A NOTE ON 2-UNIVERSAL LINKS

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Abstract. We show that no Montesinos knot (link) can be 2-universal.

1. Introduction

The main theorem of this note, that no Montesinos knot can be 2-universal (Corollary 3.1), contrasts with the existence of 2-universal knots as shown in [3]. These two combined results are somewhat surprising, for most known universal knots (links) are Montesinos'.

Our main result follows easily from a result about factorization of branched coverings through cyclic coverings (Lemma 2.2), which is interesting in its own and very useful.

Also we obtain a result on non simply connectednes of 'regular-like' branched coverings (Corollary 3.2), as another application of Lemma 2.2.

2. Branched coverings through cyclic coverings

An *m-fold branched covering* $\varphi : M^3 \to N^3$ is a proper open map between 3-manifolds such that there is a 1-subcomplex $k \subset N$ (the *branching* of φ) with $\varphi| : M - \varphi^{-1}(k) \to N - k$ a finite m-fold covering space. For the purposes of this paper, $k \subset N$ will be a properly embedded submanifold; that is, k is a link in N. We say that 'φ is *branched along* k', and write $\varphi : M \to (N, k)$.

Given a component $\tilde{k} \subset \varphi^{-1}(k) \subset M$, the homological local degree $deg(\varphi, x)$ is the same for all $x \in \tilde{k}$; this common number is called the *ramification index* of $\tilde{k}$.

A *meridian of a component* $k_1 \subset k \subset N$ is a class $\mu \in \pi_1(N - k)$ that can be represented as $\mu = [a * m * \bar{a}]$ where m is the boundary of a disk D such that $D \cap k = Int(D) \cap k_1 =$ one point, and a is an arc in $N - k$ connecting the base point with a point of m. Notice that meridians of the same component are conjugate. A *meridian of* k is a meridian of a component of k.

An m-fold branched covering $\varphi : M \to (N, k)$ determines (and is determined) by a representation $\omega_\varphi : \pi_1(N - k) \to S_m$ into the symmetric group on m symbols S_m. If $\omega_\varphi(\mu)$ is a product of disjoint cycles of order $c_1, c_2, \dots$ for μ a meridian of a component k_1 of k, then the components of

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the preimage $\varphi^{-1}(k_1)$ have ramification indices $c_1, c_2, \ldots$. We say that φ is a *branched covering of index dividing* n, if $\omega_\varphi(\mu)^n$ is the identity permutation for all meridians μ of k.

Let $k \subset S^3$ be a link; let us denote by $BC(n;k)$ the set of closed, connected, orientable 3-manifolds M such that there exists a branched covering $\varphi : M \to (S^3, k)$ of index dividing n. The link k is called n*-universal* if $BC(n;k)$ coincides with the set of all closed, connected, orientable 3-manifolds. It is common to call *universal link* a 0-universal link.

We let $p : B_n(k) \to (S^3, k)$ be the n-fold cyclic covering branched along all components of k; that is, the induced representation ω_p sends each meridian of k to an n-cycle in $Z_n \leq S_n$. The following lemma helps to organize the details in the proof of Lemma 2.2, and is proved for knots in [4], § 4 of Chap. 2. The proof goes essentially the same for links, and we include it here for completeness.

Lemma 2.1 ([4]). *Let* $k \subset S^3$ *be a link, and write* $\langle \mu^n \rangle_\pi$ *for the normal closure of* $\{\mu^n : \mu$ *is a meridian of* $k\}$ *in* $\pi_1(S^3 - k)$*. Then* $\pi_1(S^3-k)/\langle \mu^n \rangle_\pi$ *is a semi-direct product*

$$\frac{\pi_1(S^3-k)}{\langle \mu^n \rangle_\pi} \cong Z_n \ltimes \pi_1(B_n(k))$$

where the generator of Z_n *is the class of any meridian of* k *and acts on* $\pi_1(B_n(k))$ *as the isomorphism induced by an order* n *symmetry of* $B_n(k)$ *with quotient* (S^3, k)*.*

Proof. Let $k \subset S^3$ be a link of c components and let $H \leq \pi_1(S^3 - k)$ be the kernel of the composition $\pi_1(S^3-k) \xrightarrow{Ab} H_1(S^3-k) \cong Z^c \xrightarrow{\varepsilon} Z \xrightarrow{\rho} Z_n$, where Ab is the Abelianization map, ε is the augmentation $\varepsilon(x_i)_i = \sum_i x_i$, and ρ is reduction $\pmod n$. Notice that $H_1(S^3-k) \cong Z^c$ has a basis of meridians $\mu_1, \ldots, \mu_c$, one for each component of k. We have that $H \cong p_\# \pi_1(B_n(k) - p^{-1}(k))$ where $p : B_n(k) \to (S^3, k)$ is the n-fold cyclic covering branched along all components of k. If μ is a meridian of k, then $p^{-1}(\mu)$ is a closed curve which represents, up to conjugation, the element $\mu^n \in H$, and we obtain the fundamental group of $B_n(k)$ adding the 'branching relations', $\pi_1(B_n(k)) \cong H/\langle \mu^n \rangle_H$, where $\langle \mu^n \rangle_H$ is the normal closure in H of $\{\mu^n : \mu$ is a meridian of $k\}$. Notice that $\langle \mu^n \rangle_H = \langle \mu^n \rangle_\pi$, for $\nu^{-1}\mu^n\nu = (\nu^{-1}\mu\nu)^n$ is the n-th power of a meridian, for each μ, ν meridians of k. Therefore the sequence

$$1 \to \frac{H}{\langle \mu^n \rangle_H} \to \frac{\pi_1(S^3-k)}{\langle \mu^n \rangle_\pi} \xrightarrow{\xi} \frac{\pi_1(S^3-k)}{H} \cong Z_n \to 1$$

is exact. The map ξ has a section $\pi_1(S^3-k)/H \to \pi_1(S^3-k)/\langle \mu^n \rangle_\pi$, and therefore $\pi_1(S^3-k)/\langle \mu^n \rangle_\pi \cong Z_n \ltimes \pi_1(B_n(k))$ where the generator $\bar{\mu}$ of Z_n acts on $\pi_1(B_n(k))$ as the isomorphism induced by an order n homeomorphism of $B_n(k)$ with quotient (S^3, k). $\square$

Lemma 2.2. *Let $k \subset S^3$ be a link, and let $\varphi : M \to (S^3, k)$ be an m-fold branched covering of index dividing n. Then there exists a commutative square of branched coverings*

$$\begin{tikzcd} & \tilde{M} \arrow[dl, "q"'] \arrow[dr, "\psi"] & \\ M \arrow[dr, "\varphi"'] & & B_n(k) \arrow[dl, "p"] \\ & (S^3, k) & \end{tikzcd}$$

where p is the n-fold cyclic covering of (S^3, k) branched along all components of k, ψ is an m-fold (unbranched) covering space, and q is an n-fold covering branched along the components of $\varphi^{-1}(k) \subset M$ with ramification index less than n.

Proof. Let $\omega : \pi_1(S^3 - k) \to S_m$ be the representation determined by the covering $\varphi : M \to (S^3, k)$. The covering subgroup of φ is $U = \omega^{-1}(St(1)) \cong \varphi_{\#}\pi_1(M - \varphi^{-1}(k))$ where $St(1) \leq S_m$ is the subgroup of permutations fixing the symbol 1. Since $\omega(\mu)^n$ is the identity permutation for each meridian μ of k, the representation ω factors

$$\begin{tikzcd} \pi_1(S^3 - k) \arrow[rr, "\omega"] \arrow[dr] & & S_m \\ & \pi_1(S^3 - k)/\langle \mu^n \rangle_\pi \arrow[ur, "\bar{\omega}"'] & \end{tikzcd}$$

From the previous lemma we know $\pi_1(S^3 - k)/\langle \mu^n \rangle_\pi \cong Z_n \ltimes \pi_1(B_n(k))$, and, by restriction, we get $\tau = \bar{\omega}| : \pi_1(B_n(k)) \to S_m$ a representation which perhaps is not transitive. This τ induces an m-fold (unbranched) covering space $\psi : \tilde{M} \to B_n(k)$ such that $\tilde{M}$ is connected if and only if τ is transitive. The covering subgroup of ψ is $\bar{U} = \tau^{-1}(St(1)) = \pi_1(B_n(k)) \cap \bar{\omega}^{-1}(St(1)) = (H \cap U)/\langle \mu^n \rangle_H \cong \psi_{\#}\pi_1(\tilde{M})$, if $\tilde{M}$ is connected. As in the proof of the previous lemma, $H \cong \pi_1(B_n(k) - p^{-1}(k))$. We then see that $U \cap H \cong p_{\#}\psi_{\#}\pi_1(\tilde{M} - \psi^{-1}(p^{-1}(k)))$. Therefore $\tilde{M}$ is the pullback of φ and p as in [2], and the lemma follows. If $\tilde{M}$ is not connected, we perform the same analysis on subgroups for each component K of $\tilde{M}$; that is, we analyze $\psi| : K \to (S^3, k)$ for each component K and obtain that $\tilde{M}$ is again a pullback, and the lemma follows.

□

Remark. The previous lemma and its proof show that getting an m-fold covering $\varphi : M \to (S^3, k)$ of index dividing n is the same as finding a special representation $\pi_1(B_n(k)) \to S_m$. This point of view is exploited in [6] to construct 'dihedral-like' coverings of Montesinos knots. We thank the referee for pointing out that the construction of Lemma 2.2 is a standard pullback.

3. Branched coverings of fixed index

Let $k \subset S^3$ be a Montesinos link. Then $B_2(k)$ is an orientable Seifert manifold with orbit surface the 2-sphere, $(O, 0; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$, or an orientable Seifert manifold with orbit surface a non-orientable surface of (non-orientable) genus g, $(O, -g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$. See [5].

Corollary 3.1. *If k is a Montesinos link, then k is not 2-universal.*

Proof. If $\varphi : M \to (S^3, k)$ is an m-fold branched covering of index dividing 2, then from Lemma 2.2 we obtain $\psi : \tilde{M} \to B_2(k)$ an m-fold (unbranched) covering space, and $q : \tilde{M} \to M$ a 2-fold branched covering. Since $B_2(k)$ is a Seifert manifold, we see that $\tilde{M}$ is also a Seifert manifold. Since q is 2-fold, q is a regular covering; therefore there exists an involution of $\tilde{M}$ with quotient M. We conclude that M is a Seifert orbifold ([1]), and that $BC(2; k)$ is not the set of all closed, connected, orientable 3-manifolds. Therefore k is not 2-universal. □

Remark. In particular, from the previous corollary, we see: A hyperbolic 2-bridge knot, which is known to be 12-universal, cannot be 2-universal; the Borromean rings, known to be 4-universal, are not 2-universal.

Corollary 3.2. *Let $k \subset S^3$ be a link such that order of $\pi_1(B_n(k))$, perhaps infinite, does not divide m. Let $\varphi : M \to (S^3, k)$ be an m-fold branched covering with induced representation $\omega : \pi_1(S^3 - k) \to S_m$ such that $\omega(\mu)$ is a product of disjoint n-cycles for each meridian μ of k. Then M is not simply connected.*

Proof. From Lemma 2.2 we obtain $\psi : \tilde{M} \to B_n(k)$ an m-fold (unbranched) covering space, and $q : \tilde{M} \to M$ an n-fold covering. By hypotesis there are no components of $\varphi^{-1}(k) \subset M$ with ramification index less than n; therefore q is a covering (unbranched) space, and $q_\# : \pi_1(K) \to \pi_1(M)$ is an embedding for each component K of $\tilde{M}$. If $\pi_1(B_n(k))$ is infinite, each component of $\tilde{M}$ has infinite fundamental group and the corollary follows. If $\pi_1(B_n(k))$ is finite, since its order does not divide m, at least one component of $\tilde{M}$ is not simply connected, for the index of $\pi_1(K)$ in $\pi_1(B_n(k))$ is a divisor of m for each component K of $\tilde{M}$; the corollary follows. □

References

[1] F. Bonahon and L. Siebenmann. The classification of Seifert fibred 3-orbifolds. Low-dimensional topology (Chelwood Gate, 1982), 19–85, London Math. Soc. Lecture Note Ser., 95, Cambridge Univ. Press, Cambridge, 1985.

[2] J. Hempel. Branched coves over strongly amphicheiral links. Topology 29 (1990), 247–255.

[3] H. M. Hilden, M. T. Lozano and J. M. Montesinos. On 2-universal knots. Bol. Soc. Mat. Mexicana (3) 10 (2004).

[4] F. González-Acuña. On homology spheres. Thesis. Princeton 1969.

[5] J. M. Montesinos. Variedades de Seifert que son recubridores cíclicos ramificados de dos hojas. Bol. Soc. Mat. Mexicana (2) 18 (1973), 1–32.

[6] V. Núñez and J. Rodríguez–Viorato. Dihedral coverings of Montesinos knots. Bol. Soc. Mat. Mexicana (3) 10, (2004).

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