi_1(G')$.

Proof.
Let $p : M \to F$ be the orbit projection of M and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

Recall $F_0 = p(M_0)$ and $\{v_j\}$ is a basis of orientation reversing curves for $\pi_1(F)$. By Lemma 3.3.11, there exists a homeomorphism $\overline{\varphi}_0 : G_0 \to F_0$, where $F_0 = p(M_0)$ and $G_0 = \tilde{p}(\varphi^{-1}(M_0))$. Then there exists a basis $\{v_j', q_i'\}$, where $j = 1, \ldots, g$ and $i = 1, \ldots, r$, for $\pi_1(G_0)$ such that $\overline{\varphi}_0(v_j') = v_j$ and $\overline{\varphi}_0(q_i') = q_i$, for all $j = 1, \ldots, g$ and for $i = 1, \ldots, r$.

Recall the valuation homomorphism of M, $e : \pi_1(F) \to \mathbb{Z}_2$, is trivial. Let $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ be the valuation homomorphism of $\tilde{M}$; by Lemma 3.3.12 we have that $\varphi| : \tilde{p}^{-1}(q_i') \to p^{-1}(q_i)$ is a covering, $\tilde{e}(v_j') = \tilde{e}(q_i') = +1$, for $j = 1, \ldots, g$ and $i = 1, \ldots, r$.

From Lemma 3.3.9 it follows we have a basis $\{\tilde{h}, \tilde{q}_i\}$ for $\pi_1(\tilde{p}^{-1}(q_i'))$ such that $\varphi_\#(\tilde{h}) = h^n$ and $\varphi_\#(\tilde{q}_i) = q_i h^{-k_i}$.

Analogously, there is a basis $\{\tilde{v}_j, \tilde{h}\}$ for $\pi_1(\tilde{p}^{-1}(v_j'))$ such that $\varphi_\#(\tilde{h}) = h^n$ and $\varphi_\#(\tilde{v}_j) = v_j h^{-s_j}$, for $j = 1, \ldots, g$. Note that, by construction, $\tilde{v}_j$ and $\tilde{q}_i$ intersect each fiber of $\tilde{p}^{-1}(v_j')$ and $\tilde{p}^{-1}(q_i')$, respectively, in exactly one point.

Since h commutes with v_j and q_i, then:

$$\begin{array}{rcl} \varphi_{\#}\left(\tilde{q}_1\cdots\tilde{q}_r(\prod\tilde{v}_j^2)^{-1}\right) & \simeq & q_1h^{-k_1}\cdots q_rh^{-k_r}(\prod(v_jh^{-s_j})^2)^{-1} \\ & \simeq & h^{-\sum k_i+2\sum s_j}q_1\cdots q_r(\prod v_j^2)^{-1}, \text{ (recall } \sum k_i-2\sum s_j=0.) \\ & \simeq & q_1\cdots q_r(\prod v_j^2)^{-1}, \\ & \simeq & 1. \end{array}$$

Thus $\tilde{q}_1\cdots\tilde{q}_r(\prod\tilde{v}_j^2)^{-1}\simeq 1$ for $\varphi_{\#}$ is injective.

Then the curves $\tilde{q}_1,\ldots,\tilde{q}_r$ span a surface G_0' in M_0. After some isotopies of G_0' in $\tilde{M}$ fixing $\partial G_0'$, we obtain G_0' is an orbit surface. After filling the holes of $\tilde{M}_0$, G_0' gives rise to G' as required. □

Lemma 3.3.17 *Suppose $M=(NnII,g;\beta_1/\alpha_1,\ldots,\beta_r/\alpha_r)$ is a Seifert manifold. Assume h is a regular fiber of M. Let $\omega:\pi_1(M_0)\to S_n$ be a representation such that $\omega(h)=\varepsilon_n$, where $\varepsilon_n=(1,2,\ldots,n)$. By Remark 3.3.2, $\omega:\pi_1(M_0)\to S_n$ is defined by*

$$\begin{array}{rcl} \omega(h) & = & \varepsilon_n \\ \omega(q_i) & = & \varepsilon_n^{k_i}, \forall i=1,\ldots,r, \\ \omega(v_1) & = & \varepsilon_n^{s_1}, \\ \omega(v_j) & = & \rho_j, \forall j=2,\ldots,g; \end{array}$$

where $\sum k_i-2s_1=0$ and ρ_j is a reflection, for $j=2,\ldots,g$. Assume $\rho_j(1)=t_j$, for $j=2,\ldots,g$.

Let $\varphi:\tilde{M}\to M$ be the covering defined by ω.

Then there are an orbit surface G_0' of $\tilde{M}_0$ and a basis $\tilde{v}_1,\ldots,\tilde{v}_g$ for $\pi_1(G_0')$ and curves $\tilde{q}_i$ in the boundary of G_0' such that $\varphi_{\#}(\tilde{q}_i)=q_ih^{-k_i}$, $\varphi_{\#}(\tilde{v}_1)=v_1h^{-(s_1)}$ and $\varphi_{\#}(\tilde{v}_j)=v_jh^{-(t_j-1)}$, for all $j=2\ldots,g$.

In particular, we have an orbit surface G' of $\tilde{M}$ such that $\tilde{v}_1,\ldots,\tilde{v}_g$ is a basis for $\pi_1(G')$.

Proof.

Let $p:M\to F$ be the orbit projection of M and let $\tilde{p}:\tilde{M}\to G$ be the orbit projection of $\tilde{M}$.

Recall $F_0=p(M_0)$ and $\{v_j\}$ is a basis of orientation reversing curves for $\pi_1(F)$. By Lemma 3.3.11, there exists a homeomorphism $\overline{\varphi}_0:G_0\to F_0$, where $F_0=p(M_0)$ and $G_0=\tilde{p}(\varphi^{-1}(M_0))$. Then there exists a basis $\{v_j',q_i'\}$, where $j=1,\ldots,g$ and $i=1,\ldots,r$, for $\pi_1(G_0)$ such that $\overline{\varphi}_0(v_j')=v_j$ and $\overline{\varphi}_0(q_i')=q_i$, for all $j=1,\ldots,g$ and for $i=1,\ldots,r$.

Recall also the valuation homomorphism of M, $e:\pi_1(F)\to\mathbb{Z}_2$, is defined by $e(v_1)=+1$ and $e(v_j)=-1$, for $j=2,\ldots,g$. Let $\tilde{e}:\pi_1(G)\to\mathbb{Z}_2$ be the valuation homomorphism of

$\tilde{M}$; by Lemma 3.3.12 we have that $\varphi| : \tilde{p}^{-1}(q_i') \to p^{-1}(q_i)$ is a covering, $\tilde{e}(v_1') = \tilde{e}(q_i') = +1$, for $i = 1, \ldots, r$, and $\tilde{e}(v_j') = -1$, if $j = 2, \ldots, g$.

By Lemma (3.3.9), we have basis $\{\tilde{h}, \tilde{v_1}\}$ and $\{\tilde{h}, \tilde{q}_i\}$ for $\pi_1(\tilde{p}^{-1}(v_1'))$ and $\pi_1(\tilde{p}^{-1}(q_i'))$, respectively, such that $\varphi_\#(\tilde{h}) = h^n$, $\varphi_\#(\tilde{v_1}) = v_1 h^{-s_1}$ and $\varphi_\#(\tilde{q_i}) = q_i h^{-k_i}$. Note that there is also a basis $\{\tilde{v_j}, \tilde{h}\}$ for $\pi_1(\tilde{p}^{-1}(v_j'))$ such that $\varphi_\#(\tilde{h}) = h^n$ and $\varphi_\#(\tilde{v_j}) = v_j h^{-(t_j-1)}$, for $j = 2, \ldots, g$, for Lemma 3.3.10. By construction, $\tilde{v_j}$ and $\tilde{q}_i$ intersect each fiber of $\tilde{p}^{-1}(v_j')$ and $\tilde{p}^{-1}(q_i')$, respectively, in exactly one point.

Since h anticommutes with v_1, then $h^{-(t_j-1)}v_j = v_j h^{(t_j-1)}$ and $h^{-2s_1}v_j = v_j h^{2s_1}$. Consequently $h^{-(t_j-1)}v_j h^{-(t_j-1)} = v_j$, $h^{-2s_1}v_j^2 = v_j^2 h^{-2s_1}$ and

$$\begin{aligned}
\varphi_\# \left(\tilde{q}_1 \cdots \tilde{q}_r (\textstyle\prod_{j=1}^g \tilde{v}_j^2)^{-1}\right) &\simeq q_1 h^{-k_1} \cdots q_r h^{-k_r} ((v_1 h^{-s_1})^2 \textstyle\prod_{j=2}^g v_j h^{-(t_j-1)} v_j h^{-(t_j-1)})^{-1} \\
&\simeq h^{-\sum k_i + 2s_1} q_1 \cdots q_r (\textstyle\prod_{j=1}^g v_j^2)^{-1}, \text{ (recall } \textstyle\sum k_i - 2s_1 = 0.) \\
&\simeq q_1 \cdots q_r (\textstyle\prod v_j^2)^{-1}, \\
&\simeq 1.
\end{aligned}$$

Thus $\tilde{q}_1 \cdots \tilde{q}_r (\prod \tilde{v}_j^2)^{-1} \simeq 1$ for $\varphi_\#$ is injective.

Then the curves $\tilde{q}_1, \ldots, \tilde{q}_r$ span a surface G_0' in M_0. After some isotopies of G_0' in $\tilde{M}$ fixing $\partial G_0'$, we obtain G_0' is an orbit surface. After filling the holes of $\tilde{M}_0$, G_0' gives rise to G' as required. □

Lemma 3.3.18 *Suppose $M = (NnIII, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ is a Seifert manifold with orbit projection $p : M \to F$. Assume h is a regular fiber of M. Let $\omega : \pi_1(M_0) \to S_n$ be a representation such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2, \ldots, n)$. By Remark 3.3.2, $\omega : \pi_1(M_0) \to S_n$ is defined by*

$$\begin{aligned}
\omega(h) &= \varepsilon_n \\
\omega(q_i) &= \varepsilon_n^{k_i}, \forall i = 1, \ldots, r, \\
\omega(v_1) &= \varepsilon_n^{s_1}, \\
\omega(v_2) &= \varepsilon_n^{s_2}, \quad \text{and} \\
\omega(v_j) &= \rho_j, \forall j = 3, \ldots, g;
\end{aligned}$$

where $\sum k_i - 2s_1 - 2s_2 = 0$ and ρ_j is a reflection, for $j = 3, \ldots, g$. Assume $\rho_j(1) = t_j$, for $j = 2, \ldots, g$.

Let $\varphi : \tilde{M} \to M$ be the covering defined by ω.

Then there are an orbit surface G_0' of $\tilde{M}_0$ and a basis $\tilde{v}_1, \ldots, \tilde{v}_g$ for $\pi_1(G_0')$ and curves $\tilde{q}_i$ in the boundary of G_0' such that $\varphi_\#(\tilde{q}_i) = q_i h^{-k_i}$, $\varphi_\#(\tilde{v_1}) = v_1 h^{-(s_1)}$, $\varphi_\#(\tilde{v_2}) = v_2 h^{-(s_2)}$, $\varphi_\#(\tilde{v_j}) = v_j h^{-(t_j-1)}$, for all $j = 3 \ldots, g$.

In particular, we have an orbit surface G' of $\tilde{M}$ such that $\tilde{v}_1, \ldots, \tilde{v}_g$ is a basis for $\pi_1(G')$.

Proof.
Let $p : M \to F$ be the orbit projection of M and let $\tilde{p} : \tilde{M} \to G$ be the orbit projection of $\tilde{M}$.

Recall $F_0 = p(M_0)$ and $\{v_j\}$ is a basis of orientation reversing curves for $\pi_1(F)$. By Lemma 3.3.11, there exists a homeomorphism $\overline{\varphi}_0 : G_0 \to F_0$, where $F_0 = p(M_0)$ and $G_0 = \tilde{p}(\varphi^{-1}(M_0))$. Then there exists a basis $\{v'_j, q'_i\}$, where $j = 1, \ldots, g$ and $i = 1, \ldots, r$, for $\pi_1(G_0)$ such that $\overline{\varphi}_0(v'_j) = v_j$ and $\overline{\varphi}_0(q'_i) = q_i$, for all $j = 1, \ldots, g$ and for $i = 1, \ldots, r$.

The valuation homomorphism of M, $e : \pi_1(F) \to \mathbb{Z}_2$, is defined by $e(v_1) = e(V_2) = +1$ and $e(v_j) = -1$, for $j = 3, \ldots, g$. Let $\tilde{e} : \pi_1(G) \to \mathbb{Z}_2$ be the valuation homomorphism of $\tilde{M}$; by Lemma 3.3.12 we have $\varphi| : \tilde{p}^{-1}(q'_i) \to p^{-1}(q_i)$ is a covering, $\tilde{e}(v'_1) = \tilde{e}(v'_2) = \tilde{e}(q'_i) = +1$, for $i = 1, \ldots, r$, and $\tilde{e}(v'_j) = -1$, if $j = 3, \ldots, g$.

By Lemma 3.3.9, we have basis $\{\tilde{h}, \tilde{v}_1\}$, $\{\tilde{h}, \tilde{v}_2\}$ and $\{\tilde{h}, \tilde{q}_i\}$ for $\pi_1(\tilde{p}^{-1}(v'_1))$, $\pi_1(\tilde{p}^{-1}(v'_2))$ and $\pi_1(\tilde{p}^{-1}(q'_i))$, respectively, such that $\varphi_\#(\tilde{h}) = h^n$, $\varphi_\#(\tilde{v}_1) = v_1 h^{-s_1}$, $\varphi_\#(\tilde{v}_2) = v_2 h^{-s_2}$ and $\varphi_\#(\tilde{q}_i) = q_i h^{-k_i}$. Note that by Lemma 3.3.10 there is also a basis $\{\tilde{v}_j, \tilde{h}\}$ for $\pi_1(\tilde{p}^{-1}(v'_j))$ such that $\varphi_\#(\tilde{h}) = h^n$ and $\varphi_\#(\tilde{v}_j) = v_j h^{-(t_j-1)}$, for $j = 3, \ldots, g$. By construction, $\tilde{v}_j$ and $\tilde{q}_i$ intersect each fiber of $\tilde{p}^{-1}(v'_j)$ and $\tilde{p}^{-1}(q'_i)$, respectively, in exactly one point.

Note that

$$
\begin{aligned}
\varphi_\# \left(\tilde{q}_1 \cdots \tilde{q}_r (\textstyle\prod_{j=1}^g \tilde{v}_j^2)^{-1}\right) &\simeq q_1 h^{-k_1} \cdots q_r h^{-k_r} ((v_1 h^{-s_1})^2 \textstyle\prod_{j=2}^g v_j h^{-(t_j-1)} v_j h^{-(t_j-1)})^{-1} \\
&\simeq h^{-\sum k_i + 2s_1} q_1 \cdots q_r (\textstyle\prod_{j=1}^g v_j^2)^{-1}, \text{ (recall } \textstyle\sum k_i - 2s_1 = 0.) \\
&\simeq q_1 \cdots q_r (\textstyle\prod v_j^2)^{-1}, \\
&\simeq 1;
\end{aligned}
$$

because h commutes with v_1, v_2 and q_i; and h anticommutes with v_j, for $j = 3, \ldots, g$.

Thus $\tilde{q}_1 \cdots \tilde{q}_r (\prod \tilde{v}_j^2])^{-1} \simeq 1$ because $\varphi_\#$ is injective.

Then the curves $\tilde{q}_1, \ldots, \tilde{q}_r$ span a surface G'_0 in M_0. After some isotopies of G'_0 in $\tilde{M}$ fixing $\partial G'_0$, we obtain G'_0 is an orbit surface. After filling the holes of $\tilde{M}_0$, G'_0 gives rise to G' as required. □

Theorem 3.3.15 *Let $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ be a Seifert manifold, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$. Let h be a regular fiber of M. Write $M_0 = \overline{M - \sqcup_{i=1}^r V_i}$, where each V_i is a fibered neighborhood of an exceptional fiber or a fibered neighborhood of a regular fiber, for $i = 1, \ldots, r$, and V_i is homeomorphic (under a fiber preserving homeomorphism) to the torus $T(\beta_i/\alpha_i)$. Assume $n \in \mathbb{N}$. Let $\omega : \pi_1(M_0) \to S_n$*

be a representation such that $\omega(h) = \varepsilon_n$, where $\varepsilon_n = (1, 2, \ldots, n)$. Then

$$\begin{aligned} \omega(q_i) &= \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \quad \textit{and} \\ \omega(v_j) &= \tau_j, \end{aligned}$$

where $\{h, v_j, q_i\}$ is a stardad system of generators of $\pi_1(M_0)$, and τ_j is a power of ε_n if v_j commutes with h, or a reflection if v_j anticommutes with h.

Let $\varphi : \tilde{M} \to M$ be the covering of M branched along fibers determined by ω. Then $\tilde{M}$ is in the same class of M and the Seifert symbol of $\tilde{M}$ is:

$$(Xx, g; \frac{B_1}{A_1}, \ldots, \frac{B_r}{A_r}),$$

with

$$B_i = \frac{\beta_i + k_i\alpha_i}{gcd\{n, \beta_i + k_i\alpha_i\}},$$

$$A_i = \frac{n\alpha_i}{gcd\{n, \beta_i + k_i\alpha_i\}},$$

where $gcd\{n, \beta_i + k_i\alpha_i\}$ denotes the greatest common divisor of n and $\beta_i + k_i\alpha_i$.

Proof.
By Remark 3.3.2, ω is defined as stated. Also $\tilde{M}$ is in the same class of M because of Corollary 3.3.1.

Suppose that F, of genus g, is the orbit surface of M. Recall $F_0 = p(M_0)$, $\tilde{M}_0 = \varphi^{-1}(M_0)$ and $G_0 = \tilde{p}(\tilde{M}_0)$, where $\tilde{p} : \tilde{M} \to G$ is the orbit projection of $\tilde{M}$.

Let G be the orbit surface of $\tilde{M}$.

By Lemma 3.3.11, there exists a homeomorphism $\overline{\varphi}_0 : G_0 \to F_0$. Thus ∂G_0 has r components because ∂F_0 has r components. Therefore $\partial \tilde{M}_0$ has r components.

Note that we can obtain M from M_0 by glueing solid tori U_i to T_i with homeomorphisms $f_i : \partial U_i \to T_i$ such that $f_i(m_i) = q_i^{\alpha_i} h^{\beta_i}$, where m_i is a meridian of ∂V_i..

Let G' be the orbit surface of $\tilde{M}$ obtained in Lemmas 3.3.13, 3.3.14, 3.3.15, 3.3.16, 3.3.17 and 3.3.18. Recall that Lemmas 3.3.13, 3.3.14), 3.3.15, 3.3.16, (3.3.17) and (3.3.18) give us a basis $\{\tilde{v}_j\}$ for $\pi_1(G)$ and curves $\tilde{q}_i$ in G, such that, $\varphi_\#(\tilde{q}_i) = q_i h^{-k_i}$.

Now we compute B_i and A_i.

Because of $m_i \sim q_i^{\alpha_i} h^{\beta_i}$, we have that $\omega(m_i) = \omega(q_i^{\alpha_i} h^{\beta_i}) = \varepsilon^{\beta_i + k_i\alpha_i}$. Let $d_i = gcd\{n, \beta_i + k_i\alpha_i\}$. Note that the order of $\omega(m_i)$ is n/d_i and that $\varphi^{-1}(m_i)$ has d_i components. Let $\tilde{m}_i$ be a component of $\varphi^{-1}(m_i)$, then

$$\varphi(\tilde{m}_i) = m_i^{n/d_i} = q_i^{n\alpha_i/d_i} h^{n\beta_i/d_i}. \tag{3.4}$$

On the other hand, $\tilde{m}_i = \tilde{q}_i^{A_i} \tilde{h}^{B_i}$ for some A_i and B_i positive integer numbers such that $gcd\{A_i, B_i\} = 1$, then

$$\varphi(\tilde{m}_i) = (q_i h^{-k_i})^{A_i} h^{nB_i} = q_i^{A_i} h^{-A_i k_i + nB_i}. \tag{3.5}$$

Equating (3.4) and (3.5) we get that

$$B_i = \frac{\beta_i + k_i\alpha_i}{gcd\{n, \beta_i + k_i\alpha_i\}}, \text{ and}$$

$$A_i = \frac{n\alpha_i}{gcd\{n, \beta_i + k_i\alpha_i\}}.$$

□

Chapter 4

Heegaard genera of coverings of Seifert manifolds branched along fibers

4.1 Heegaard genera of Seifert manifolds

Theorem 4.1.1 *[B-Z]*
Let $M = (Oo, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ be a Seifert manifold; assume $\alpha_i > 1$, and $1 \leq i \leq r$.

i) *If $M = (O, 0; 1/2, 1/2, \ldots, 1/2, \beta_r/(2\lambda + 1))$, with $\lambda > 0$, r even and $r \geq 4$, then $rank(\pi_1(M)) = r - 2 \leq h(M) \leq r - 1$.*

ii) *Suppose that M does not belong to the case (i) and $r \geq 3$, then $rank(\pi_1(M)) = h(M) = 2g + r - 1$.*

ii') *If $g > 0$ and $r = 2$, then $rank(\pi_1(M)) = h(M) = 2g + 1$.*

iii) *If $r = 1$, then $rank(\pi_1(M)) = h(M) = 2g$ if $\beta_1 = \pm 1$.*
Otherwise, $rank(\pi_1(M)) = h(M) = 2g + 1$.

iii') *If $r = 0$, then $rank(\pi_1(M)) = h(M) = 2g$ if $\beta_1 = \pm 1$.*
Otherwise $rank(\pi_1(M)) = h(M) = 2g + 1$.

Theorem 4.1.2 *[B-Z]*
Let $M = (On, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ be a Seifert Manifold; suppose $\alpha_i > 1$ and $1 \leq i \leq r$.

i) *If $r \geq 2$, then $h(M) = g + r - 1$.*

ii) *Suppose r=1.*

- **(a)** *If $\beta_1 = \pm 1$, then $h(M) = g$.*
- **(b)** *If $\beta_1 \neq \pm 1$ is even, then $h(M) = g + 1$.*

iii) *If $r = 0$, then $h(M) = g$ if $\beta_1 = \pm 1$; otherwise, $h(M) = g + 1$.*

Remark 4.1.1 *In Theorem 4.1.2, if $\beta_1 \neq \pm 1$ is odd, Boileau and Zieschang claimed but did not prove that $h(M) = g + 1$. According to [**Nu1**] this claim is correct.*

Theorem 4.1.3 *[**Nu**] Let M be a non-orientable Seifert manifold.*

(i) *If $M = (No, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$, where $\alpha_i > 1$, then*

  (a) *If $r \geq 2$, then $h(M) = 2g + r - 1$.*

  (b) *Suppose $r = 1$. If β_1 is even, then $h(M) = 2g + 1$. If $\beta_1 = 1$, then $h(M) = 2g$.*

  (c) *Suppose $r = 0$. If β_1 is even then $h(M) = 2g + 1$. If β_1 is odd, then $h(M) = 2g$.*

  Also, if $r = 1$ and $\beta_1 \neq 1$ is odd, then $2g \leq h(M) \leq 2g + 1$.

(ii) *If $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$, where $Xx \in \{NnI, NnII, NnIII\}$, and $\alpha_i > 1$; then:*

  (a) *If $r \geq 2$, then $h(M) = g + r - 1$.*

  (b) *Suppose $r = 1$. If β_1 is even, then $h(M) = g + 1$. If $\beta_1 = 1$, then $h(M) = g$.*

  (c) *Suppose $r = 0$. If β_1 is even, then $h(M) = g + 1$. If β_1 is odd, then $h(M) = g$.*

  Also, if $r = 1$ and $\beta_1 \neq 1$ is odd, then $g \leq h(M) \leq g + 1$.

4.2 Heegaard genera of coverings

Let M be a Seifert manifold with orbit projection $p : M \to F$. Assume $\varphi : \tilde{M} \to M$ is a covering of M branched along fibers. In this section we compare the Heegaard genus of $\tilde{M}$, $h(\tilde{M})$, with the Heegaard genus of M, $h(M)$. We always will assume that M is not in the following list:

(a) $M = (On, 1; \beta/\alpha)$, $\alpha \geq 1$

(b) $M = (Oo, 0; \beta_1/\alpha_1, \beta_2/\alpha_2)$, $\alpha_i \geq 1$

(c) $M = (Oo, 0; \beta_1/2, \beta_2/2, \beta_3/m)$

(d) $M = (Oo, 0; \beta_1/2, \beta_2/3, \beta_3/3)$

(e) $M = (Oo, 0; \beta_1/2, \beta_2/3, \beta_3/4)$

(f) $M = (Oo, 0; \beta_1/2, \beta_2/3, \beta_3/5)$

We take out the cases $(a) - (f)$ because these manifolds have finite fundamental group and in this cases S^3 is the universal covering of M. Thus $h(M) > h(S^3) = 0$ if $\pi_1(M) \neq 1$.

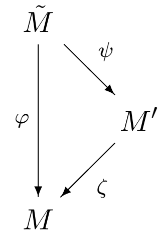
(g) $M = (Oo, 0; 1/2, 1/2, \ldots, 1/2, \beta_r/(2\lambda + 1))$, with $\lambda > 0$, r even and $r \geq 4$.

(h) $M = (Zz, g; \beta/\alpha)$, with $Zz \in \{No, NnI, NnII, NnIII\}$, $\beta \neq 1$ odd and $\alpha \geq 2$. (Non-orientable Seifert manifolds with exactly one exceptional fiber and $\beta \neq 1$ odd.)

We rule out (g) y (h) because we can not compute $h(M)$ precisely. In case (g), we only know $r - 2 \leq h(M) \leq r - 1$ and in case (h), $h(M)$ satisfies $2g \leq h(M) \leq 2g + 1$.

Let M be a Seifert manifold and $\{h_i\}_{i=1}^r$ be a set of fibers of M which contains all the exceptional fibers and a finite number of regular fibers. Recall each fiber has a neighborhood V_i fiber preserving homeomorphic to a solid fibered torus $T(\beta_i/\alpha_i)$ be the fibered solid torus homeomorphic to V_i, for $i = 1, \ldots, r$. Note that α_i and β_i are coprime numbers and $\alpha_i \geq 1$. Define $M_0 = \overline{M - \cup V_i}$.

Suppose $\varphi : \tilde{M} \to M$ is a covering of M branched along fibers and $\tilde{M}$ is connected. By Theorem (3.3.1), we know that there are $\psi : \tilde{M} \to M'$ and $\zeta : M' \to M$ branched coverings such that the following diagram is commutative



Also if ω_ψ and ω_ζ are the representations associated to ψ and ζ, respectively, we have that $\omega_\psi(h') = \varepsilon_t$ and $\omega_\zeta(h) = (1)$, where (1) is the identity permutation in S_n and $\varepsilon_t = (1, 2, \ldots, t)$; h and h' are regular fibers of M and M', respectively.

Thus we will only consider representations $\omega(\pi_1(M_0)) \to S_n$ such that $\omega(h) = (1)$ and $\omega(h) = \varepsilon_n$, where h is a regular fiber of M.

Along this section we use the following notation:

- M is a Seifert manifold with orbit projection $p : M \to F$, and h is a regular fiber of M.

- The surface F has genus g. Let $\{h_i\}_{i=1}^r$ be a set of fibers of M which contains all the exceptional fibers and some regular fibers. Recall each fiber has a neighborhood V_i fiber preserving homeomorphic to a fibered solid torus $T(\beta_i/\alpha_i)$, for $i = 1, \ldots, r$.

- $\{v_j\}$ is a basis for $\pi_1(F)$ and we assume v_j is orientation reversing if F is non-orientable, for each j.

- $M_0 = \overline{M - \cup_{i=1}^{r} V_i}$.

  Note that ∂M_0 has r components; $T_1, \ldots, T_r$

- $q_i = p(T_i)$.
- $\omega : \pi_1(M_0) \to S_n$ is a transitive representation.
- The identity permutation in S_n is denoted by (1) and the standard $n-$cycle $(1, \ldots, n)$ is denoted by ε_n.
- $\varphi : \tilde{M} \to M$ is the covering branched along fibers of M associated to the representation $\omega : \pi_1(M_0) \to S_n$ and $\tilde{p} : \tilde{M} \to G$ is the orbit projection of $\tilde{M}$.
- The surface G has genus $\tilde{g}$.
- The natural number n is always greater than 2. Otherwise, if $n = 1$ then φ would be a homeomorphism.
- The Heegaard genus of M is denoted by $h(M)$.

4.2.1 Heegaard genera when $\omega(h) = (1)$

Let $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ be a Seifert manifold, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$. Suppose that $\omega : \pi_1(M_0) \to S_n$ is a transitive representation defined by

$$\begin{aligned} \omega(h) &= (1), \\ \omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \quad \textit{and} \\ \omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j}; \end{aligned}$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively.

By Theorem 3.3.8,

a) If F is non-orientable, $\tilde{M}$ is the manifold

$$(Yy, \tilde{g}; \frac{B_{1,1}}{A_{1,1}}, \ldots, \frac{B_{1,\ell_1}}{A_{1,\ell_1}}, \ldots, \frac{B_{r,1}}{A_{r,1}}, \ldots, \frac{B_{r,\ell_r}}{A_{r,\ell_r}}),$$

where $Yy \in \{Oo, On, No, NnI, NnII, NnIII\}$ and it is determined by Theorems 3.3.3, 3.3.5, 3.3.6 and (3.3.7). If G is orientable, then

$$\tilde{g} = 1 - \frac{n(2-g) + \sum_{i=1}^{r} \ell_i - nr}{2}.$$

If G is non-orientable, then

$$\tilde{g} = n(g-2) + 2 + nr - \sum_{i=1}^{r} \ell_i.$$

b) If F is orientable, then $\tilde{M}$ is the manifold

$$(Yy, \tilde{g}; \frac{B_{1,1}}{A_{1,1}}, \ldots, \frac{B_{1,\ell_1}}{A_{1,\ell_1}}, \ldots, \frac{B_{r,1}}{A_{r,1}}, \ldots, \frac{B_{r,\ell_r}}{A_{r,\ell_r}}),$$

where $Yy \in \{Oo, No\}$ and it is determined by Theorems 3.3.2 and 3.3.4); and

$$\tilde{g} = 1 + n(g-1) + \frac{nr - \sum_{i=1}^{r} \ell_i}{2}.$$

The numbers $B_{i,k}$ and $A_{i,k}$ in the Seifert symbol for $\tilde{M}$ in (a) and (b) are:

$$B_{i,k} = \frac{order(\sigma_{i,k}) \cdot \beta_i}{gcd\{\alpha_i, order(\sigma_{i,k})\}}, \text{ and}$$

$$A_{i,k} = \frac{\alpha_i}{gcd\{\alpha_i, order(\sigma_{i,k})\}},$$

where $gcd\{\alpha_i, order(\sigma_{i,k}\}$ denotes the greatest common divisor of α_i and $order(\sigma_{i,k})$.

We hightlight the following equations for future reference.

$$\text{Note that } n \geq \ell_i \geq 1, \text{ for all } i = 1, \ldots, r, \tag{4.1}$$

because ℓ_i is the number of disjoint cycles of $\omega(q_i)$ and

$$A_{i,k} = 1, \text{ if and only if, } \alpha_i | order(\sigma_{i,k}) \tag{4.2}$$

since the definition of $A_{i,k}$.

Let a be a positive number. Assume $n > 1$. Then

$$n(a-2) + 2 \geq a \text{ if and only if } a \geq 2 \tag{4.3}$$

and

$$2 + 2n(a-1) \geq 2a \text{ if and only if } a \geq 1. \tag{4.4}$$

Lemma 4.2.1 *Let $M = (Xx, g; \beta_1/1)$, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$. Consider a transitive representation $\omega : \pi_1(M_0) \to S_n$ defined by*

$$\begin{aligned} \omega(h) &= (1), \\ \omega(q_1) &= \sigma_1 \cdots \sigma_{\ell_1}, \quad \textit{and} \\ \omega(v_j) &= \rho_1^j \cdots \rho_{s_j}^j, \end{aligned}$$

where $\sigma_{i,1}\cdots\sigma_{i,\ell_i}$ and $\rho_{j,1}\cdots\rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively.

By Theorem 3.3.8, $\tilde{M} = (Yy, \tilde{g}; B_1/A_1, \cdots, B_{\ell_1}/A_{\ell_1})$, with $B_k = order(\sigma_k)\cdot\beta_1$ and $A_k = 1$, for $k = 1, \ldots, \ell_1$. Let $p : M \to F$ be the orbit projection of M. Let g be the genus of F. Then:

(a) *If F is non-orientable, then $h(\tilde{M}) = n(g-2) + n - \ell_1 + 3$.*

(b) *If F is orientable, then $h(\tilde{M}) = 2n(g-1) + n - \ell_1 + 3$*

Proof.
By Theorem 3.2.1, we can assume $\tilde{M} = (Yy, \tilde{g}; n\beta_1/1)$. Note that $n\beta_1 \neq 1$ for $n \geq 2$ and β_1 is an integer number. Also $n\beta_1$ is even if β_1 is even, this implies that we can compute $h(\tilde{M})$, if $\tilde{M}$ is non-orientable.

(a) Suppose F is non-orientable.

(i) If G is non-orientable, then $\tilde{g} = n(g-2) + 2 + n - \ell_1$, by Lemma 3.3.8. Since $n\beta_1 \neq 1$, then

$$h(\tilde{M}) = \tilde{g} + 1 = n(g-2) + n - \ell_1 + 3.$$

(ii) If G is orientable, by Lemma 3.3.8, $2\tilde{g} = n(g-2) + 2 + n - \ell_1$. Thus

$$h(\tilde{M}) = 2\tilde{g} + 1 = n(g-2) + n - \ell_1 + 3,$$

for $n\beta_1 \neq 1$.

Therefore

$$h(\tilde{M}) = 2\tilde{g} + 1 = n(g-2) + n - \ell_1 + 3.$$

(b) Suppose F is orientable. Then G is orientable and by Lemma 3.3.8 we know $2\tilde{g} = 2n(g-1) + n - \ell_1 + 2$. Since $n\beta_1 \neq 1$ we obtain

$$h(\tilde{M}) = 2\tilde{g} + 1 = 2\tilde{g} = 2n(g-1) + n - \ell_1 + 3.$$

□

Corollary 4.2.1 *Let $M = (Xx, g; \beta_1/1)$, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$. Consider a transitive representation $\omega : \pi_1(M_0) \to S_n$ defined by*

$$\begin{aligned} \omega(h) &= (1), \\ \omega(q_1) &= \sigma_1\cdots\sigma_{\ell_1}, \quad y \\ \omega(v_j) &= \rho_1^j\cdots\rho_{s_j}^j, \end{aligned}$$

where $\sigma_{i,1}\cdots\sigma_{i,\ell_i}$ *and* $\rho_{j,1}\cdots\rho_{j,s_j}$ *are the disjoint cycle decompositions of* $\omega(q_i)$ *and* $\omega(v_j)$*, respectively.*

Let $\varphi:\tilde{M}\to M$ *be the covering of* M *branched along fibers associated to* ω*. Then* $h(\tilde{M})\geq h(M)$

Proof.
Consider the following cases:

First case. F is non-orientable. By Lemma 4.2.1, $h(\tilde{M})=2\tilde{g}+1=n(g-2)+n-\ell_1+3$. Recalling Equations 4.3 and 4.1 we conclude $h(\tilde{M})\geq h(M)$.

Second case. F is orientable. Then $h(\tilde{M})=2\tilde{g}+1=2\tilde{g}=2n(g-1)+n-\ell_1+3$ for Lemma 4.2.1. By Equation 4.4 we obtain $h(\tilde{M})\geq h(M)$.

Lemma 4.2.2 *Let* $M=(Xx,g;\beta_1/\alpha_1)$ *with* $\alpha\geq 2$*. Consider a transitive representation* $\omega:\pi_1(M_0)\to S_n$ *defined by*

$$\begin{aligned}\omega(h) &= (1),\\ \omega(q_1) &= \sigma_1\cdots\sigma_{\ell_1},\quad y\\ \omega(v_j) &= \rho_1^j\cdots\rho_{s_j}^j,\end{aligned}$$

where $\sigma_{i,1}\cdots\sigma_{i,\ell_i}$ *and* $\rho_{j,1}\cdots\rho_{j,s_j}$ *are the disjoint cycle decompositions of* $\omega(q_i)$ *and* $\omega(v_j)$*, respectively.*

Let $\varphi:\tilde{M}\to M$ *be covering associated to* ω*. By Theorem 3.3.8* $\tilde{M}=(Yy,\tilde{g};B_1/A_1,\cdots,B_{\ell_1}/A_{\ell_1})$*, where*

$$B_k=\frac{order(\sigma_k)\cdot\beta_1}{gcd\{\alpha_1,order(\sigma_k)\}}$$

and

$$A_k=\frac{\alpha_1}{gcd\{\alpha_1,order(\sigma_k)\}}.$$

Recall $gcd\{\alpha_1,order(\sigma_k)\}$ *denotes the greatest common divisor of* α_1 *and* $order(\sigma_k)$*.*

Let $k_1=\#\{\sigma_k:\alpha_1\nmid order(\sigma_k)\}$*. Then:*

(a) *Assume* F *is non-orientable.*

1. *Suppose* $k_1=0$*. If* $\beta_1=1$*,* $n=\alpha_1$ *and* $\omega(q_1)=(1,2,\ldots,\alpha_1)$*, then* $h(\tilde{M})=n(g-2)+n-\ell_1+2$*. Otherwise,* $h(\tilde{M})=n(g-2)+n-\ell_1+3$*.*
2. *Suppose* $k_1=1$*. Then* $h(\tilde{M})=n(g-2)+n-\ell_1+3$
3. *Suppose* $k_1\geq 2$*, then* $h(\tilde{M})=n(g-2)+n-\ell_1+k_1+1$*.*

(b) *Assume* F *is orientable.*

1. *Suppose* $k_1 = 0$. *If* $\beta_1 = 1$, $n = \alpha_1$ *and* $\omega(q_1) = (1, 2, \ldots, \alpha_1)$, *then* $h(\tilde{M}) = 2n(g-1) + n - \sum \ell_1 + 2$. *Otherwise,* $h(\tilde{M}) = 2n(g-1) + n - \ell_1 + 3$.
2. *Suppose* $k_1 = 1$, *then* $h(\tilde{M}) = 2n(g-1) + n - \ell_1 + 3$.
3. *Suppose* $k_1 \geq 2$, *then* $h(\tilde{M}) = 2n(g-1) + n - \ell_1 + k_1 + 1$.

Proof.
Note that $A_i = 1$ if and only if $\alpha_1 | order(\sigma_i)$. Thus k_1 is the number of exceptional fibers of $\tilde{M}$. Let G be the orbit surface of $\tilde{M}$ and let $\tilde{g}$ of G.

(a) Suppose F is non-orientable.

1. Assume $k_1 = 0$. Then $\alpha_1 | order(\sigma_k)$, for all $k = 1, \ldots, \ell_1$.. Thus there are integer numbers $p_k > 0$ such that $order(\sigma_k) = p_k \alpha_1$. Hence, by Theorem 3.2.1 we can assume that $\tilde{M} = (Yy, \tilde{g}; B/1)$, where $B = \beta_1 \sum p_k$. Also, if β_1 is even then B is even; then it is possible to compute the Heegaard genus of $\tilde{M}$ when β_1 is even. Note that $B = 1$ if and only if $\beta_1 = 1$, $n = \alpha_1$ and $\omega(q_1) = (1, 2, \ldots, \alpha_1)$.
   (i) If G is non-orientable, then $\tilde{g} = n(g-2) + 2 + n - \ell_1$ due to Theorem 3.3.8 Therefore, from Theorems 4.1.1,4.1.2 and 4.1.3 we obtain that $h(\tilde{M}) = \tilde{g} = n(g-2) + n - \ell_1 + 2$, if $\beta_1 = 1$, $n = \alpha_1$ and $\omega(q_1) = (1, 2, \ldots, \alpha_1)$; Otherwise, $h(\tilde{M}) = \tilde{g} + 1 = n(g-2) + n - \ell_1 + 3$.
   (ii) If G is orientable, then $2\tilde{g} = n(g-2) + 2 + n - \ell_1$ due to Theorem 3.3.8. Therefore, from Theorem 4.1.1, 4.1.2 and 4.1.3 we obtain that $h(\tilde{M}) = \tilde{g} = n(g-2) + n - \ell_1 + 2$, if $n = \alpha_1$ and $\omega(q_1) = (1, 2, \ldots, \alpha_1)$; Otherwise, $h(\tilde{M}) = \tilde{g} + 1 = n(g-2) + n - \ell_1 + 3$.
2. Assume $k_1 = 1$. By renumbering the indices, if necessary, we can assume that $A_1 \geq 2$ and $A_m = 1$, for each $m = 2, \ldots, \ell_1$. Then there are integer numbers $p_m > 0$ such that $order(\sigma_m) = p_m \alpha_1$, for all $m \in \{2, \ldots, \ell_1\}$. Thus, by Theorem 3.2.1 we have that $\tilde{M} = (Yy, \tilde{g}; B/A_1)$, where

$$\begin{array}{rcl} B & = & B_1 + \beta_1 A_1 \sum p_m \\ & = & \frac{\beta_1(order(\sigma_1) + \alpha_1 \sum p_m)}{gcd\{\alpha_1, order(\sigma_1)\}} \end{array}$$

   Note that B is an even number if β_1 is even. Then we alwasy can compute the Heegaard genus of $\tilde{M}$.

   Suppose that $B = 1$. Then $gcd\{\alpha_1, order(\sigma_1)\} = \beta_1(order(\sigma_1) + \alpha_1 \sum p_m)$. From this fact we obtain $\beta_1 | \alpha_1$ and $(order(\sigma_1) + \alpha_1 \sum p_m) | order(\sigma_1)$, consequently, $\beta_1 = 1$ and $\alpha_1 \sum p_m = 0$. Since $\alpha_1 > 0$ we conclude $\sum p_m = 0$. Thus $p_m = 0$. This contradicts our assumption of $p_m > 0$.
   Therefore $B \neq 1$.
   (i) If G is non-orientable, then $\tilde{g} = n(g-2) + n - \ell_1 + 1$. Hence by Theorems 4.1.1, 4.1.2 and 4.1.3 we obtain $h(\tilde{M}) = 2\tilde{g} + 1 = n(g-2) + n - \ell_1 + 3$.

(ii) If G is orientable, then $2\tilde{g} = n(g-2) + n - \ell_1 + 1$. By Theorems 4.1.1, 4.1.2 and 4.1.3 we conclude $h(\tilde{M}) = \tilde{g} + 1 = n(g-2) + n - \ell_1 + 3$.

3. Assume $k_1 \geq 2$. Recall k_1 is the number of exceptional fibers of $\tilde{M}$.

(i) If G is non-orientable, from Theorem 3.3.8 we obtain that $\tilde{g} = n(g-2) + n - \ell_1 + 2$. By Theorems 4.1.1, 4.1.2 and 4.1.3 we conclude $h(\tilde{M}) = \tilde{g} + k_1 - 1 = n(g-2) + n - \ell_1 + k_1 + 1$.

(ii) If G is orientable, by Theorem 3.3.8 we know that $2\tilde{g} = n(g-2) + n - \ell_1 + 2$. Since k_1 is the number of exceptional fibers of $\tilde{M}$ we have $h(\tilde{M}) = 2\tilde{g} + k_1 - 1 = n(g-2) + n - \ell_1 + k_1 + 1$.

(b) Suppose F is orientable, then G is orientable and $2\tilde{g} = 2n(g-1+n-\ell_1) + 2$ due to Theorem 3.3.8.

1. If $k_1 = 0$, then $\alpha_1 | o(\sigma_k)$, for all $k = 1, \ldots, \ell_1$.. Thus there are integer numbers $p_k > 0$ such that $order(\sigma_k) = p_k\alpha_1$. Hence, by Theorem 3.2.1 we can assume that $\tilde{M} = (Yy, \tilde{g}; B/1)$, where $B = \beta_1 \sum p_k$. Also, if β_1 is even then B is even; then it is possible to compute the Heegaard genus of $\tilde{M}$ when β_1 is even. Note that $B = 1$ if and only if $\beta_1 = 1$, $n = \alpha_1$ and $\omega(q_1) = (1, 2, \ldots, \alpha_1)$. Therefore $h(\tilde{M}) = 2\tilde{g} = 2n(g-1)+n-\ell_1+2$, if $n = \alpha_1$ and $\omega(q_1) = (1, 2, \ldots, \alpha_1)$. Otherwise, $h(\tilde{M}) = 2\tilde{g}+1 = 2n(g-1)+n-\ell_1+3$.
2. If $k_1 = 1$, by renumbering the indices, if necessary, we can suppose that $A_1 \geq 2$ and $A_m = 1$, for each $m = 2, \ldots, \ell_1$. Then there exist integer numbers $p_m > 0$ such that $order(\sigma_m) = p_m\alpha_1$, for all $m \in \{2, \ldots, \ell_1\}$. By Theorem (3.2.1), we can assume $\tilde{M} = (Yy, \tilde{g}; B/A_1)$, where

$$\begin{aligned} B &= B_1 + \beta_1 A_1 \sum p_m \\ &= \frac{\beta_1(order(\sigma_1) + \alpha_1 \sum p_m)}{gcd\{\alpha_1, order(\sigma_1)\}} \end{aligned}$$

Note that B is an even number if β_1 is even. Then we always can compute the Heegaard genus of $\tilde{M}$.

Suppose that $B = 1$. Then $gcd\{\alpha_1, order(\sigma_1)\} = \beta_1(order(\sigma_1) + \alpha_1 \sum p_m)$. From this fact we obtain $\beta_1 | \alpha_1$ and $(order(\sigma_1) + \alpha_1 \sum p_m) | order(\sigma_1)$, consequently, $\beta_1 = 1$ and $\alpha_1 \sum p_m = 0$. Since $\alpha_1 > 0$ we conclude $\sum p_m = 0$. Thus $p_m = 0$ and we obtain a contradiction to our assumption $p_m > 0$.

Therefore $B \neq 1$ and $h(\tilde{M}) = 2\tilde{g} + 1 = 2n(g-1) + n - \ell_1 + 3$.
3. If $k_1 \geq 2$, then $h(\tilde{M}) = 2\tilde{g} + k_1 - 1$ since k_1 is the number of exceptional fibers. Therefore $h(\tilde{M}) = 2n(g-1) + n - \ell_1 + k_1 + 1$. □

Corollary 4.2.2 *Let $M = (Xx, g; \beta_1/\alpha_1)$ where $Xx \in \{Oo, On.No.NnI, NnII, NnIII\}$ and $\alpha_1 \geq 2$. Consider a transitive representation $\omega : \pi_1(M_0) \to S_n$ defined by*

$$\begin{aligned} \omega(h) &= (1), \\ \omega(q_1) &= \sigma_1 \cdots \sigma_{\ell_1}, \quad y \\ \omega(v_j) &= \rho_1^j \cdots \rho_{s_j}^j, \end{aligned}$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively.

Let $\varphi : \tilde{M} \to M$ be covering associated to ω. Then $h(\tilde{M}) \geq h(M)$.

Proof.

Recall F and G are the orbit surfaces of M and $\tilde{M}$, respectively. Let k_1 be as in previous lemma.

(a) Suppose F is non-orientable. Then $g \geq 2$ because $g = 1$ implies M has finite fundamental group.

1. Assume $k_1 = 0$. If $\beta_1 = 1$, $n = \alpha_1$ and $\omega(q_1) = (1, \ldots, \alpha_1)$, then $h(\tilde{M}) = n(g-2) + n - \ell_1 + 2$, by Lemma 4.2.2. Notice that $h(M) = g$ because $\beta = 1$. From Equation 4.3 we get that $n(g-2) + 2 \geq g$. Equation 4.1 yields to $n \geq \ell_1$. Therefore $h(\tilde{M}) \geq h(M)$.

   If $\beta_1 \neq 1$ or $n \neq \alpha_1$ or $\omega(q_1) \neq (1, \ldots, \alpha_1)$, then $h(\tilde{M}) = n(g-2) + n - \ell_1 + 3$. Recalling Equations 4.3 and 4.1 we obtain that $n(g-2) + 2 \geq g$ and $n - \ell_1 \geq 0$. Therefore $h(\tilde{M}) \geq g + 1 \geq h(M)$.
2. Assume $k_1 = 1$. From Lemma 4.2.2 we know that $h(\tilde{M}) = n(g-2) + n - \ell_1 + 3$. Using again Equations 4.3 and 4.1 we conclude $h(\tilde{M}) \geq g + 1 \geq h(M)$.
3. Assume $k_1 \geq 2$. Then $h(\tilde{M}) = n(g-2) + n - \ell_1 + k_1 + 1$ because of Lemma 4.2.2. Since $k_1 \geq 2$, Equation 4.3 implies that $n(g-2) + k_1 \geq g$. By Equation 4.1, we conclude that $h(\tilde{M}) \geq h(M)$ as we stated.

(b) Suppose F is orientable. Note that F is not S^2, otherwise M would be a Seifert manifold with finite fundamental group and we do not want M with finite fundamental group. Thus $g \geq 1$.

1. Suppose $k_1 = 0$. If $\beta = 1$, $n = \alpha_1$ and $\omega(q_1) = (1, \ldots, \alpha_1)$, then $h(\tilde{M}) = 2n(g-1) + n - \ell_1 + 2$ for Lemma 4.2.2. Also $h(M) = 2g$ because $\beta = 1$. Since $g \geq 1$, using Equation 4.4 we obtain that $2n(g-1) + 2 \geq 2g$. From Equation 4.1 we conclude $h(\tilde{M}) \geq h(M)$.
   If $\beta \neq 1$ or $n \neq \alpha_1$ or $\omega(q_1) \neq (1, \ldots, \alpha_1)$, then $h(\tilde{M}) = 2n(g-1) + n - \ell_1 + 3$. By Equations 4.4 and 4.1, we conclude $h(\tilde{M}) \geq 2g + 1 \geq h(M)$.

2. Suppose $k_1 = 1$. In this case, $h(\tilde{M}) = 2n(g-1)+n-\ell_1+3$. Hence Equations 4.4 and (4.1) let us cocnlude $h(\tilde{M}) \geq 2g+1 \geq h(M))$.
3. Suppose $k_1 \geq 2$. From Lemma 4.2.2 we obtain that $h(\tilde{M}) = 2n(g-1)+n-\ell_1+k_1+1$. Equation (4.4) yields to $2n(g-1)+k_1 \geq 2g$. From Equation 4.1 we obtain $h(\tilde{M}) \geq h(M)$. □

Lemma 4.2.3 *Let $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$, where $Xx \in \{Oo, On, No, NnI, NnII,$. $\alpha_i \geq 2$, for each $i \in \{1, \ldots, r\}$, and $r \geq 2$ (A Seifert manifold with at least two exceptional fibers). Consider a transitive representation $\omega : \pi_1(M_0) \to S_n$ defined by*

$$\begin{aligned} \omega(h) &= (1), \\ \omega(q_i) &= \sigma_{i,1}\cdots\sigma_{i,\ell_i}, \text{ for } i = 1, \ldots, r \text{ and} \\ \omega(v_j) &= \rho_{j,1}\cdots\rho_{j,s_j}, \end{aligned}$$

where $\sigma_{i,1}\cdots\sigma_{i,\ell_i}$ and $\rho_{j,1}\cdots\rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively.

Let $\varphi : \tilde{M} \to M$ be the covering associated to ω. By Theorem (3.3.8),

$$\tilde{M} = (Yy, \tilde{g}; \frac{B_{1,1}}{A_{1,1}}, \ldots, \frac{B_{1,\ell_1}}{A_{1,\ell_1}}, \ldots, \frac{B_{r,1}}{A_{r,1}}, \ldots, \frac{B_{r,\ell_r}}{A_{r,\ell_r}}),$$

where

$$B_{i,k} = \frac{order(\sigma_{i,k})\cdot\beta_i}{gcd\{\alpha_i, order(\sigma_{i,k})\}}, \text{ and}$$

$$A_{i,k} = \frac{\alpha_i}{gcd\{\alpha_i, order(\sigma_{i,k})\}}.$$

Let $k_i = \#\{\sigma_{i,s} \in \omega(q_i) : \alpha_i \nmid order(\sigma i, s)\}$. By renumbering the indices, if necessary, we can assume that $\omega(q_i) = \sigma_1\cdots\sigma_{k_i}\cdots\sigma_{\ell_1}$ in such way that $\alpha_i \nmid order(\sigma_{i,k})$, for $k = 1, \ldots, k_i$.

(a) *Assume F is non-orientable.*

1. *Suppose $\sum_{i=1}^{r} k_i = 0$. Note that $\alpha_i | order(\sigma i, s)$, for $i = 1, \ldots, r$ and for $s = 1, \ldots, \ell_i$.. Assume that $p_{i,s}$ are integer numbers such that $order(\sigma_{i,s}) = p_{i,s}\alpha_i$. Write $B = \sum_{i=1}^{r}\sum_{s=1}^{\ell_i} p_{i,s}\beta_i$.*

   Then $h(\tilde{M}) = n(g-2)+nr-\sum\ell_i+2$, if $B = \pm 1$; Otherwise, $h(\tilde{M}) = n(g-2)+nr-\sum\ell_i+3$.
2. *Suppose $\sum_{i=1}^{r} k_i = 1$. By renumbering indices, if necessary, in this case we can assume that $\alpha_1 \nmid order(\sigma_{1,1})$, $\alpha_1 | order(\sigma_{1,s})$, for $s = 2, \ldots, \ell_1$, and $\alpha_i | order(\sigma_{i,s})$, for $i = 2, \ldots, r$ and for $s = 1, \ldots, \ell_i$. Assume $p'_{1,s}$, for $s = 2, \ldots, \ell_1$ and $p_{i,s}$, for $i = 2, \ldots, r$ and for $s = 1, \ldots, \ell_i$, are*

integers numbers such that $order(\sigma_{1,s}) = p'_{1,s}\alpha_1$, *for* $s = 2, \ldots, \ell_1$, *and* $order(\sigma_{i,s}) = p_{i,s}\alpha_i$, *for* $i = 2, \ldots, r$ *and for* $s = 1, \ldots, \ell_i$. *Define*

$$B = B_{1,1} + A_{1,1}(\beta_1 \sum_{s=2}^{\ell_1} p'_{1,s} + \sum_{i=2}^{r}\sum_{s=1}^{\ell_i} p_{i,s}\beta_i).$$

Then $h(\tilde{M}) = n(g-2) + nr - \sum \ell_i + 2$, *if* $B = \pm 1$; *Otherwise,* $h(\tilde{M}) = n(g-2) + nr - \sum \ell_i + 3$.

3. *Suppose* $\sum_{i=1}^{r} k_i \geq 2$. *Then* $h(\tilde{M}) = n(g-2) + nr - \sum \ell_i + \sum k_i + 1$.

(b) *Assume* F *is orientable.*

1. *Suppose* $\sum_{i=1}^{r} k_i = 0$. *Note that* $\alpha_i | order(\sigma i, s)$, *for* $i = 1, \ldots, r$ *and for* $s = 1, \ldots, \ell_i$.. *Let* $p_{i,s}$ *be integer numbers such that* $order(\sigma_{i,s}) = p_{i,s}\alpha_i$. *Define* $B = \sum_{i=1}^{r}\sum_{s=1}^{\ell_i} p_{i,s}\beta_i$. *Then* $h(\tilde{M}) = 2n(g-1) + nr - \sum \ell_i + 2$, *if* $B = \pm 1$; *Otherwise,* $h(\tilde{M}) = 2n(g-1) + nr - \sum \ell_i + 3$.

2. *Suppose* $\sum_{i=1}^{r} k_i = 1$. *We can assume that* $\alpha_1 \nmid order(\sigma_{1,1})$, $\alpha_1 | order(\sigma_{1,s})$, *for* $s = 2, \ldots, \ell_1$, *and* $\alpha_i | order(\sigma_{i,s})$, *for* $i = 2, \ldots, r$ *and for* $s = 1, \ldots, \ell_i$. *Assume that* $p'_{1,s}$, *for* $s = 2, \ldots, \ell_1$ *and* $p_{i,s}$, *for* $i = 2, \ldots, r$ *and for* $s = 1, \ldots, \ell_i$, *are integers numbers such that* $order(\sigma_{1,s}) = p'_{1,s}\alpha_1$, *for* $s = 2, \ldots, \ell_1$, *and* $order(\sigma_{i,s}) = p_{i,s}\alpha_i$, *for* $i = 2, \ldots, r$ *and for* $s = 1, \ldots, \ell_i$. *Write*

$$B = B_{1,1} + A_{1,1}(\beta_1 \sum_{s=2}^{\ell_1} p'_{1,s} + \sum_{i=2}^{r}\sum_{s=1}^{\ell_i} p_{i,s}\beta_i).$$

Then $h(\tilde{M}) = 2n(g-1) + nr - \sum \ell_i + 2$, *if* $B = \pm 1$. *Otherwise,* $h(\tilde{M}) = 2n(g-1) + nr - \sum \ell_i + 3$.

3. *Suppose* $\sum_{i=1}^{r} k_i \geq 2$. *Then* $h(\tilde{M}) = 2n(g-1) + nr - \sum \ell_i + \sum k_i + 1$.

Proof.

Note that $\sum k_i$ is the number of exceptional fibers of $\tilde{M}$ because $A_{i,k} = \frac{\alpha_i}{gcd\{\alpha_i, order(\sigma_{i,k})\}} = 1$ if and only if $\alpha_i | order(\sigma_{i,k})$. We proceed case by case.

(a) Suppose F is non-orientable.

1. Assume $\sum k_i = 0$. Recall $p_{i,s}$ are integer numbers such that $order(\sigma_{i,s}) = p_{i,s}\alpha_i$. From definition of $B_{i,k}$, $A_{i,k}$ and from Theorem 3.2.1 we can assume that $\tilde{M} = (Yy, \tilde{g}; B/1)$, where $B = \sum_{i=1}^{r}\sum_{s=1}^{\ell_i} p_{i,s}\beta_i$.

   (i) If G is non-orientable, then $\tilde{g} = n(g-2) + nr - \sum \ell_i + 2$. Therefore $h(\tilde{M}) = \tilde{g} = n(g-2) + nr - \sum \ell_i + 2$, if $B = \pm 1$. Otherwise, $h(\tilde{M}) = \tilde{g} + 1 = n(g-2) + nr - \sum \ell_i + 3$.

   (ii) If G is orientable then $2\tilde{g} = n(g-2) + nr - \sum \ell_i + 2$. Then $h(\tilde{M}) = 2\tilde{g} = n(g-2) + nr - \sum \ell_i + 2$, if $B = \pm 1$. Otherwise, $h(\tilde{M}) = 2\tilde{g} + 1 = n(g-2) + nr - \sum \ell_i + 3$.

2. Assume $\sum k_i = 1$. Recall $B = B_{1,1}+A_{1,1}(\beta_1 \sum_{s=2}^{\ell_1} p'_{1,s}+\sum_{i=2}^{r}\sum_{s=1}^{\ell_i} p_{i,s}\beta_i$, where $p'_{1,s}$, for $s = 2,\ldots,\ell_1$ and $p_{i,s}$, for $i = 2,\ldots,r$ and for $s = 1,\ldots,\ell_i$, are integers numbers such that $order(\sigma_{1,s}) = p'_{1,s}\alpha_1$, for $s = 2,\ldots,\ell_1$, and $order(\sigma_{i,s}) = p_{i,s}\alpha_i$, for $i = 2,\ldots,r$ and for $s = 1,\ldots,\ell_i$. Then

$$\tilde{M} = (Yy, \tilde{g}; B_{1,1}/A_{1,1}, B_{1,2}/1, \ldots, B_{1,\ell_1}/1, \ldots, B_{r,1}/1, \ldots, B_{r,\ell_r}/1).$$

By Theorem 3.2.1 and Definition of $B_{i,k}$, we can consider $\tilde{M} = (Yy, \tilde{g}; B/A_{1,1})$.

(i) If G is non-orientable, then $\tilde{g} = n(g-2)+nr-\sum \ell_i + 2$. Thus $h(\tilde{M}) = \tilde{g} = n(g-2)+nr-\sum \ell_i+2$, if $B = \pm 1$. Otherwise, $h(\tilde{M}) = \tilde{g}+1 = n(g-2)+nr-\sum \ell_i + 3$.

(ii) If G is orientable, then $2\tilde{g} = n(g-2)+nr-\sum \ell_i+2$ and we can conclude that $h(\tilde{M}) = n(g-2)+nr-\sum \ell_i+2$, if $B = \pm 1$. Otherwise, $h(\tilde{M}) = n(g-2)+nr-\sum \ell_i + 3$.

3. Assume $\sum k_i \geq 2$. Note that if G is non-orientable then $\tilde{g} = n(g-2)+nr-\sum \ell_i+2$, and if G is orientable then $2\tilde{g} = n(g-2)+nr-\sum \ell_i+2$. Since $\sum k_i$ is the number of exceptional fibers then $h(\tilde{M}) = \tilde{g}+\sum k_i - 1$, if F is non-orientable and $h(\tilde{M}) = 2\tilde{g}+\sum k_i-1$, if F is orientable. Then it is clear that $h(\tilde{M}) = n(g-2)+nr-\sum \ell_i+\sum k_i+1$.

(b) Suppose F is orientable. Then $2\tilde{g} = 2n(g-1)+nr-\sum \ell_i+2$, by Theorem 3.3.8.

1. Assume $\sum k_i = 0$. Recall $p_{i,s}$ are integer numbers such that $order(\sigma_{i,s}) = p_{i,s}\alpha_i$. From definition of $B_{i,k}$, $A_{i,k}$ and from Theorem 3.2.1 we obtain that $\tilde{M} = (Yy, \tilde{g}; B/1)$, where $B = \sum_{i=1}^{r}\sum_{s=1}^{\ell_i} p_{i,s}\beta_i$. Thus $h(\tilde{M}) = 2\tilde{g} = 2n(g-1)+nr-\sum \ell_i+2$, if $B = \pm 1$. Otherwise, $h(\tilde{M}) = 2\tilde{g}+1 = 2n(g-1)+nr-\sum \ell_i+3$.
2. Assume $\sum k_i = 1$. Recall $B = B_{1,1}+A_{1,1}(\beta_1 \sum_{s=2}^{\ell_1} p'_{1,s}+\sum_{i=2}^{r}\sum_{s=1}^{\ell_i} p_{i,s}\beta_i$, where $p'_{1,s}$, for $s = 2,\ldots,\ell_1$ and $p_{i,s}$, for $i = 2,\ldots,r$ and for $s = 1,\ldots,\ell_i$, are integers numbers such that $order(\sigma_{1,s}) = p'_{1,s}\alpha_1$, for $s = 2,\ldots,\ell_1$, and $order(\sigma_{i,s}) = p_{i,s}\alpha_i$, for $i = 2,\ldots,r$ and for $s = 1,\ldots,\ell_i$. Then $\tilde{M} = (Yy, \tilde{g}; B_{1,1}/A_{1,1}, B_{1,2}/1, \ldots, B_{1,\ell_1}/1, \ldots, B_{r,1}/1, \ldots, B_{r,\ell_r}/1)$. By Theorem 3.2.1 and Definition of $B_{i,k}$, we can consider $\tilde{M} = (Yy, \tilde{g}; B/A_{1,1})$. Thus $h(\tilde{M}) = 2\tilde{g} = 2n(g-1)+nr-\sum \ell_i+2$, if $B = \pm 1$. Otherwise, $h(\tilde{M}) = 2\tilde{g}+1 = 2n(g-1)+nr-\sum \ell_i+3$.
3. Assume $\sum k_i \geq 2$. Then $h(\tilde{M}) = 2n(g-1)+nr-\sum \ell_i+\sum k_i+1$ for $\sum k_i$ is the number of exceptional fibers of $\tilde{M}$. □

Corollary 4.2.3 *Let $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ where $Xx \in \{Oo, On, No, NnI, NnII, N$ and $g \neq 0$, and $\alpha_i \geq 2$, for each $i \in \{1,\ldots,r\}$, and $r \geq 2$ (A Seifert manifold with at least two exceptional fibers and orbit surface different from S^2). Consider* *a transitive representation $\omega : \pi_1(M_0) \to S_n$ defined by*

$$\begin{aligned} \omega(h) &= (1), \\ \omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \ \textit{for } i = 1, \ldots, r \ \textit{ and} \\ \omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j}, \end{aligned}$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively.

Let $\varphi : \tilde{M} \to M$ be the covering associated to ω. Then $h(\tilde{M}) \geq h(M)$.

Proof.

Let r be the number of exceptional fibers of M. Since M has at least two exceptional fibers, then $h(M) = 2g + r - 1$ or $h(M) = g + r - 1$, if F is orientable or not, respectively. Let k_i be as in previous lemma. Recall $\sum k_i$ is the number of exceptional fibers of $\tilde{M}$. Again we proceed case by case.

(a) If F is non-orientable. Recall $\tilde{g} = n(g-2) + 2 + nr - \sum_{i=1}^{r} \ell_i$, if G is non-orientable; otherwise, if G is orientable we have $2\tilde{g} = n(g-2) + 2 + nr - \sum_{i=1}^{r} \ell_i$.

1. If $\sum k_i = 0$, then $h(\tilde{M}) \geq n(g-2) + nr - \sum \ell_i + 2$. Recall $\alpha_i \geq 2$ and $\alpha_i | order(\sigma_{i,k})$, for all i, k, then each cycle of $\omega(q_i)$ has order at least 2. Thus $\ell_i \leq \frac{n}{2}$. Also $\ell_i \leq n-1$ since $n - 1 \geq \frac{n}{2}$, if $n \geq 2$. Then $\sum_{i=1}^{r-2} \ell_i \leq (n-1)(r-2)$.
   Hence
   $$\sum_{i=1}^{r} \ell_i \leq (n-1)(r-2) + \frac{n}{2} + \frac{n}{2}$$
   because $\ell_{r-1} \leq \frac{n}{2}$ and $\ell_r \leq \frac{n}{2}$.

   Note that $(n-1)(r-2) + n = (n-1)(r-1) + 1$.

   Since $\tilde{g} - h(M) = (n-1)(g-2) + (n-1)r - \sum \ell_i + 1$ and $h(\tilde{M}) \geq \tilde{g}$, then
   $$\tilde{g} - h(M) \geq (n-1)(g-2) + (n-1)(r-1) - \sum \ell_i + 1 \geq 0.$$
   Therefore $h(\tilde{M}) \geq h(M)$.
2. If $\sum k_i = 1$, then $\tilde{g} - h(M) = (n-1)(g-2) + (n-1)r - \sum \ell_i + 1$.

   Recall $h(\tilde{M}) \geq \tilde{g}$ and ℓ_1 is the number of cycles of $\omega(q_1)$.

   From previous lemma, we can suppose $\alpha_{1,1} \nmid order(\sigma_{1,1})$,$\alpha_{1,1} | order(\sigma_{1,s})$, for $s = 2, \ldots, \ell_1$, and $\alpha_i | order(\sigma_{i,k})$, for $i = 2, \ldots, r$ and for $s = 1, \ldots, \ell_i$.

Then $order(\sigma_{1,s}) \geq 2$, if $s \neq 1$; and $order(\sigma_{i,s})$, for $i = 2, \ldots, r$ and for all s.

Assume $n \geq 3$, in this case we have that $\ell_i \leq \frac{n}{2} \leq n-1$, for all $i = 2, \ldots, r$, since $order(\sigma_{i,k}) \geq 2$, for $i \geq 2$. Thus $\sum_{i=3}^{r} \ell_i \leq (n-1)(r-3)$.

Now note that

$$\ell_1 \leq \frac{n - order(\sigma_{1,1})}{2} + 1$$

for $\omega(q_1)$ contains the cycle $\sigma_{1,1}$ and the cycles $\sigma_{j,k}$, for $j = 2, \ldots, r$, but the cycles $\sigma_{j,k}$, for $j = 2, \ldots, r$, have order at least 2 then we have at most $\frac{n-order(\sigma_{1,1})}{2} + 1$ cycles in $\omega(q_1)$. Also we have the following inequality $\frac{n-order(\sigma_{1,1})}{2} + 1 \leq \frac{n-1}{2} + 1$; it follows for $order(\sigma_{1,1}) \geq 1$. Thus $l_1 \leq \frac{n-1}{2} + 1$.

Then

$$\sum_{i=1}^{r} \ell_i \leq (n-1)(r-3) + \frac{n}{2} + \frac{n-1}{2} + 1 = (n-1)(r-3) + n + \frac{1}{2}$$

because $\ell_2 \leq \frac{n}{2}$ and $\ell_1 \leq \frac{n-1}{2} + 1$. Since $(n-1)(r-3) + n + \frac{1}{2} \leq (n-1)(r-1) + 1$ we obtain

$$(n-1)(r-1) + 1 - \sum_{i=1}^{r} \ell_i \geq 0.$$

Recalling $\tilde{g} - h(M) = (n-1)(g-2) + (n-1)r - \sum \ell_i + 1$ we conclude that $h(\tilde{M}) \geq \tilde{g} \geq h(M)$.

If $n = 2$, then $\tilde{M}$ has exactly one exceptional fiber if and only if $M = (Xx, g; \beta_1/\alpha_1, \beta_2/2, \ldots, \beta_r/2)$, where $\alpha_1 > 2$ y $\omega(q_i) = (1, 2)$, for $i = 1, \ldots, r$. Thus $\tilde{M} = (Yy, \tilde{g}; B_{1,1}, A_{1,1}, \beta_2/1, \ldots, \beta_r/1)$. It is easy to see in this case that $\sum \ell_i = r$ Then $\tilde{g} - h(M) = g - 1$. Recall $g \neq 0$. Therefore $h(\tilde{M}) \geq h(M)$.

3. If $\sum k_i \geq 2$, notice that

$$h(\tilde{M}) - h(M) = (n-1)(g-2) + (n-1)r - \left(\sum \ell_i - \sum k_i\right)$$

The inequality

$$\ell_i \leq \frac{n - \sum_{i=1}^{k_i} order(\sigma_{i,s})}{2} + k_i$$

follows since ℓ_i is the number of cycles of $\omega(q_j)$ and $order(\sigma_{i,j}) \geq 2$ for $j = k+1, \ldots, r$; note also

$$\frac{n - \sum_{i=1}^{k_i} order(\sigma_{i,s})}{2} + k_i \leq \frac{n-1}{2} + k_i$$

since $\sum_{i=1}^{k_i} order(\sigma_{i,s} \geq 1$.

Then $\sum \ell_i - \sum k_i \leq \frac{(n-1)r}{2}$. On the other hand, $\frac{r}{2} \leq r-1$ for $r \geq 2$. Thus $\frac{(n-1)(r-1)}{2} - (\sum_{i=1}^{r} \ell_i - \sum_{i=1}^{r} k_i) \geq 0$ and we obtain

$$(n-1)(r-1) - (\sum_{i=1}^{r} \ell_i - \sum_{i=1}^{r} k_i) \geq 0.$$

Therefore $h(\tilde{M}) \geq h(M)$.

(b) Assume F is orientable. In this case, G is orientable and $2\tilde{g} = 2n(g-1) + nr - \sum \ell_i$.

1. If $\sum k_i = 0$, then $h(\tilde{M}) \geq n(g-2) + nr - \sum \ell_i + 2$. Recall $\alpha_i \geq 2$ and $\alpha_i | order(\sigma_{i,k})$, for all i, k, then each cycle of $\omega(q_i)$ has order at least 2. Thus $\ell_i \leq \frac{n}{2}$. Also $\ell_i \leq n-1$ since $n-1 \geq \frac{n}{2}$, if $n \geq 2$. Then $\sum_{i=1}^{r-2} \ell_i \leq (n-1)(r-2)$.
   Hence
   $$\sum_{i=1}^{r} \ell_i \leq (n-1)(r-2) + \frac{n}{2} + \frac{n}{2}$$
   because $\ell_{r-1} \leq \frac{n}{2}$ and $\ell_r \leq \frac{n}{2}$.

   It is clear that $(n-1)(r-2) + n = (n-1)(r-1) + 1$.

   Since $\tilde{g} - h(M) = 2(n-1)(g-1) + (n-1)r - \sum \ell_i + 1$ and $h(\tilde{M}) \geq \tilde{g}$, then
   $$2\tilde{g} - h(M) \geq 2(n-1)(g-1) + (n-1)(r-1) - \sum \ell_i + 1 \geq 0.$$
   Therefore $h(\tilde{M}) \geq h(M)$.
2. If $\sum k_i = 1$, Recall $h(\tilde{M}) \geq \tilde{g}$. Then
   $$2\tilde{g} - h(M) = 2(n-1)(g-1) + (n-1)r - \sum \ell_i + 1.$$

   By previous Lemma, we can suppose $\alpha_{1,1} \nmid order(\sigma_{1,1})$,$\alpha_{1,1} | order(\sigma_{1,s})$, for $s = 2, \ldots, \ell_1$, and $\alpha_i | order(\sigma_{i,k})$, for $i = 2, \ldots, r$ and for $s = 1, \ldots, \ell_i$. Then $order(\sigma_{1,s}) \geq 2$, if $s \neq 1$; and $order(\sigma_{i,s})$, for $i = 2, \ldots, r$ and for all s.

Assume $n \geq 3$, in this case we have that $\ell_i \leq \frac{n}{2} \leq n-1$, for all $i = 2,\ldots,r$, since $order(\sigma_{i,k}) \geq 2$, for $i \geq 2$. Thus $\sum_{i=3}^{r} \ell_i \leq (n-1)(r-3)$. Now note that

$$\ell_1 \leq \frac{n - order(\sigma_{1,1})}{2} + 1 \leq \frac{n-1}{2} + 1.$$

The first inequality $\ell_1 \leq \frac{n-order(\sigma_{1,1})}{2} + 1$ follows for ℓ_1 is the number of cycles in $\omega(q_1)$; in $\omega(q_1)$ we have the cycle $\sigma_{1,1}$ and the cycles $\sigma_{j,k}$, for $j = 2,\ldots,r$, but the cycles $\sigma_{j,k}$ have order at least 2, for $j = 2,\ldots,r$, then we have at most $\frac{n-order(\sigma_{1,1})}{2} + 1$ cycles in $\omega(q_1)$. The second inequality $\frac{n-order(\sigma_{1,1})}{2} + 1 \leq \frac{n-1}{2} + 1$ follows because $order(\sigma_{1,1}) \geq 1$.

Then

$$\sum_{i=1}^{r} \ell_i \leq (n-1)(r-3) + \frac{n}{2} + \frac{n-1}{2} + 1 = (n-1)(r-3) + n + \frac{1}{2}$$

for $\ell_2 \leq \frac{n}{2}$ and $\ell_1 \leq \frac{n-1}{2}+1$. Since $(n-1)(r-3)+n+\frac{1}{2} \leq (n-1)(r-1)+1$ we obtain

$$(n-1)(r-1) + 1 - \sum_{i=1}^{r} \ell_i \geq 0.$$

Therefore $h(\tilde{M}) \geq \tilde{g} \geq h(M)$.

If $n = 2$, then $\tilde{M}$ has exactly one exceptional fiber if and only if $M = (Xx, g; \beta_1/\alpha_1, \beta_2/2, \ldots, \beta_r/2)$, where $\alpha_1 > 2$ y $\omega(q_i) = (1,2)$, for $i = 1\ldots,r$. Thus $\tilde{M} = (Yy, \tilde{g}; B_{1,1}, A_{1,1}, \beta_2/1, \ldots, \beta_r/1)$. It is easy to see in this case that $\sum \ell_i = r$. Then $2\tilde{g} - h(M) = 2(g-1)+1$. Because of the fact $g \neq 0$, we conclude $h(\tilde{M}) \geq h(M)$.

3. If $\sum k_i \geq 2$, then

$$h(\tilde{M}) - h(M) = 2(n-1)(g-1) + (n-1)r - (\sum_{i=1}^{r} \ell_i - \sum_{i=1}^{r} k_i)$$

Note that

$$\ell_i \leq \frac{n - \sum_{i=1}^{k_i} order(\sigma_{i,s})}{2} + k_i$$

because ℓ_i is the number of cycles of $\omega(q_j)$ and $order(\sigma_{i,j}) \geq 2$ for $j = k+1,\ldots,r$; note also

$$\frac{n - \sum_{i=1}^{k_i} order(\sigma_{i,s})}{2} + k_i \leq \frac{n-1}{2} + k_i$$

since $\sum_{i=1}^{k_i} order(\sigma_{i,s} \geq 1$.

Therefore $\frac{(n-1)(r-1)}{2} - (\sum_{i=1}^{r} \ell_i - \sum_{i=1}^{r} k_i) \geq 0$.

Since $r \geq 2$, then $\frac{r}{2} \leq r - 1$. Thus

$$(n-1)(r-1) - (\sum_{i=1}^{r} \ell_i - \sum_{i=1}^{r} k_i) \geq 0.$$

Therefore $h(\tilde{M}) \geq h(M)$.

We can summarize the previous Corollary in the following Theorem.

Theorem 4.2.1 *Let $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ where $Xx \in \{Oo, On, No, NnI, NnII,$ and $g \neq 0$. Let $n \in \mathbb{N}$ and $\omega : \pi_1(M_0) \to S_n$ be a transitive representation defined by*

$$\begin{aligned} \omega(h) &= (1), \\ \omega(q_i) &= \sigma_{i,1} \cdots \sigma_{i,\ell_i}, \forall i = 1, \ldots, r \quad and \\ \omega(v_j) &= \rho_{j,1} \cdots \rho_{j,s_j}, \end{aligned}$$

where $\sigma_{i,1} \cdots \sigma_{i,\ell_i}$ and $\rho_{j,1} \cdots \rho_{j,s_j}$ are the disjoint cycle decompositions of $\omega(q_i)$ and $\omega(v_j)$, respectively, and $\{h, v_j, q_i\}$ is a standard system of generators of $\pi_1(M_0)$.

Then $h(\tilde{M}) \geq h(M)$.

Proof.

The result follows from Corollaries (4.2.1), (4.2.2) and (4.2.3). □

4.2.2 Heegaard genus when $\omega(h) = \varepsilon_n$

Recall $\varepsilon_n = (1, 2, \ldots, n) \in S_n$. Given a Seifert manifold $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$, with orbit projection $p : M \to F$, where F has genus g, and given a representation $\omega : \pi_1(M_0) \to S_n$ defined by

$$\begin{aligned} \omega(h) &= \varepsilon_n, \\ \omega(q_i) &= \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \quad and \\ \omega(v_j) &= \tau_j, \end{aligned}$$

τ_j is a power of the n-cycle ε_n, if $e(v_j) = +1$ or τ_j is a reflection ρ_j, if $e(v_j) = -1$. Then, if $\varphi : \tilde{M} \to M$ is the covering determined by ω, by

Theorem 3.3.15 we have that $\tilde{M} = (Xx, g; B_1/A_1, \ldots, B_r/A_r)$ where

$$B_i = \frac{\beta_i + k_i\alpha_i}{gcd\{n, \beta_i + k_i\alpha_i\}}$$

and

$$A_i = \frac{n\alpha_i}{gcd\{n, \beta_i + k_i\alpha_i\},}$$

where $gcd\{n, \beta_i + k_i\alpha_i\}$ denotes the greatest common divisor of n and $\beta_i + k_i\alpha_i$ Note that $\alpha_i \geq 2$ implies that $A_i \geq 2$.

Lemma 4.2.4 *Let $M = (Xx, g; \beta_1/\alpha_1)$ be a Seifert manifold, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$ where $\alpha_1 \geq 1$. Suppose that $n \in \mathbb{N}$ and $\omega : \pi_1(M_0) \to S_n$ is a representation defined by*

$$\begin{array}{rcl} \omega(h) & = & \varepsilon_n, \\ \omega(q_1) & = & \varepsilon^{k_1}, \quad \textit{and} \\ \omega(v_j) & = & \tau_j, \end{array}$$

where τ_j is a power of ε_n, if v_j commutes with h; otherwise, if v_j anticommutes with h, τ_j is a reflection ρ_j.
Suppose $\varphi : \tilde{M} \to M$ is the covering determined by ω.

- *Assume $\beta_1 \nmid n$ or $\beta_1 = \pm 1$, then $h(\tilde{M}) = h(M)$.*
- *Assume $\beta_1 \neq 1$ and $\beta_1 | n$, then $h(\tilde{M}) = g$, if F is orientable; otherwise, $h(\tilde{M}) = 2g$, if F is orientable. Furthermore, $h(\tilde{M}) < h(M)$.*

Proof.
Observe that $\tilde{M} = (Xx, g; B_1/A_1)$, with $B_1 = \frac{\beta_1}{gcd\{n,\beta_1\}}$ and $A = \frac{n\alpha_1}{gcd\{n,\beta_1\}}$. It is clear that $B_1 = 1$ if and only if $\beta_1 | n$.

- If $\beta_1 \nmid n$, then $\beta_1 \neq 1$, $B_1 \neq 1$ and

$$h(M) = h(\tilde{M}) = \begin{cases} 2g+1, & \text{if } F \text{ is orientable, or} \\ g+1, & \text{otherwise.} \end{cases}$$

  If $\beta_1 = \pm 1$, then $B_1 = 1$. Thus $h(\tilde{M}) = h(M) = g$. Therefore $h(\tilde{M}) = h(M)$.
- Suppose $\beta_1 \neq 1$ and $\beta_1 | n$. Thus $\tilde{M} = (Xx, g; \frac{1}{A_1})$.
  (a) If F is non-orientable, then $h(M) = g + 1$ (of course, when M is non-orientable we ask β_1 be even, in order, to compute $h(M)$; recall if β_1 is odd we can not compute $h(M)$). On the other hand, $h(\tilde{M}) = g$. Therefore $h(\tilde{M}) < h(M)$.
  (b) If F is orientable, then $h(M) = 2g + 1$ and $h(\tilde{M}) = 2g$. Therefore $h(\tilde{M}) < h(M)$.

□

Lemma 4.2.5 *Let $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ be a Seifert manifold, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$ such that $\alpha_i \geq 2$ and $r \geq 2$. Consider a representation $\omega : \pi_1(M_0) \to S_n$ defined by*

$$\begin{aligned} \omega(h) &= \varepsilon_n, \\ \omega(q_i) &= \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \quad \textit{and} \\ \omega(v_j) &= \tau_j, \end{aligned}$$

such that τ_j is a power of ε_n, if v_j commutes with h; otherwise, τ_j is a reflection ρ_j, if v_j anticommutes with h.
Let $\varphi : \tilde{M} \to M$ be the covering associated to ω. Then $h(\tilde{M}) = h(M)$.

Proof.

Let F and G be the orbit surfaces of M and $\tilde{M}$, respectively. If g is the genus of F, then G also has genus g since F and G are homeomorphic because of Theorem (3.3.15). Note that $\alpha_i \geq 2$ implies that $A_i \geq 2$, thus the number of exceptional fibers of $\tilde{M}$ is equal to r. Therefore $h(\tilde{M}) = h(M)$. □

Now we are able to prove the following theorem.

Theorem 4.2.2 *Consider $M = (Xx, g; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r)$ a Seifert manifold, where $Xx \in \{Oo, On, No, NnI, NnII, NnIII\}$ and assume $\omega : \pi_1(M_0) \to S_n$ is a representation defined by*

$$\begin{aligned} \omega(h) &= \varepsilon_n, \\ \omega(q_i) &= \varepsilon_n^{k_i}, \forall i = 1, \ldots, r \quad \textit{and} \\ \omega(v_j) &= \tau_j, \end{aligned}$$

such that τ_j is a power of ε_n if v_j commutes with h; otherwise, τ_j is a reflection ρ_j, if v_j anticommutes with h.

Suppose $\varphi : \tilde{M} \to M$ is the covering determined by ω. If $M = (Xx, g; \beta/\alpha)$, with $\alpha \geq 2$ (recall $\beta \neq 1$ is even if M is non-orientable) and $\beta | n$, then $h(\tilde{M}) < h(M)$. Otherwise, $h(\tilde{M}) = h(M)$.

Proof.

The result follows from Lemma (4.2.4) and Lemma (4.2.5). □

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