ON GENERA OF COVERINGS OF TORUS BUNDLES

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Abstract. After showing that a covering space of surface bundles over S^1 factors as a 'covering of fibers' followed by a 'power covering', we prove that, for torus bundles, power coverings do not lower Heegaard genus, and that fiber coverings lower the genus only in special cases.

1. Introduction.

Any closed connected 3-manifold M is the union of two handlebodies with pairwise disjoint interiors. The minimal genus of the handlebodies among all such decompositions is called the *Heegaard genus* of M, or simply the *genus* of M and is denoted by $g(M)$. The *rank* of a group is the cardinality of a minimal set of generators for the group. The rank of the fundamental group of M is called *the rank* of M, and gives a lower bound $rank(\pi_1(M)) \leq g(M)$. For a covering space of 3-manifolds $\varphi : \tilde{M} \to M$, we say that φ *lowers the genus* if $g(\tilde{M}) < g(M)$.

There are two famous questions. First, for a 3-manifold M, is the genus of M equal to the rank of M? And, secondly, if the fundamental group of M contains a finite index subgroup of a given rank, can the rank of the subgroup be smaller than the rank of M? In terms of covering spaces, we can pose the second question as: Is there a finite-sheeted covering space of M that lowers the genus? and, how large is the lowering?

P. Shalen states two conjectures ([8]):

(1) For closed connected orientable hyperbolic 3-manifolds rank equals genus.
(2) For closed connected orientable hyperbolic 3-manifolds a finite-sheeted covering space lowers the genus at most by one.

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With respect to Conjecture (1), we know nowadays that the genus can be arbitrarily larger than the rank (for non-hyperbolic manifolds see [7]; for hyperbolic manifolds see [2]).

With respect to Conjecture (2), one needs to establish for a given manifold, first, if there are indeed covering spaces that lower the genus, and, then, to determine how large is the lowering. For hyperbolic manifolds very few examples of genus-lowering covering spaces are known (see [8], Section 4.5). For non-hyperbolic manifolds we can distinguish cases:

(1) If $\pi_1(M)$ is finite and non-trivial, then the universal cover of M lowers the genus.
(2) If M is a torus bundle over S^1, we show in this work that only in special cases M admits genus-lowering covering spaces.
(3) For Seifert manifolds with orbit surface of genus g,
    (a) If $g \neq 0$, there are only few examples of genus-lowering covering spaces (see Section 5 and [5]).
    (b) If $g = 0$, there are examples, but it is an open problem to determine all possible genus-lowering covering spaces.
(4) If M is a graph-manifold, it is an open problem to determine if there are genus-lowering covering spaces of M.

The paper is organized as follows. In Section 2, after some algebraic remarks, we determine the structure of the covering spaces of surface bundles, namely, we show that any covering space of surface bundles is a product of a covering of fibers followed by a power covering (Corollary 2.3). This reduces the problem of finding coverings that lower the genus to finding either power coverings or fiber coverings that lower the genus. We also enlist results on torus bundles from [6] that are used throughout the paper. In Section 3 we show that power coverings of torus bundles do not lower the genus. In Section 4 we determine the structure of the coverings of fibers of torus bundles (Theorem 4.1) and characterize the examples of coverings of fibers of torus bundles that lower the genus (Theorem 4.12). Of interest is Corollary 4.9 where we explicitly describe the subgroups of the fundamental group of the torus which correspond to finite covering spaces. We include in Section 5 the examples of covering spaces of Seifert manifolds that lower the genus mentioned above. These coverings of Seifert manifolds are cyclic, and give examples for the remark in Section 4.6 of [8].

2. Preliminaries.

If X is a set, we write $S(X)$ for the symmetric group on the symbols in X. If $\#(X) = n$, we write $S_n = S(X)$. If $G \leq S(X)$, and $i \in X$, we write $St_G(i) = \{\sigma \in G : \sigma(i) = i\}$; also write $St(i) = St_{S(X)}(i)$.

Lemma 2.1. *Assume that $G \leq S_n$ is a transitive group and $K \lhd G$. Let $A_1, ..., A_m$ be the orbits of K. Then for each $i \in \{1, \ldots, m\}$ and each $\sigma \in G$, $\sigma \cdot A_i = A_s$ for some s. In particular $\#A_1 = \cdots = \#A_m$.*

Proof. For $\sigma \in G$ and $i \in \{1, \ldots, m\}$,

$$\begin{aligned} K \cdot (\sigma \cdot A_i) &= (K \cdot \sigma) \cdot A_i \\ &= (\sigma \cdot K) \cdot A_i, \quad \text{for, } K \text{ is normal,} \\ &= \sigma \cdot (K \cdot A_i) \\ &= \sigma \cdot A_i, \qquad\quad \text{for, } A_i \text{ is an orbit of } K. \end{aligned}$$

Thus $\sigma \cdot A_i$ is a union of orbits of K. Since A_i is an orbit, $A_i \neq \emptyset$. Pick some $a \in A_i$; then $\sigma(a) \in A_s$ for some s, and $A_s \subset \sigma \cdot A_i$.

Assume that there is a t such that $A_t \subset \sigma \cdot A_i$. Choose $b \in A_i$ such that $\sigma(b) \in A_t$. Since there is a $\tau_1 \in K$ such that $\tau_1(a) = b$, then $\sigma(b) = \sigma(\tau_1(a)) = \tau_2(\sigma(a))$ for some other $\tau_2 \in K$ for, K is normal. Since A_s is an orbit of K, then $\tau_2(\sigma(a)) \in A_s$. Then $A_s \cap A_t \neq \emptyset$, and therefore $A_s = A_t$. We conclude that $\sigma \cdot A_i = A_s$.

Since G is transitive, for each s there is a $\sigma \in G$ such that $\sigma \cdot A_1 = A_s$. It follows that $\#A_1 = \cdots = \#A_m$. □

Lemma 2.2. *Assume that $G \leq S_n$ is a transitive group and $K \lhd G$. Then there exist homomorphisms $q : G \to S_m$ and $\gamma : q^{-1}(St(1)) \to S_{n/m}$ such that $St_G(1) \subset q^{-1}(St(1))$, and $q(K) = 1$, and $\gamma|K$ is transitive.*

Proof. By Lemma 2.1, G is imprimitive with the orbits of K, $A_1, ..., A_m$, a set of imprimitivity blocks. We assume that $1 \in A_1$. Then we have these homomorphisms: $q : G \to S(\{A_1, \ldots, A_m\}) = S_m$ which is induced by the quotient $p : A_1 \cup \cdots \cup A_m \to \{A_1, \ldots, A_m\}$ such that $p(a) = A_i \Leftrightarrow a \in A_i$, and $\gamma : q^{-1}(St(A_1)) \to S(A_1) = S_{n/m}$ which is given by restriction $\gamma(\sigma) = \sigma|A_1$. □

2.1. **Coverings of surface bundles.**

Corollary 2.3. *Let $F \hookrightarrow M \to S^1$ be a surface bundle over S^1, and let $\varphi : \widetilde{M} \to M$ be an n-fold covering space. Then there is a commutative diagram of covering spaces of surface bundles over S^1*

$$\begin{tikzcd}
\widetilde{M} \arrow[dd, "\varphi"'] \arrow[dr, "\varphi_\gamma"] & \\
& N \arrow[dl, "\varphi_q"] \\
M &
\end{tikzcd}$$

such that φ_q and φ_γ are m-fold and n/m-fold covering spaces, respectively, and $\varphi_q^{-1}(F) = F_1 \sqcup \cdots \sqcup F_m$ with $\varphi_q| : F_i \to F$ a homeomorphism for $i = 1, \ldots, m$, and $\varphi_\gamma^{-1}(\tilde{F})$ is connected for $\tilde{F}$ any fiber of N.

Proof. Recall that, if $F \hookrightarrow M \to S^1$ is a surface bundle, then $\pi_1(M)$ is isomorphic to a semi-direct product $\pi_1(F) \rtimes \mathbb{Z}$. In particular $\pi_1(F) \lhd \pi_1(M)$. Then $\omega(\pi_1(F)) \lhd Image(\omega)$ where $\omega : \pi_1(M) \to S_n$ is the representation associated to φ. Lemma 2.2 applies.

□

Remark 2.4. Note that the coverings φ_γ or φ_q in Corollary 2.3 might be homeomorphisms.

Let $F \hookrightarrow M \to S^1$ be a surface bundle. Then there is a homeomorphism $h : F \to F$, the *monodromy* of M, such that

$$M = \frac{F \times I}{(x, 0) \sim (h(x), 1)}.$$

The *infinite cyclic covering* of M, $u : F \times \mathbb{R} \to M$, is the covering corresponding to the subgroup $\pi_1(F) \leq \pi_1(M)$, and has covering translations generated by $t : F \times \mathbb{R} \to F \times \mathbb{R}$, such that $t(x, \lambda) = (h(x), \lambda + 1)$.

Remark 2.5. Let $F \hookrightarrow M \to S^1$ be the surface bundle of Corollary 2.3.

(1) The covering $\varphi_q : N \to M$ of the corollary is constructed by taking m copies $F \times I \times \{1\}, \ldots, F \times I \times \{m\}$, and then glueing $((x, 0), i) \sim ((h(x), 1), i+1)$ where the indices are taken mod m; that is,

$$N = \frac{F \times I \times \{1\} \sqcup \cdots \sqcup F \times I \times \{m\}}{((x, 0), i) \sim ((h(x), 1), i + 1)}.$$

Note that N is an F-bundle with monodromy h^m.

(2) The covering $\varphi_\gamma : \widetilde{M} \to N$ of the corollary is constructed by taking the covering space $\psi : \widetilde{F} \to F$ associated to $\omega|\pi_1(F) : \pi_1(F) \to S_{n/m}$, and some monodromy $\widetilde{g} : \widetilde{F} \to \widetilde{F}$ such that the diagram

$$\begin{tikzcd} \tilde{F} \arrow[r, "\tilde{g}"] \arrow[d, "\psi"'] & \tilde{F} \arrow[d, "\psi"] \\ F \arrow[r, "g"] & F \end{tikzcd}$$

commutes, where g is the monodromy of N. And

$$\widetilde{M} = \frac{\widetilde{F} \times I}{(x,0) \sim (\widetilde{g}(x), 1)}.$$

The covering projection is obtained from $\psi \times 1$.

(3) The covering $\varphi_q : N \to M$ in the corollary is characterized by a commutative diagram of covering spaces

$$\begin{tikzcd} F \times \mathbb{R} \arrow[dd, "u"'] \arrow[dr, "v"] & \\ & N \arrow[dl, "\varphi_q"] \\ M & \end{tikzcd}$$

where u and v are infinite cyclic coverings. We call this type of covering space a *power covering*.

(4) The covering $\varphi_\gamma : \tilde{M} \to N$ in the corollary is characterized by a commutative diagram of covering spaces

$$\begin{tikzcd} \tilde{F} \times \mathbb{R} \arrow[r, "\psi \times 1"] \arrow[d, "\tilde{u}"'] & F \times \mathbb{R} \arrow[d, "u"] \\ \tilde{M} \arrow[r, "\varphi_\gamma"] & M \end{tikzcd}$$

where ψ is as in (2), and u and $\tilde{u}$ are infinite cyclic coverings. We call this type of covering space a *covering of fibers*.

Notice that, since $\psi \times 1$ sends each fundamental region of $\tilde{u}$ onto a fundamental region of u, the covering $\psi \times 1$ commutes with the covering translations of $\tilde{u}$ and u.

2.2. **Torus bundles.** Let M be a torus bundle over S^1. Then

$$M \cong \frac{T^2 \times I}{(x,0) \sim (A(x),1)},$$

for some homeomorphism $A : T^2 \to T^2$. We write $M = M_A$ for the torus bundle with monodromy A. Throughout this paper we fix a basis $\pi_1(T^2) \cong \langle x, y : [x,y] \rangle$. With respect to this basis, the homeomorphism A can be identified with an integral invertible matrix, namely, its induced isomorphism $A_{\#} : \pi_1(T) \to \pi_1(T)$. We consider here only orientable torus bundles, that is, $A \in SL(2,\mathbb{Z})$.

It is known that the Heegaard genus of M_A is two or three. Also the fundamental group of M_A is a semi-direct product

$$\pi_1(M_A) \cong \pi_1(T^2) \rtimes \mathbb{Z} \cong \langle x, y, t : x^t = x^\alpha y^\gamma, y^t = x^\beta y^\delta, [x,y] = 1 \rangle,$$

where $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. The first homology group of M_A is of the form

$$H_1(M_A) \cong \mathbb{Z} \oplus Coker(A - I) \cong \mathbb{Z} \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$$

where $n_1 | n_2$.

By [6], the following conditions are equivalent:

- M_A is a double branched covering of S^3.
- $n_1 = 1, 2$.
- A is conjugate to $\left(\begin{smallmatrix} -1 & -a \\ b & ab-1 \end{smallmatrix}\right)$ in $GL(2,\mathbb{Z})$ for some integers a and b.

Notice that $\left(\begin{smallmatrix} -1 & -a \\ b & ab-1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} -1 & -b \\ a & ab-1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; that is, the integers a and b are interchangeable.

If the Heegaard genus of M_A is two, then M_A is a double branched covering of the 3-sphere. Also, $g(M_A) = 2$ if and only if A is conjugate to $\left(\begin{smallmatrix} -1 & -1 \\ b & b-1 \end{smallmatrix}\right)$ in $GL(2,\mathbb{Z})$ (see [6]).

If $A = \left(\begin{smallmatrix} -1 & -a \\ b & ab-1 \end{smallmatrix}\right)$, write $M_{a,b} = M_A$. Then the numbers a and b are complete invariants of M_A. That is, $M_{a_1,b_1} \cong M_{a_2,b_2}$ if and only if the sets $\{a_1, b_1\} = \{a_2, b_2\}$, except for $M_{1,6} \cong M_{2,3}$ (see [6], Theorem 4).

3. Power coverings of torus bundles.

In this section we prove

Theorem 3.1. *Let $\eta : \widetilde{M} \to M$ be a finite power covering space of torus bundles. If the Heegaard genus of M is three, then the Heegaard genus of $\widetilde{M}$ is also three.*

Proof. Let us assume that $M = M_A$ for some $A \in SL(2,\mathbb{Z})$, and fix $\eta : \widetilde{M} \to M_A$ an n-fold power covering. Then $\widetilde{M} \cong M_{A^n}$. In the following lemmata we check that, if the Heegaard genus of M_A is three and $n \geq 2$, then the genus of M_{A^n} is at least three, by computing the rank of $\pi_1(M_{A^n})$ or of $H_1(M_{A^n})$, which are lower bounds for the genus. $\square$

Recall that the first homology group of M_{A^n} is

$$H_1(M_{A^n}) \cong \mathbb{Z} \oplus Coker(A^n - I).$$

We repeatedly use the fact that

$$A^n - I = (A - I)(A^{n-1} + \cdots + A + I)$$

For reference purposes, we upgrade the following easy remark to a lemma.

Lemma 3.2. *Let C be an $m \times m$ integral matrix such that there is an integer $s \neq \pm 1$ with $s|C_{i,j}$ for each i, j. Then the rank of $Coker(C)$ is m.*

Proof. The Smith normal form of C is $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_m \end{pmatrix}$ with $t_i|t_{i+1}$. Since $s|C_{i,j}$ for each i, j, and t_1 is the greatest common divisor of the $C_{i,j}$, then $s|t_1$, and the lemma follows.

$\square$

Lemma 3.3. *If M_A is not a double branched cover of the 3-sphere, and $n \geq 2$, then the rank of $H_1(M_{A^n})$ is three.*

Proof. By [6], the Smith normal form of $A - I$ is $\begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix}$ with $n_1|n_2$, and $n_1 \neq 1, 2$.

Then $H_1(M_{A^n}) \cong \mathbb{Z} \oplus Coker\left((A - I)(A^{n-1} + \cdots + A + I)\right) \cong \mathbb{Z} \oplus Coker(\begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} B)$ for some matrix B. Now n_1 divides all entries of the matrix $\begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} B$, and Lemma 3.2 applies. $\square$

We assume now that M_A is a double branched covering of S^3. Then we may assume that $A = \left(\begin{smallmatrix} -1 & -a \\ b & ab-1 \end{smallmatrix}\right)$. If the Heegaard genus of M_A is three, then $|a|, |b| \neq 1$.

Lemma 3.4. *If one of a or b is zero, and $n \geq 2$, then the Heegaard genus of M_{A^n} is three.*

Proof. We may assume that $b = 0$. Notice that

$$A^n = \begin{cases} \begin{pmatrix} -1 & -na \\ 0 & -1 \end{pmatrix}, & \text{if } n \text{ is odd} \\ \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}, & \text{if } n \text{ is even} \end{cases}$$

If also $a = 0$, then $A^n - I = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} -2 & 0 \\ 0 & -2 \end{smallmatrix}\right)$. Then Lemma 3.2 applies, and the rank of $H_1(M_{A^n})$ is three. Assume then that $a \neq 0$.

Claim 3.5. *If $B = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$, and $|\alpha| \geq 2$, then the rank of $H_1(M_B)$ is three.*

Proof of Claim 3.5. The Smith normal form of $B - I$ is $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then Lemma 3.2 applies to $H_1(M_B) \cong \mathbb{Z} \oplus Coker(B - I)$. □

Claim 3.6. *If $B = \left(\begin{smallmatrix} -1 & -\alpha \\ 0 & -1 \end{smallmatrix}\right)$, and $|\alpha| \geq 2$, then the rank of $\pi_1(M_B)$ is three.*

Proof of Claim 3.6. The fundamental group of M_B is

$$\pi_1(M_B) = \langle x, y, t : txt^{-1}x, tyt^{-1}yx^{\alpha}, xyx^{-1}y^{-1} \rangle.$$

Write $r_1 = txt^{-1}x$, $r_2 = tyt^{-1}yx^{\alpha}$, and $r_3 = xyx^{-1}y^{-1}$.

Also write $d_1 = \partial/\partial x$, $d_2 = \partial/\partial y$, and $d_3 = \partial/\partial t$, the Fox derivatives. Then

$$d_1(r_2) = tyt^{-1}yP, \quad d_2(r_2) = tyt^{-1} + t, \quad d_3(r_2) = -tyt^{-1} + 1$$

where $P = 1 + x + \cdots + x^{\alpha-1}$ if $\alpha > 0$, and $P = -(x^{-1} + x^{-2} + \cdots + x^{\alpha})$ if $\alpha < 0$. Also

$$d_1(r_1) = txt^{-1} + t, \quad d_2(r_1) = 0, \quad d_3(r_1) = -txt^{-1} + 1$$

and

$$d_1(r_3) = -xyx^{-1} + 1, \quad d_2(r_3) = -xyx^{-1}y^{-1} + x, \quad d_3(r_3) = 0$$

We define a ring homomorphism $\rho : \mathbb{Z}\pi_1(M_B) \to \mathbb{Z}_{\alpha}$, from the group ring of $\pi_1(M_B)$ into the ring of integers $\pmod{\alpha}$. First define

$$\rho(x) = 1, \quad \rho(y) = 1, \quad \rho(t) = -1.$$

Extending $\rho : \pi_1(M_B) \to \mathbb{Z}_\alpha$ multiplicatively, we see that $\rho(r_i) = 1$ for $i = 1, 2, 3$.

Then we can extend ρ to the ring group, $\rho : \mathbb{Z}\pi_1(M_B) \to \mathbb{Z}_\alpha$, as a ring homomorphism sending 1 to 1. Also ρ sends all Fox derivatives of the relators r_1, r_2 and r_3 into zero.

By [3], we conclude that $\text{rank}(\pi_1(M_B)) = 3$. □

To finish the proof of Lemma 3.4, we note that claims 3.5 and 3.6 give the number 3 as a lower bound for the genus of M_{A^n} when n is even or odd, respectively. It follows that the genus of M_{A^n} is three.

□

The remaining case is $|a|, |b| \geq 2$ for $A = \left(\begin{smallmatrix} -1 & -a \\ b & ab-1 \end{smallmatrix}\right)$.

Lemma 3.7. *If $|a|, |b| \geq 2$, and $n \geq 2$, then the rank of $H_1(M_{A^n})$ is three.*

Proof. If a and b are both even integers, then the Smith normal form of $A - I$ is $\left(\begin{smallmatrix} -2 & 0 \\ 0 & ab/2-2 \end{smallmatrix}\right)$, and, as in the proof of Lemma 3.3, Lemma 3.2 applies to $Coker(A^n - I)$.

Then assume that a and b are not both even. Note that, then, $|ab| \geq 6$.

We define $f(0) = f(1) = 1$, and

$$f(n) = \begin{cases} f(n-1) - f(n-2), & \text{if } n \text{ is odd} \\ abf(n-1) - f(n-2), & \text{if } n \text{ is even} \end{cases}$$

For convenience we define $f(-1) = 0$.

Claim 3.8. *If $n \geq 1$, then* $A^n = \begin{pmatrix} -f(2n-2) & -af(2n-1) \\ bf(2n-1) & f(2n) \end{pmatrix}$.

Proof of Claim 3.8. This follows by an easy induction on n. □

Claim 3.9. *If $n \geq 2$, then* $A^n + A^{n-1} + \cdots + A + I =$

$$\begin{cases} f(n)\begin{pmatrix} -abf(n-2) & -af(n-1) \\ bf(n-1) & abf(n) \end{pmatrix}, & \textit{if } n \textit{ is odd} \\[2ex] f(n)\begin{pmatrix} -f(n-2) & -af(n-1) \\ bf(n-1) & f(n) \end{pmatrix}, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof of Claim 3.9. One can check directly that the lemma holds for $n = 1, 2$, and 3.

Assume that $n \geq 3$.

First Case: "n is even". Say $n = 2k$.

Then, by Lemma 3.8,

$$\begin{aligned} A^n = & \; A^k A^k \\ = & \begin{pmatrix} -f(n-2) & -af(n-1) \\ bf(n-1) & f(n) \end{pmatrix}^2 \\ = & \begin{pmatrix} f(n-2)^2 - abf(n-1)^2 & af(n-2)f(n-1) - af(n-1)f(n) \\ -bf(n-2)f(n-1) + bf(n-1)f(n)) & -abf(n-1)^2 + f(n)^2 \end{pmatrix}. \end{aligned}$$

Write $B = A^n + A^{n-1} + \cdots + A + I$. Then, by induction on n,

$$B = A^n + \; f(n-1) \begin{pmatrix} -abf(n-3) & -af(n-2) \\ bf(n-2) & abf(n-1) \end{pmatrix}$$

for, $n-1$ is odd. Equating elements

$$\begin{aligned} B_{1,1} = & \; f(n-2)^2 - abf(n-1)^2 - abf(n-1)f(n-3) \\ = & \; f(n-2)^2 - abf(n-1)(f(n-1) + f(n-3)) \\ = & \; f(n-2)^2 - abf(n-1)f(n-2), \quad \text{for, } n-1 \text{ is odd} \\ = & \; -f(n-2)(abf(n-1) - f(n-2)) \\ = & \; -f(n-2)f(n), \quad \text{for, } n \text{ is even} \end{aligned}$$

$$\begin{aligned} B_{1,2} = & \; af(n-2)f(n-1) - af(n-1)f(n) - af(n-1)f(n-2) \\ = & \; -af(n-1)f(n) \end{aligned}$$

$$\begin{aligned} B_{2,1} = & \; -bf(n-2)f(n-1) + bf(n-1)f(n) + bf(n-2)f(n-1) \\ = & \; bf(n-1)f(n) \end{aligned}$$

$$\begin{aligned} B_{2,2} = & \; -abf(n-1)^2 + f(n)^2 + abf(n-1)^2 \\ = & \; f(n)^2, \end{aligned}$$

That is

$$A^n + A^{n-1} + \cdots + A + I = \; f(n) \begin{pmatrix} -abf(n-2) & -af(n-1) \\ bf(n-1) & abf(n) \end{pmatrix}.$$

Second Case: "n is odd". Say $n = 2k + 1$.

Now, by Lemma 3.8,

$$\begin{aligned}
A^n &= A^{k+1}A^k \\
&= \begin{pmatrix} -f(n-1) & -af(n) \\ bf(n) & f(n+1) \end{pmatrix} \begin{pmatrix} -f(n-3) & -af(n-2) \\ bf(n-2) & f(n-1) \end{pmatrix} \\
&= \begin{pmatrix} f(n-3)f(n-1) - abf(n-2)f(n) & af(n-1)f(n-2) - af(n-1)f(n) \\ -bf(n-3)f(n) + bf(n-2)f(n+1) & -abf(n-2)f(n) + f(n-1)f(n+1) \end{pmatrix}.
\end{aligned}$$

Write $B = A^n + A^{n-1} + \cdots + A + I$. Then, by induction on n,

$$B = A^n + \ f(n-1) \begin{pmatrix} -f(n-3) & -af(n-2) \\ bf(n-2) & f(n-1) \end{pmatrix}$$

for, $n-1$ is even. Equating

$$\begin{aligned}
B_{1,1} = &\ f(n-3)f(n-1) - abf(n-2)f(n) - f(n-3)f(n-1) \\
= &\ -abf(n-2)f(n)
\end{aligned}$$

$$\begin{aligned}
B_{1,2} = &\ af(n-2)f(n-1) - af(n-1)f(n) - af(n-2)f(n-1) \\
= &\ -af(n-1)f(n)
\end{aligned}$$

$$\begin{aligned}
B_{2,1} = &\ -bf(n-3)f(n) + bf(n-2)f(n+1) + bf(n-2)f(n-1) \\
= &\ -bf(n-3)f(n) + bf(n-2)(f(n+1) + f(n-1)) \\
= &\ -bf(n-3)f(n) + bf(n-2)abf(n) \quad \text{for, } n+1 \text{ is even} \\
= &\ bf(n)(abf(n-2) - f(n-3)) \\
= &\ bf(n)f(n-1) \quad \text{for, } n-1 \text{ is even} \\
B_{2,2} = &\ -abf(n-2)f(n) + f(n-1)f(n+1) + f(n-1)^2 \\
= &\ -abf(n-2)f(n) + f(n-1)(f(n+1) + f(n-1)) \\
= &\ -abf(n-2)f(n) + f(n-1)abf(n) \quad \text{for, } n+1 \text{ is even} \\
= &\ abf(n)(f(n-1) - f(n-2)) \\
= &\ abf(n)^2 \quad \text{for, } n \text{ is odd}
\end{aligned}$$

That is

$$A^n + A^{n-1} + \cdots + A + I = \ f(n) \begin{pmatrix} -abf(n-2) & -af(n-1) \\ bf(n-1) & abf(n) \end{pmatrix}.$$

□

Claim 3.10. *If $|ab| \geq 6$, and $n \geq 2$, then $|f(n)| > 1$.*

Proof of Claim 3.10. Consider the 'generating function' $G(z) = \sum_{i=0}^{\infty} f(i)z^i$. The even summands of $G(z)$ are given by $\frac{1}{2}(G(z) + G(-z))$, and the odd summands by $\frac{1}{2}(G(z) - G(-z))$. Then, by the definition of the sequence f,

$$\frac{1}{2}(G(z) + G(-z)) - \frac{1}{2}abz(G(z) - G(-z)) + \frac{1}{2}z^2(G(z) + G(-z)) = 1$$

and
$$\frac{1}{2}(G(z)-G(-z))-\frac{1}{2}z(G(z)+G(-z))+\frac{1}{2}z^2(G(z)-G(-z))=0,$$
that is,
$$(1-abz+z^2)G(z)+(1+abz+z^2)G(-z)=2$$
and
$$(1-z+z^2)G(z)+(-1-z-z^2)G(-z)=0.$$
Thus
$$G(-z)=\frac{1-z+z^2}{1+z+z^2}G(z)$$
and, substituting,
$$G(z)=\frac{1+z+z^2}{1+(2-ab)z^2+z^4}.$$

Write
$$\varphi=\frac{-(2-ab)+\sqrt{(2-ab)^2-4}}{2},\quad \hat{\varphi}=\frac{-(2-ab)-\sqrt{(2-ab)^2-4}}{2},$$
which are the solutions of $1+(2-ab)t+t^2=0$. Notice that $\varphi\hat{\varphi}=1$, and $\varphi+\hat{\varphi}=-(2-ab)$. Then
$$G(z)=\frac{1+z+z^2}{(1-\varphi z^2)(1-\hat{\varphi}z^2)}.$$
Since $|ab|\geq 6$, we have $\varphi\neq\hat{\varphi}$. We can write
$$\alpha=-\frac{1+\varphi+\varphi z}{\hat{\varphi}-\varphi},\quad \beta=\frac{1+\hat{\varphi}+\hat{\varphi}z}{\hat{\varphi}-\varphi},$$
then
$$G(z)=\frac{\alpha}{1-\varphi z^2}+\frac{\beta}{1-\hat{\varphi}z^2}.$$
Now
$$\frac{1}{1-\varphi z^2}=\sum_{i=0}^{\infty}\varphi^i z^{2i},\quad \frac{1}{1-\hat{\varphi}z^2}=\sum_{i=0}^{\infty}\hat{\varphi}^i z^{2i},$$
then
$$G(z)=\sum_{i=0}^{\infty}\frac{\hat{\varphi}^i-\varphi^i+\hat{\varphi}^{i+1}-\varphi^{i+1}}{\hat{\varphi}-\varphi}z^{2i}+\sum_{i=0}^{\infty}\frac{\hat{\varphi}^i-\varphi^i}{\hat{\varphi}-\varphi}z^{2i+1},$$
that is,
$$f(n)=\begin{cases}\dfrac{\hat{\varphi}^k-\varphi^k+\hat{\varphi}^{k+1}-\varphi^{k+1}}{\hat{\varphi}-\varphi} & \text{if } n=2k, n\geq 0\\[2ex] \dfrac{\hat{\varphi}^k-\varphi^k}{\hat{\varphi}-\varphi} & \text{if } n=2k+1, n\geq 1\end{cases}$$

Recall that $(x^m - y^m)/(x-y) = \sum_{i=0}^{m-1} x^{m-1-i}y^i$. We have that, by definition, $\varphi > \hat\varphi$. Also, since $\varphi\hat\varphi = 1$, either both $\varphi, \hat\varphi > 0$ or both $\varphi, \hat\varphi < 0$.

Assume that n is odd, say, $n = 2k+1$. Then if both $\varphi, \hat\varphi > 0$, it follows that $\varphi > 1$, and

$$f(n) = \frac{\hat\varphi^k - \varphi^k}{\hat\varphi - \varphi} = \sum_{i=0}^{k-1} \varphi^{k-1-i}\hat\varphi^i > \varphi^{k-1} > 1, \text{ if } k > 1.$$

If both $\varphi, \hat\varphi < 0$, then $\hat\varphi < -1$, and

$$f(n) = \frac{\hat\varphi^k - \varphi^k}{\hat\varphi - \varphi} = \sum_{i=0}^{k-1} \hat\varphi^{k-1-i}\varphi^i = \hat\varphi^{k-1} + \sum_{i=1}^{k-1} \hat\varphi^{k-1-i}\varphi^i.$$

If k is odd, then $f(n) > \hat\varphi^{k-1} > 1$ when $k > 1$. If k is even, then $f(n) < \hat\varphi^{k-1} < -1$ when $k > 0$, and then $|f(n)| > 1$.

In any case $|f(n)| > 1$ for n odd and $n \geq 5$. We check directly $|f(3)| = |2 - ab| \geq 4$ for, $|ab| \geq 6$.

Now if n is even, say, $n = 2k$, then

$$\begin{aligned} f(n) &= \frac{\hat\varphi^k - \varphi^k}{\hat\varphi - \varphi} + \frac{\hat\varphi^{k+1} - \varphi^{k+1}}{\hat\varphi - \varphi} \\ &= \sum_{i=0}^{k-1} \hat\varphi^{k-1-i}\varphi^i + \sum_{j=0}^{k} \hat\varphi^{k-j}\varphi^j. \end{aligned}$$

Thus, if $\varphi, \hat\varphi > 0$, $f(n) > \varphi^k > 1$ for n even and ≥ 2.

If φ and $\hat\varphi$ both are negative numbers, then $\hat\varphi < -1 < \varphi < 0$. Notice that, $\varphi < 0$ implies $2 - ab > 0$. Since $|ab| \geq 6$, we have $|2 - ab| \geq 4$. Thus $\sqrt{(2-ab)^2 - 4} \geq \sqrt{4^2 - 4} = \sqrt{12}$.

Then

$$\hat\varphi = \frac{-(2-ab) - \sqrt{(2-ab)^2 - 4}}{2} \leq \frac{-4 - \sqrt{12}}{2} = -2 - \sqrt{3} < -3$$

and since $1 = \varphi\hat\varphi > \varphi(-3)$ we have that

$$0 > \varphi > -1/3 \quad \text{and} \quad \hat\varphi < -3.$$

We compute

$$
\begin{aligned}
f(n) &= \sum_{i=0}^{k-1} \hat{\varphi}^{k-1-i}\varphi^i + \sum_{j=0}^{k} \hat{\varphi}^{k-j}\varphi^j \\
&= \varphi\hat{\varphi} \sum_{i=0}^{k-1} \hat{\varphi}^{k-1-i}\varphi^i + \sum_{j=0}^{k} \hat{\varphi}^{k-j}\varphi^j \\
&= \varphi\hat{\varphi}\hat{\varphi}^{k-1} \sum_{i=0}^{k-1} \varphi^{2i} + \hat{\varphi}^k \sum_{j=0}^{k} \varphi^{2j} \\
&= \hat{\varphi}^k \sum_{i=0}^{k-1} \varphi^{2i+1} + \hat{\varphi}^k \sum_{j=0}^{k} \varphi^{2j} \\
&= \hat{\varphi}^k \sum_{i=0}^{2k} \varphi^i \\
&= \hat{\varphi}^k \frac{\varphi^{2k+1}-1}{\varphi-1}.
\end{aligned}
$$

The distance from φ to 1 is $|\varphi - 1| < 1 + 1/3 = 4/3$ for, $0 > \varphi > -1/3$; and the distance $|\varphi^{2k+1} - 1| > 1$ for, $\varphi^{2k+1} < 0$. Thus

$$
|f(n)| = |\hat{\varphi}^k| \frac{|\varphi^{2k+1}-1|}{|\varphi - 1|} > \frac{3}{4}|\hat{\varphi}^k| > \frac{3}{4}3^k > 1
$$

for $k \geq 1$.

Therefore, if $\varphi, \hat{\varphi} < 0$, $|f(n)| > 1$ for each n even and $n \geq 2$.

In any case, $|f(n)| > 1$ for each $n \geq 2$. $\square$

To finish the proof of Lemma 3.7, we have that, by claims 3.9 and 3.10, $A^n - I = f(n)B$ for some matrix B, and $|f(n)| > 1$ if $n \geq 2$. By Lemma 3.2, it follows that $H_1(M_{A^n}) = \mathbb{Z} \oplus Coker(A^n - I)$ has rank three for $n \geq 2$. $\square$

Remark 3.11. Notice that, following the proof of previous lemma, we see that for a torus bundle $M \cong M_{1,b}$, the power coverings of M have genus three if $|b| \geq 6$.

4. Fiber coverings of torus bundles.

Let $T_A \hookrightarrow M_A \to S^1$ be a torus bundle over S^1 with monodromy $A \in SL(2,\mathbb{Z})$. Write $\widetilde{M}_A = T_A \times \mathbb{R}$ for the infinite cyclic covering $u : \widetilde{M}_A \to M_A$ corresponding to the subgroup $\pi_1(T_A) \leq \pi_1(M_A)$. We identify $\pi_1(T_A)$ with $\pi_1(\widetilde{M}_A)$ through the isomorphism induced by the inclusion $T_A \hookrightarrow \widetilde{M}_A$. The group of covering transformations of u is generated by $\varphi_A : \widetilde{M}_A \to \widetilde{M}_A$ given by $\varphi_A(z,s) = (A(z), s+1)$. The induced homomorphism $(\varphi_A)_\# : \pi_1(\widetilde{M}_A) \to \pi_1(\widetilde{M}_A)$ acts as the matrix A, and we abuse notation writing $\varphi_A = (\varphi_A)_\#$. Then we have an action of the ring group $\mathbb{Z}\langle t\rangle$ on $\pi_1(\widetilde{M}_A)$ given by $t \cdot c = \varphi_A(c)$ for each $c \in \pi_1(\widetilde{M}_A)$, where $\langle t\rangle$ is the infinite cyclic group generated by t. The structure of $\mathbb{Z}\langle t\rangle$-module on $\pi_1(\widetilde{M}_A)$ obtained by this action is denoted by H_A. Notice that $\pi_1(M_A) \cong H_A \rtimes_{\varphi_A} \langle t\rangle$.

For a covering of fibers of torus bundles, the notation $\eta : M_B \to M_A$ is reserved, and implies the following statement: If T_A is a fiber of M_A and $T_B = \eta^{-1}(T_A)$, then the diagram

$$\begin{tikzcd} T_B \arrow[r, "B"] \arrow[d, "\eta|"'] & T_B \arrow[d, "\eta|"] \\ T_A \arrow[r, "A"] & T_A \end{tikzcd}$$

commutes.

Theorem 4.1. *Let $L \leq H_A$ be an additive subgroup. Then the following are equivalent.*

1. *L is a $\mathbb{Z}\langle t\rangle$-submodule of H_A of index n.*
2. *There are $B \in SL_2(\mathbb{Z})$ a matrix, and $h_1 : H_B \to L$ a $\mathbb{Z}\langle t\rangle$-isomorphism such that $h_1 \rtimes 1 : H_B \rtimes_{\varphi_B} \langle t\rangle \to H_A \rtimes_{\varphi_A} \langle t\rangle$ is a monomorphism with image of index n.*
3. *There are $B \in SL_2(\mathbb{Z})$ a matrix and a commutative diagram of fiber preserving covering spaces*

$$\begin{tikzcd} \widetilde{M}_B \arrow[r] \arrow[d, "u"'] & \widetilde{M}_A \arrow[d, "v"] \\ M_B \arrow[r] & M_A \end{tikzcd}$$

where the vertical arrows, u and v, are the infinite cyclic coverings of M_B and M_A, respectively, and the horizontal arrows are n-fold coverings which are coverings of fibers.

Proof. "(1) $\Rightarrow$ (2)". Let $L \leq H_A$ be an additive subgroup.

Assume that L is a $\mathbb{Z}\langle t\rangle$-submodule of H_A of index n. Write $H_A = \langle x, y\rangle$; then $L = \langle a_1, a_2\rangle = \langle x^p y^q, x^s y^r\rangle \leq H_A$ with $det\left(\begin{smallmatrix} p & s \\ q & r \end{smallmatrix}\right) = n$. Now write $B = A|L : L \to L \in SL(2, \mathbb{Z})$, which is the matrix A written in terms of the basis $\{a_1, a_2\}$.

If T_A is the fiber of M_A, let $\eta : T_B \to T_A$ be the covering space corresponding to L, regarded as a subgroup of $\pi_1(T_A)$. Let $b_1, b_2 \in \pi_1(T_B)$ be the elements such that $\eta_\#(b_i) = a_i$ $(i = 1, 2)$. Then H_B is generated by b_1 and b_2, regarded as elements of $\pi_1(\widetilde{M_B})$. We define $h_1 : H_B \to L$ as the linear extension of $b_1 \mapsto a_1$ and $b_2 \mapsto a_2$. Then h_1 is an isomorphism, and, since $B = A|L$, it follows that h_1 is a $\mathbb{Z}\langle t\rangle$-morphism.

We have inclusions

$$H_A \xrightarrow{k} H_A \rtimes_{\varphi_A} \langle t\rangle \xleftarrow{\ell} \langle t\rangle,$$

$$H_B \xrightarrow{i} H_B \rtimes_{\varphi_B} \langle t\rangle \xleftarrow{j} \langle t\rangle.$$

Write $f = (k|L) \circ h_1$, and consider the commutative diagram

$$\begin{tikzcd}
& H_B \rtimes_{\varphi_B} \langle t\rangle \arrow[ddd, dashed, "h"] & \\
H_B \arrow[ur, "i"] \arrow[d, bend right, "h_1"'] \arrow[ddr, "f"] & & \langle t\rangle \arrow[ul, "j"'] \arrow[ddl, "\ell"] \\
L \arrow[dr, bend right, "k"'] & & \\
& H_A \rtimes_{\varphi_A} \langle t\rangle &
\end{tikzcd}$$

Now $\ell(t)f(m)\ell(t^{-1}) = f(\varphi_A(t)(m))$. Indeed, we compute $\ell(t)f(m)\ell(t^{-1}) = (1, t)(h_1(m), 1)(1, t^{-1}) = (\varphi_A(t)(h_1(m)), 1)$, and also $f(\varphi_A(t)(m)) = (h_1(\varphi_A(t)(m)), 1)$. Since h_1 is a $\mathbb{Z}\langle t\rangle$-morphism, we obtain $\varphi_A(t)(h_1(m)) = h_1(\varphi_A(t)(m))$.

Then, by the Universal Property of Semi-direct Products, the arrow $h = f \rtimes_{\varphi_B} \ell$ such that $h(m,t) = (h_1(m), t)$ is a homomorphism. If $h(m,t^a) = h(n,t^b)$, then $h_1(m) = h_1(n)$ and $t^a = t^b$; it follows that $m = n$, and $a = b$. Thus h is a monomorphism.

Now the functions

$$\frac{H_A \rtimes_{\varphi_A} \langle t\rangle}{L \rtimes_{\varphi_B} \langle t\rangle} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \frac{H_A}{L}$$

given by

$$\alpha((a,t^k)\cdot L \rtimes_{\varphi_B} \langle t\rangle) = \varphi_A(t^{-k})(a)\cdot L$$

and

$$\beta(a\cdot L) = (a,1)\cdot (L \rtimes_{\varphi_B} \langle t\rangle)$$

satisfy $\alpha\beta = 1$, and $\beta\alpha = 1$.

Therefore the indices $n = [H_A : L] = [H_A \rtimes_{\varphi_A} \langle t\rangle : L \rtimes_{\varphi_B} \langle t\rangle]$.

"(2) $\Rightarrow$ (1)". Now assume that (2) holds. If $h_1(b) = a$, then $(\varphi_A(t)(a), 1) = (1,t)(a,1)(1,t^{-1}) = h(t)h(b)h(t^{-1}) = (h_1 \rtimes 1)((1,t)(b,1)(1,t^{-1})) = (h_1 \rtimes 1)(\varphi_B(t)(b), 1)$. Thus $\varphi_A(t)(a) = h_1(\varphi_B(h_1(b))) = \varphi_B(t)(a)$.

Then $\varphi_A(t)(L) \subset L$ and $\varphi_A(t)(a) = \varphi_B(t)(a)$ for each $a \in L$.

As above, the functions

$$\frac{H_A \rtimes_{\varphi_A} \langle t\rangle}{L \rtimes_{\varphi_B} \langle t\rangle} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \frac{H_A}{L}$$

are bijections, and thus $[H_A : L] = n$.

"(2) $\Rightarrow$ (3)". By (2), notice that $B = A|L$. Then the matrix B solves the 'lifting problem'

$$\begin{tikzcd} H_B \arrow[r, "B"] \arrow[d] & H_B \arrow[d] \\ H_A \arrow[r, "A"] & H_A \end{tikzcd}$$

where the vertical arrows are inclusions. Thus, if $\psi : \tilde{T} \to T$ is the covering space of the torus corresponding to $H_B \leq H_A$, then B also solves the topological lifting problem

$$\begin{tikzcd}
\tilde{T} \arrow[r, "B"] \arrow[d, "\psi"'] & \tilde{T} \arrow[d, "\psi"] \\
T \arrow[r, "A"] & T
\end{tikzcd}$$

One can then define $\eta : M_B \to M_A$ an n-fold covering of fibers.

Thus we obtain a diagram

$$\begin{tikzcd}
\tilde{T} \times \mathbb{R} \arrow[r, "\psi\times 1"] \arrow[d, "u"'] & T \times \mathbb{R} \arrow[d, "v"] \\
M_B \arrow[r, "\eta"] & M_A
\end{tikzcd}$$

where the vertical arrows are infinite cyclic coverings, as required.

"(3) $\Rightarrow$ (1)". Since we have that the diagram

$$\begin{tikzcd}
\widetilde{M_B} \arrow[r, "\psi\times 1"] \arrow[d, "u"'] & \widetilde{M_A} \arrow[d, "v"] \\
M_B \arrow[r] & M_A
\end{tikzcd}$$

commutes, we see that $\psi \times 1$ is compatible with the covering transformations of $\widetilde{M_B}$ and $\widetilde{M_A}$; that is, $(\psi \times 1) \circ \varphi_B = \varphi_A \circ (\psi \times 1)$. It follows that H_B is a $\mathbb{Z}\langle t\rangle$-submodule of H_A. $\square$

Corollary 4.2. *Let M_A be a torus bundle and let $\varphi : \tilde{M} \to M_A$ be an n-fold covering of fibers. If $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, then $\tilde{M} = M_B$, where*

$$B = \begin{pmatrix} \dfrac{det\left(\begin{smallmatrix} p\alpha+q\beta & s \\ p\gamma+q\delta & r \end{smallmatrix}\right)}{n} & \dfrac{det\left(\begin{smallmatrix} s\alpha+r\beta & s \\ s\gamma+r\delta & r \end{smallmatrix}\right)}{n} \\ \\ \dfrac{det\left(\begin{smallmatrix} p & p\alpha+q\beta \\ q & p\gamma+q\delta \end{smallmatrix}\right)}{n} & \dfrac{det\left(\begin{smallmatrix} p & s\alpha+r\beta \\ q & s\gamma+r\delta \end{smallmatrix}\right)}{n} \end{pmatrix},$$

and $\left(\begin{smallmatrix} p & s \\ q & r \end{smallmatrix}\right)$ has determinant n, and corresponds to the subgroup $H_B = \langle x^p y^q, x^s y^r \rangle \leq H_A$ determined by φ.

Proof. Write $a_1 = x^p y^q$, and $a_2 = x^s y^r$. As in the proof of "(1)⇒(2)" in the theorem, B is the matrix A written in terms of the basis a_1, a_2. If $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $Aa_1 = a_1^a a_2^c$, $Aa_2 = a_1^b a_2^d$ translates into the linear systems

$$\begin{array}{rclcrcl} pa + sc & = & p\alpha + q\beta & \quad & pb + sd & = & s\alpha + r\beta \\ qa + rc & = & p\gamma + q\delta & \quad & qb + rd & = & s\gamma + r\delta. \end{array}$$

Solving for a, b, c, d gives B the form of the statement.

□

Remark 4.3. For an integral matrix $\left(\begin{smallmatrix} p & s \\ q & r \end{smallmatrix}\right)$ with determinant $n \geq 2$, we have a function $\widetilde{\ } : GL(2,\mathbb{Z}) \to GL(2,\mathbb{Z})$ such that $A \mapsto \widetilde{A}$, where, if $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, then $\widetilde{A}$ is as the matrix B in Corollary 4.2. Also if $A \in SL(2,\mathbb{Z})$, then $\widetilde{A} \in SL(2,\mathbb{Z})$. This function is multiplicative: $\widetilde{A_1 \cdot A_2} = \widetilde{A_1} \cdot \widetilde{A_2}$, and preserves inverses: $\widetilde{A^{-1}} = \widetilde{A}^{-1}$. That is, it is a group homomorphism. We do not know if this last property make any geometrical sense in terms of the covering spaces involved.

Lemma 4.4. *Let X, Y and Z path-connected topological spaces. If in the following pushout commutative diagram*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\varphi"'] & Z \arrow[d, "\psi"] \\ Y \arrow[r, "h"'] & P \end{tikzcd}$$

the arrow $\varphi : X \to Y$ is a covering space and the arrow $g : X \to Z$ is a homeomorphism, then $\psi : Z \to P$ is a covering space and $h : Y \to P$ is a homeomorphism.

In particular φ and ψ have the same number of sheets.

Proof. We may assume that

$$P = \frac{Y \sqcup Z}{g(x) \sim \varphi(x), \forall x \in X}$$

and that ψ and h are the corresponding inclusions followed by the quotient $\pi : Y \sqcup Z \to P$.

Note that, if $w \in P$, then $w = \{a\} \cup \{g(x) : x \in \varphi^{-1}(a)\}$ for some $a \in Y$. Then h and ψ clearly are surjective, and h is one-to-one.

For $U \subset P$, if $h^{-1}(U)$ is open, since the square above commutes and g is a homeomorphism, it follows that $\psi^{-1}(U)$ also is open; therefore $\pi^{-1}(U) \cap Y$ and $\pi^{-1}(U) \cap Z$ are open in Y and Z, respectively. It follows that $\pi^{-1}(U)$ is open in $Y \sqcup Z$, and, therefore, U is open in P. Thus h is a homeomorphism.

Now for $W \subset P$, if $\psi^{-1}(W)$ is open, since φ is an identification, it follows that $h^{-1}(W)$ is open, and thus W is open in P. That is, ψ is an identification.

If $w \in P$, then there is a fundamental neighborhood $V \subset Y$ of $h^{-1}(p)$ for the covering φ. Then $h(V)$ is a fundamental neighborhood of p for ψ. Thus ψ is a covering space.

□

Proposition 4.5. *Let $\varphi : M_B \to M_A$ be an n-fold covering of fibers of torus bundles, where $B = \widetilde{A}$ as in Remark 4.3.*

For any matrix D such that B is conjugate to D in $GL(2, \mathbb{Z})$, there exists an n-fold covering of fibers $\psi : M_D \to M_C$ with $D = \widetilde{C}$ as in Remark 4.3, and $M_C \cong M_A$.

Proof. Assume that $D = gBg^{-1}$. Write T_B for the fiber of M_B. Then we have a commutative diagram

$$\begin{tikzcd}
T_D \arrow[r, "D"] \arrow[d, "g"'] & T_D \arrow[d, "g"] \\
T_B \arrow[r, "B"'] & T_B
\end{tikzcd}$$

where T_D is a torus ($T_D = g^{-1}(T_B)$). This gives a fiber preserving homeomorphism $g : M_B \to M_D$. Taking the pushout of φ and g, we obtain $h : M_A \to M$ a homeomorphism and $\psi : M_D \to M$ a covering space with $\psi g = h\varphi$ as in Lemma 4.4. Write T_A for the fiber of M_A, $T_C = h(T_A)$, and $C = hAh^{-1} : T_C \to T_C$. Then $M = M_C$. Since $\psi D = \psi g B g^{-1} = h\varphi B g^{-1} = hA\varphi g^{-1} = C\psi$, we have that ψ is a covering of fibers by Theorem 4.1, and $D = \widetilde{C}$, with p, q, s, r given by the subgroup $H_D \leq H_C$.

□

4.1. **Cyclic coverings of the torus.** An n-fold covering space $\eta : X \to Y$ is called *cyclic* if the associated representation $\omega_\eta : \pi_1(Y) \to S_n$ has image $\omega(\pi_1(Y)) \cong \mathbb{Z}_n$, a cyclic group. Write $\varepsilon_n = (1, 2, \dots, n) \in S_n$ for the standard n-cycle.

For a torus T with $\pi_1(T) = \langle a, b : [a, b]\rangle$, a cyclic covering $\eta : \tilde{T} \to T$ with $\tilde{T}$ connected, has associated representation $\zeta_\eta : \pi_1(T) \to S_n$ such that $a \mapsto \varepsilon_n^\sigma$, $b \mapsto \varepsilon_n^\tau$ with, say, $(n, \sigma) = 1$. Now ζ_η is conjugate to $\omega_\rho : \pi_1(T) \to S_n$ such that $a \mapsto \varepsilon_n$, $b \mapsto \varepsilon_n^\rho$ for some integer ρ. The covering space equivalence class of η, has a unique representative η_ρ with associated representation ω_ρ (for uniqueness we assume that, say, $\rho \in \{0, \dots, n-1\}$).

In the one-to-one correspondence between coverings of T and subgroups of $\pi_1(T)$, we have that η_ρ corresponds to the subgroup $(\eta_\rho)_\#(\pi_1(\tilde{T})) = \langle a^n, a^{-\rho} b\rangle$ (see [4], Lemma in p.5).

4.2. **Non-cyclic coverings of the torus.** For positive integers m, n, d, and i_0 such that m divides n, dm divides n, and $(d, i_0) = 1$ with $0 \leq i_0 \leq d-1$, write $\rho = i_0 n/d$. We construct a transitive representation $\omega(m, n, d, \rho)$ of $\pi_1(T) = \langle a, b : [a, b]\rangle$ into S_{mn} with image isomorphic to $\mathbb{Z}_m \oplus \mathbb{Z}_n$.

For $j = 0, \dots, m-1$, define

$$\sigma_{j+1} = (jn + 1, jn + 2, \dots, jn + n). \tag{1}$$

Then $\sigma = \sigma_1 \cdots \sigma_m$ is a product of m cycles of order n.

If $d = 1$, then $i_0 = 0$, and $\rho = 0$. For $j = 1, \dots, n$, define

$$\tau_j = (j, n + j, 2n + j, \dots, (m-1)n + j).$$

Then $\tau = \tau_1 \cdots \tau_n$ is a product of n cycles of order m, and $\omega : \pi_1(T) \to S_{mn}$ given by $a \mapsto \sigma$, and $b \mapsto \tau$, is a representation $\omega(m, n, 1, 0)$ as required.

If $d > 1$, then $i_0 > 0$, and $\rho > 0$. Define

$$\tau_{k,j} = r_k + (j-1)n$$

where r_k is the number $(k-1)\rho + 1$ reduced $\pmod n$, and $1 \leq j \leq m$, and $1 \leq k \leq d$. Then $\tau_1 = (\tau_{1,1}, \tau_{1,2}, \ldots, \tau_{d,m})$ is a cycle of order dm.

Now, for $1 \leq \ell \leq n/d - 1$, define

$$\tau_{\ell+1} = (\tau_{1,1} + \ell, \tau_{1,2} + \ell, \ldots, \tau_{d,m} + \ell).$$

Then $\tau = \tau_1\tau_2 \cdots \tau_{n/d}$ is a product of n/d cycles of order dm, and $\omega : \pi_1(T) \to S_{mn}$ given by $a \mapsto \sigma$, and $b \mapsto \tau$, is a representation $\omega(m, n, d, \rho)$ as required.

For example, for $m = 2, n = 8, d = 4, i_0 = 1$, and $\rho = 2$, the representation $\omega(2, 8, 4, 2) : \pi_1(T) \to S_{16}$ is given by

$$a \mapsto (1, 2, 3, 4, 5, 6, 7, 8)(9, 10, 11, 12, 13, 14, 15, 16)$$
$$b \mapsto (1, 9, 3, 11, 5, 13, 7, 15)(2, 10, 4, 12, 6, 14, 8, 16).$$

Lemma 4.6. *If $\omega : \pi_1(T) \to S_k$ is a transitive representation with image isomorphic to $\mathbb{Z}_m \oplus \mathbb{Z}_n$, and with m a submultiple of n, then there are non-negative integers d and ρ such that ω is conjugate to $\omega(m, n, d, \rho)$.*

Proof. Since ω is transitive, then ω is regular and $k = mn$. Since m divides n, we see that the order of, say, $\omega(a)$ is n, and that m divides the order of $\omega(b)$. We obtain $order(\omega(b)) = dm$ for some d. Also, since $x^n = 1$ for al $x \in Image(\omega)$, then dm divides n.

For $j = 0, \ldots, m-1$ write O_{j+1} for the orbit of $\omega(a)$ that contains $\omega(b)^j(1)$, and γ_{j+1} for the corresponding n-cycle of $\omega(a)$ which acts on O_{j+1} in the disjoint cycle decomposition $\omega(a) = \gamma_1 \cdots \gamma_m$. For $j = 1, \ldots, m$, take $v \in S_{mn}$ such that $v(\omega(b)^j(1)) = 1 + jn$, and $v\gamma_j v^{-1} = \sigma_j$ with σ_j as in Equation 1 above; this is possible for, the orbits of the γ_j's are disjoint. Then, if $\rho+1 = (v \cdot \omega(b) \cdot v^{-1})^{m-1}(1)$, we have that ω is conjugate to $\omega(m, n, d, \rho)$; that is, $v \cdot \omega \cdot v^{-1} = \omega(m, n, d, \rho)$.

□

Remark 4.7. Notice that the n-fold cyclic covering of the torus η_ρ as in Section 4.1, can be regarded as the associated covering of $\omega(1, n, d, \rho)$ with $d = order(\varepsilon_n^\rho) = n/(n, \rho)$.

Lemma 4.8. *Let $\eta : \tilde{T} \to T$ be an mn-fold non-cyclic covering space of the torus, T, with m a submultiple of n and $\tilde{T}$ connected. Let $\omega : \pi_1(T) \to S_{mn}$ be the representation associated to η. Then, as in Lemma 4.6, ω is conjugate to $\omega(m, n, d, \rho)$ for some integers d and ρ.*

If, say, $\omega(a)$ has order n, then for any integer r such that $r \equiv \rho \mod n$, there is a basis $\tilde{a}$, $\tilde{b}$ of $\pi_1(\tilde{T})$ such that $\eta_\#(\tilde{a}) = a^n$, and $\eta_\#(\tilde{b}) = a^{-r}b^m$.

Proof. The proof goes as the proof in [4], Lemma in p. 5. □

Corollary 4.9. *Let $\eta : \tilde{T} \to T$ be an mn-fold covering space with m a submultiple of n such that the image of its associated representation is isomorphic to $\mathbb{Z}_m \oplus \mathbb{Z}_n$ (we allow $m = 1$).*

There is an integer ρ such that m divides ρ, and such that the subgroup of $\pi_1(T)$ corresponding to η is $\eta_\#(\pi_1(\tilde{T})) = \langle a^n, a^{-\rho}b^m \rangle$.

4.3. **Coverings of torus bundles that lower the genus.** Let $\eta : \tilde{T} \to T$ be a finite-sheeted covering space of the torus T with $\tilde{T}$ connected. We say that η extends to a covering of torus bundles if there is a covering of fibers between torus bundles, $\varphi : \tilde{M} \to M$, such that φ restricted to the fiber of $\tilde{M}$ equals η.

Lemma 4.10. *Let $\eta : \tilde{T} \to T$ be a covering space of the torus T with associated representation $\omega(m, n, d, \rho)$, $m \geq 1$. Write $\rho = i_0 n/d$.*

Then η extends to a covering of torus bundles $\tilde{M} \to M_A$ if and only if there are integers p, k, r, and s such that

$$A = \begin{pmatrix} p - i_0\frac{n}{dm}k & \frac{n}{m}r - i_0\frac{n}{dm}s + i_0\frac{n}{dm}(p - i_0\frac{n}{dm}k) \\ k & s + i_0\frac{n}{dm}k \end{pmatrix}$$

Proof. Recall that if $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, then A acts in $\pi_1(T)$ as $Ax = x^\alpha y^\gamma$ and $Ay = x^\beta y^\delta$.

Write $a_1 = x^n$, and $a_2 = x^{-\rho}y^m$; then, by Corollary 4.9, $L = \langle a_1, a_2 \rangle \leq \pi_1(T)$ is the subgroup corresponding to η. By Theorem 4.1, η extends to a covering of torus bundles $\tilde{M} \to M_A$ if and only if $AL \subset L$.

If A has the form of the statement of the lemma, then $Aa_1 = a_1^p a_2^{kn/m}$, $Aa_2 = a_1^r a_2^s \in L$, and we conclude that η extends to a covering of torus bundles $\tilde{M} \to M_A$.

If $AL \subset L$, then $Aa_1 = a_1^p a_2^q$ and $Aa_2 = a_1^r a_2^s$ for some integers p, q, r, s. If we write $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, then last equations are equivalent to

$$n\alpha = np - \rho q, \quad -\rho\alpha + m\beta = nr - \rho s$$

$$n\gamma = qm, \quad -\rho\gamma + m\delta = ms.$$

We see that q is of the form $q = \frac{n}{m}k$, and the lemma follows, that is,

$$\alpha = p - i_0\tfrac{n}{dm}k, \quad \beta = \tfrac{n}{m}r - i_0\tfrac{n}{dm}s + i_0\tfrac{n}{dm}(p - i_0\tfrac{n}{dm}k)$$

$$\gamma = k, \quad \delta = s + i_0\frac{n}{dm}k.$$

□

Remark 4.11. If A is as in the statement of Lemma 4.10, then

$$\begin{pmatrix} 1 & i_0\frac{n}{dm} \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & i_0\frac{n}{dm} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} p & r\frac{n}{m} \\ k & s \end{pmatrix}.$$

That is, it is rather common for a torus bundle M_A to admit a covering of fibers.

Theorem 4.12. *Let $\varphi : \tilde{M} \to M$ an mn-fold covering of torus bundles which is a covering of fibers with m a divisor of n.*

The genus $g(\tilde{M}) < g(M)$ if and only if $m < n$, and $M \cong M_A$, and $\tilde{M} \cong M_B$, where $A = \begin{pmatrix} -1 & -\frac{n}{m} \\ a & a\frac{n}{m} - 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & -1 \\ a\frac{n}{m} & a\frac{n}{m} - 1 \end{pmatrix}$ for some integer $a \neq \pm 1$.

Moreover if A has the form above, then the covering space of the torus associated to the representation $\omega(n, m, d, \rho)$ with $\rho = i_0 n/d = i_0 m\ell$, extends to an mn-fold covering of fibers of M_B onto M_A.

Proof. If matrices A and B have the form of the statement, then $2 = g(M_B) < g(M_A) = 3$. See Section 2.2.

Assume that $g(\tilde{M}) < g(M)$. Then $g(\tilde{M}) = 2$, and $g(M) = 3$. By Proposition 4.5, we may assume that $\tilde{M} = M_B$ with $B = \left(\begin{smallmatrix} -1 & -1 \\ b & b-1 \end{smallmatrix}\right)$ for some integer b, and $M = M_A$ where A is some matrix $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, and $B = \widetilde{A}$.

The submodule of the infinite cyclic covering of M_A corresponding to the covering φ is $H_B = \langle x^n, x^{-\rho}y^m \rangle$ with m a divisor of n, and $\rho = i_0 m\ell$ $(= i_0 n/d)$. See Corollary 4.9 (and Lemma 4.8).

As in Remark 4.3, with $p = n$, $q = 0$, $r = m$, and $s = -i_0 m\ell$,

$$\widetilde{A} = \begin{pmatrix} \alpha + \gamma i_0\ell & \left(-\gamma {i_0}^2\ell^2 - \alpha i_0\ell + \delta i_0\ell + \beta\right)\dfrac{m}{n} \\ \dfrac{\gamma n}{m} & \delta - \gamma i_0\ell \end{pmatrix}$$

Since we are assuming $\widetilde{A} = \left(\begin{smallmatrix} -1 & -1 \\ b & b-1 \end{smallmatrix}\right)$, we see that

$$\begin{array}{ll} \alpha = -\dfrac{bi_0\ell m + n}{n}, & \beta = -\dfrac{b{i_0}^2\ell^2 m^2 + bi_0\ell mn + n^2}{mn} \\ \gamma = \dfrac{bm}{n}, & \delta = \dfrac{bi_0\ell m + bn - n}{n} \end{array}$$

Since γ is an integer, it follows that n divides bm, say, $b = an/m$. Then

$$\begin{array}{ll} \alpha = -ai_0\ell - 1, & \beta = -a{i_0}^2\ell^2 - ai_0\ell\frac{n}{m} - \frac{n}{m} \\ \gamma = a, & \delta = ai_0\ell + a\frac{n}{m} - 1 \end{array}$$

and

$$A = \left(\begin{array}{cc} -ai_0\ell - 1 & -a{i_0}^2\ell^2 - a\frac{n}{m}i_0\ell - \frac{n}{m} \\ a & ai_0\ell + a\frac{n}{m} - 1 \end{array} \right)$$

Notice that

$$\left(\begin{array}{cc} 1 & i_0\ell \\ 0 & 1 \end{array} \right) A \left(\begin{array}{cc} 1 & i_0\ell \\ 0 & 1 \end{array} \right)^{-1} = \left(\begin{array}{cc} -1 & -\frac{n}{m} \\ a & a\frac{n}{m} - 1 \end{array} \right)$$

Since in the conjugacy class of A in $GL(2,\mathbb{Z})$, except for interchange of a and n/m, there is no other matrix in the form of Section 2.2, we see that $a \neq \pm 1$ and $m < n$, for we are assuming $g(M_A) = 3$.

By Lemma 4.10, the covering of the torus associated to the representation $\omega(m, n, d, i_0 n/d)$ with $n/d = m\ell$, extends to a covering of fibers $\tilde{M} \to M_A$ if and only if there are integers p, k, r, s such that

$$A = \left(\begin{array}{cc} p - i_0\ell k & \frac{n}{m}r - i_0\ell s + i_0\ell(p - i_0\ell k) \\ k & s + i_0\ell k \end{array} \right).$$

Defining $k = a$, $p = -1$, $s = a\frac{n}{m} - 1$, and $r = -1$, we obtain the required equality. And the theorem follows.

$\square$

Remark 4.13. A representation $\omega : \pi_1(T) \to S_{n^2}$ with image $\mathbb{Z}_n \oplus \mathbb{Z}_n$ is conjugate to $\omega(n, n, 1, 0)$. The subgroup of the corresponding covering space is $L = \langle x^n, y^n \rangle$. If $A \in SL(2,\mathbb{Z})$, then $AL \subset L$, and $\widetilde{A} = A$. Then the extension to a n^2-fold covering of fibers is of the form $M_A \to M_A$, and there is no genus lowering.

Remark 4.14. Theorem 4.12 implies that, if $A = \left(\begin{smallmatrix} -1 & -a \\ b & ab-1 \end{smallmatrix}\right)$ with $a > 0$ and $|a|, |b| \neq 1$, then for each positive integer m, the torus bundle M_A admits an (am)-fold covering space that lowers the genus.

5. Seifert manifolds

Let M be the orientable Seifert manifold with orientable orbit surface of genus g and Seifert symbol $(Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$, where α_i, β_i are integers with $\alpha_i \geq 1$ and $(\alpha_i, \beta_i) = 1$ for $i = 1, \dots, t$.

Then the fundamental group $\pi_1(M) = \langle a_1, b_1, \dots, a_g, b_g, q_1, \dots, q_t, h : q_1^{\alpha_1} h^{\beta_1} = 1, \dots, q_t^{\alpha_t} h^{\beta_t} = 1, q_1 \cdots q_t = [a_1, b_1] \cdots [a_g, b_g], [h, q_i] = [h, a_j] = [h, b_j] = 1 \rangle$ where $a_1, b_1, \dots, a_g, b_g$ represent a basis for the fundamental group of the orbit surface of M. By Lemma 1 of [4] one obtains

Lemma 5.1. *Let $M = (Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$ be a Seifert manifold. Let $r_1, \dots, r_t$ be integers such that $\alpha_i r_i + \beta_i \equiv 0 \mod n$ for $i = 1, \dots, t$, and assume that $r_1 + \cdots + r_t = 0$. Then there is an n-fold cyclic covering space*

$$(Oo, g; B_1/\alpha_1, \dots, B_t/\alpha_t) \to M$$

where the integer $B_i = (\alpha_i r_i + \beta_i)/n$ for $i = 1, \dots, t$.

Lemma 5.2. *Let M be the Seifert manifold with symbol $(Oo, g; \beta/\alpha)$ and $g \geq 0$. Then the Heegaard genus of M is*

$$h(M) = \begin{cases} 2g & \text{if } \beta = \pm 1 \\ 2g + 1 & \text{otherwise.} \end{cases}$$

Proof. One can construct a Heegaard decomposition for M of genus $2g$ if $\beta = \pm 1$, and a Heegaard decomposition for M of genus $2g + 1$ if $\beta \neq \pm 1$ (see [1]). Therefore $h(M) \leq \begin{cases} 2g & \text{if } \beta = \pm 1 \\ 2g + 1 & \text{otherwise.} \end{cases}$

Recall that $rank(H_1(M)) \leq h(M)$.

Since $H_1(M) = \langle a_1, b_1, \dots, a_g, b_g, q, h : q^\alpha h^\beta = 1, q = 1 \rangle_{Ab}$, then $H_1(M) \cong \begin{cases} \mathbb{Z}^{2g} & \text{if } \beta = \pm 1 \\ \mathbb{Z}^{2g} \oplus \mathbb{Z}_{|\beta|} & \text{otherwise} \end{cases}$ where the subindex 'Ab' indicates the image of the Abelianization homomorphism. In particular $h(M) \geq \begin{cases} 2g & \text{if } \beta = \pm 1 \\ 2g + 1 & \text{otherwise.} \end{cases}$

□

Corollary 5.3. *For any integers $g \geq 0$, $\alpha \geq 1$, and $|\beta| \geq 2$ with α and β coprime, there is a $|\beta|$-fold covering space*

$$(Oo, g; \pm 1/\alpha) \to (Oo, g; \beta/\alpha).$$

And the genus $g(Oo, g; \beta/\alpha) = g(Oo, g; \pm 1/\alpha) + 1$.

Proof. If we set $r_1 = 0$, then, using Lemma 5.1, we obtain $B_1 = \beta/|\beta| = \pm 1$, and a $|\beta|$-fold covering space $(Oo, g; \pm 1/\alpha) \to (Oo, g; \beta/\alpha)$. □

Remark 5.4. In the case M is an orientable Seifert manifold with non-orientable orbit surface, the following also holds.

Theorem 5.5 ([5]). *Let α, β be a pair of coprime integers with $\alpha \geq 1$ and $|\beta| \geq 2$; let $g < 0$, and let M be the Seifert manifold with symbol $(Oo, g; \beta/\alpha)$.*

If $g < -1$, then $\pi_1(M)$ is of infinite order and M has a finite covering space $\tilde{M} = (Oo, g; \pm 1/\alpha) \to M$ such that the Heegaard genus $h(\tilde{M}) = h(M) + 1$. Also $rank(\pi_1(\tilde{M})) = rank(\pi_1(M)) + 1$.

Also it follows from [5], that the manifolds of Corollary 5.3 and Theorem 5.5 are the only examples of (branched or unbranched) coverings of (orientable or not) Seifert manifolds that lower the Heegaard genus, in case the orbit surface is not the 2-sphere.

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