Fractional Brownian motion with small Hurst

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- B^H is H self-similar.
- Controls roughness.

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Mildly **uninteresting** answer:

$$\mathbb{E}[B_s^0 B_t^0] = 1/2 + \mathbb{1}_{\{s=t\}}/2.$$

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Definition
We say that $F^H = \{F_t^H\}_{t \geq 0}$ is a Neuman-Rosenbaum process if

$$F_t^H \overset{Law}{=} \frac{1}{\sqrt{H}} \left(B_t^H - \frac{1}{t} \int_0^t B_u^H du \right).$$

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Observation
It holds that $Var[F_t^H] \approx 1/H$.

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Definition
A collection of processes X^y, $y > 0$ converges in law as y tends to zero, to a random element G in $\mathcal{S}_0'$ (Gaussian field) in the $\mathcal{S}_0'$ sense, if

$$X_\psi \xrightarrow{Law} \langle G, \psi \rangle,$$

for all $\psi \in \mathcal{S}$, where $\langle \cdot, \cdot \rangle$ denotes dual pairing.

Small Hurst Neuman-Rosenbaum process

The following result serves as the foundation for our results

Theorem (Neuman and Rosenbaum (2018))
The Neuman-Rosenbaum process F^H converges weakly in $\mathcal{S}_0'(\mathbb{R})$ as H tends to zero, towards a centered Gaussian field G satisfying

$$\mathbb{E}[\langle G, \psi_1\rangle\langle G, \psi_2\rangle] = \int_{\mathbb{R}^2} g(t,s)\psi_1(t)\psi_2(s)dsdt,$$

where

$$g(t,s) = \frac{1}{ts}\int_0^t\int_0^s \log\left(\frac{|s-u||t-u|}{|u-v||t-s|}\right)dudv.$$

Some natural questions

Some natural question arising from this convergence:

- Can we get an exactly log-correlated limit in the domain $s, t \in \mathbb{R}^d$, with $g(t,s) = \log(1/|t-s|)$ (Hager and Neuman, 2020).

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We will focus in a particular type of functionals of F^H.

Additive functionals

Our central objects of interest are the following

- Local times at zero, $L_t^H(0)$ of F^H, defined as

$$L_t^H(0) := \int_0^t \delta_0(F_s^H)ds,$$

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  where δ_0 is the Dirac-delta function.

- Additive functionals of F^H, defined as

$$\mathcal{A}_t^H[f] := \int_0^t f(F_s^H) ds,$$

  where f is a tempered distribution.

Some language of fractional calculus

Fractional Brownian motion can be formulated in the framework of fractional calculus.

Definition
Let $y \in (0,1]$ be given. The left-sided y-fractional Riemann-Liouville integral/derivative of order y are defined as

$$I_{-}^{y}[f](t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{y-1} f(s) ds,$$

and

$$D_{-}^{y}[f](t) := I_{-}^{-y}[f](t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{-\infty}^{t} (t-s)^{-y} f(s) ds.$$

Representing the processes of interest

Let W be a standard Brownian motion defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathcal{F} = \sigma(W)$.

Theorem (Mandelbrot Van-Ness (1968))
For some $c_H \approx \sqrt{H}$ as $H \approx 0$,

$$\int_{\mathbb{R}} 1_{[0,t]}(s) dB_s^H := B_t := c_H \int_{\mathbb{R}} I_-^{H-1/2}[1_{[0,t]}](s) dW_s,$$

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It follows that when $H < 1/2$,

$$F_t^H := \int_{\mathbb{R}} D_-^{1/2-H}[\psi_t](s) W(ds),$$

with $\psi_t(y) := \frac{y}{t} 1_{[0,t]}(y)$, is a Neuman-Rosenbaum process.

Representing functionals of processes

To describe functionals, define $\mathfrak{H} := L^2(\mathbb{R})$, and define $I_q : \mathfrak{H}^{\otimes q} \to L^2(\Omega)$ by

$$I_q[f_q] := \int_{\mathbb{R}} \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_{q-1}} f(t_1, \ldots, t_q) W(dt_q) \cdots W(dt_1).$$

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Theorem (Chaos decomposition, Itô, 1951)
If $F \in L^2(\Omega)$, then there exist unique $f_q \in \mathfrak{H}^{\otimes q}$, such that

$$F = \mathbb{E}[F] + \sum_{q=1}^{\infty} I_q(f_q).$$

The term $J_q[F] := I_q(f_q)$ is called q-th chaos of F.

Chaos of additive functionals

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Lemma (Jaramillo, Nourdin, Peccati (2025+))
We have the chaos decomposition

$$\mathcal{A}_t^H[f] = \sum_{q=0}^{\infty} \frac{1}{q!} I_q \left[\int_0^t (-1)^q \langle f, \partial^q \phi_{\sigma^2_{s,H}} \rangle D_-^{1/2-H}[\psi_s]^{\otimes q} ds \right],$$

where $\sigma^2_{s,H}$ is the variance of F_s^H and ϕ_γ is Gaussian kernel of variance γ.

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Observe that if $f = \delta_0$, odd chaoses vanish.

Naive strategy

The previous slide suggests the "naive approach"

- We can find the chaos of $L_t^H(0)$.
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Naive works! Recall $\psi_t(y) := \frac{y}{t} 1_{[0,t]}(y)$.

Corollary (Jaramillo, Nourdin, Peccati (2025+))
The local time at zero for F^H satisfies

$$H^{-3/2}\left(L_t^H(0) - \mathbb{E}[L_t^H(0)]\right) \overset{L^2(\Omega)}{\to} -\frac{1}{2\sqrt{2\pi}} \int_0^t I_2\left[D_-^{1/2}[\psi_s]^{\otimes 2}\right] ds.$$

Second main result

Suppose that $\hat{f}$ is such that

$$\lim_{r \to 0} \hat{f}(xr)/\hat{f}(r) = x^{\alpha},$$

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Theorem (Jaramillo, Nourdin, Peccati (2025+))
Under mild conditions on f, we can find $c_f \in \mathbb{R}$, such that

$$H^{-(q+1+\alpha)/2} J_q[\mathcal{A}_t^H[f]] \stackrel{L^2(\Omega)}{\to} c_f I_q[\int_0^t D^{1/2}[\psi_s]^{\otimes Q} ds],$$

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In addition,

$$H^{-(q+d+\alpha)/2} \left(\mathcal{A}_t^H[f] - \sum_{q=0}^{Q-1} J_q[\mathcal{A}_t^H[f]] \right) \overset{L^2(\Omega)}{\to} c_f I_q[\int_0^t D^{1/2}[\psi_s]^{\otimes Q} ds].$$

Reasoning behind the result

Look at

$$\mathcal{A}_t^H[f] = \sum_{q=0}^{\infty} \frac{1}{q!} I_q \left[\int_0^t (-1)^q \langle f, \partial^q \phi_{\sigma^2_{s,H}} \rangle D_-^{1/2-H} [\psi_s]^{\otimes q} ds \right],$$

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(i) Use Parseval to prove that $\langle f, \partial^q \phi_r \rangle \approx r^{q+1+\alpha}$

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The bottleneck is part (*ii*). Main tools: fractional calculus.

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