CIMAT

Centro de Investigación en Matemáticas, A.C.

# Free Berry-Esseen theorem via Stein's method

Arturo Jaramillo Gil (joint work with Mario Diaz)

Centro de Investigación en Matemáticas (CIMAT)

# Objective

Let $\{\mu_{k,n}\ ;\ k,n \geq 1\}$ be a sequence of centered probability measures, and define

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**Goal**

Bound $d_{TV}(\nu_n, \mathbf{s})$ in a probabilistic way.

# Elements of free probability

Let $\mathcal{A}$ be a unital $C^*$-algebra and $\tau : \mathcal{A} \to \mathbb{C}$ a positive unital linear functional. We then say that the pair $(\mathcal{A}, \tau)$ is a *$C^*$-probability space*.

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**Definition (Freeness)**
Let $\{A_n\}_{n\geq 1}$ be a sequence of subalgebras of $\mathcal{A}$. For $a \in \mathcal{A}$, denote the centering of $a$ by $\bar{a} := a - \tau[a]$. We say that $\{A_n\}_{n\geq 1}$ are freely independent, or free, if

$$\tau[\bar{a}_1 \bar{a}_2 \cdots \bar{a}_k] = 0, \tag{1}$$

for $a_1, \ldots, a_k$ alternating algebras.

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**Definition (Free convolution)**
Sums of free random variables yields the free convolution $\boxplus$.

# Recipie for cooking up a free Stein identity

Our building block is

**Definition**
Let $\{P_\theta^*\}_{\theta \geq 0}$ be operators **over measures**, defined by

$$P_\theta^*[\mu] := Law(e^{-\theta} X + \sqrt{1 - e^{-2\theta}} Y),$$

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Let $\langle \cdot, \cdot \rangle$ denote dual pairing of measures and functions. The operator $P^*_\theta$ 'interpolates' from $\mu$ standardized to $\mathbf{s}$, as

$$\langle \mathbf{s}, h \rangle - \langle \mu, h \rangle = \langle P^*_\infty[\mu], h \rangle - \langle P^*_0[\mu], h \rangle$$

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Deriving and integrating, we get

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**Lemma**

$$\frac{d}{d\theta}\langle P_\theta^*[\mu], h\rangle = \langle P_\theta^*[\mu] \otimes P_\theta^*[\mu], \mathcal{L}_\boxplus[h]\rangle,$$

*where $\mathcal{L}_\boxplus[h]$ is the real function in $\mathbb{R}^2$:*

$$\mathcal{L}_\boxplus[h](x, y) := xDh(x) - \partial Dh,$$

*where $D$ denotes derivative and $\partial$ the non-commutative derivative*

$$\partial g(x, y) := (g(x) - g(y))/(x - y).$$

# What have we achieved in the free case?

For $h$ regular enough,

$$\langle \boldsymbol{s}, h\rangle - \langle \mu, h\rangle = \int_0^\infty \langle P_\theta^*[\mu] \otimes P_\theta^*[\mu], \mathcal{L}_{\boxplus}[h]\rangle d\theta.$$

**Lemma (Non-commutative Stein's lemma)**
*A law $\nu$ is semicircular if and only if*

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*for $h$ smooth with second bounded derivative.*

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As a consequence,

$$\langle \boldsymbol{s}, h\rangle - \langle \mu, h\rangle = \int_0^\infty \langle P_\theta^*[\mu] \otimes P_\theta^*[\mu] - P_\infty^*[\mu] \otimes P_\infty^*[\mu], \mathcal{L}_\boxplus[h]\rangle d\theta.$$

# What have we achieved in the free case? Pt II

Writing what we have differently,

$$|\langle \mathbf{s}, h\rangle - \langle \mu, h\rangle| = |\langle \mathcal{S}^*_{\boxplus}[\mu], \mathcal{L}_{\boxplus}[h]\rangle|,$$

where

$$\mathcal{S}^*_{\boxplus}[\mu] := \int_0^\infty (P^*_\theta[\mu] \otimes P^*_\theta[\mu] - P^*_\infty[\mu] \otimes P^*_\infty[\mu]) d\theta.$$

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Compare this with the classical case, stating

$$|\langle \gamma, h\rangle - \langle \mu, h\rangle| = |\langle \mu, \mathcal{L}[\mathcal{S}[h]]\rangle|,$$

**New bottleneck**

Bound uniformly $|\langle \rho, \mathcal{L}_{\boxplus}[g]\rangle|$, with $\rho = \mathcal{S}^*_{\boxplus}[\mu]$.

# Dealing with the bottleneck when $\mu = \nu_n$

Consider

$$\nu_n := \mu_{1,n} \boxplus \cdots \boxplus \mu_{n,n}.$$

Assume $Var[\nu_n] = 1$ and that $\mu_{k,n}$ have **small supports**.

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Consider

$$\nu_n := \mu_{1,n} \boxplus \cdots \boxplus \mu_{n,n}.$$

Assume $Var[\nu_n] = 1$ and that $\mu_{k,n}$ have **small supports**. If $\xi_{1,n}, \ldots, \xi_{n,n}$ are free with law equal to $\int_{\mathbb{R}_+} (P_\theta^* - P_\infty^*)[\mu_{k,n}] d\theta$, then

$$\langle \mathcal{S}_\boxplus^*[\nu_n], \mathcal{L}_\boxplus[h] \rangle = \tau[S_n Dh(S_n)] - \langle \mathcal{S}_\boxplus^*[\nu_n], \partial Dh \rangle,$$

with $S_n := \xi_{1,1} + \cdots + \xi_{n,n}$.

# An important technicallity

**Corollary (Superconvergence by Bercovici and Voiculescu)**
*For n large, Supp*$(\nu_n) \subset [-3,3]$ *and Supp*$(\mathcal{S}^*_{\boxplus}[\nu_n]) \subset [-5,5]$.

If the test function was holomorphic, the Cauchy formula yields

$$\langle \mathcal{S}^*_{\boxplus}[\nu_n], \mathcal{L}_{\boxplus}[h] \rangle = \frac{1}{2\pi i} \int_{\mathcal{R}} h(z)(\tau[S_n g_z(S_n)] - \langle \mathcal{S}^*_{\boxplus}[\nu_n], \partial g_z \rangle) dz,$$

where $\mathcal{R}$ strictly containing $[-5,5]$ and

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where $\mathcal{R}$ strictly containing $[-5, 5]$ and

$$g(x) := (z - x)^{-2}.$$

But for total variation, $h$ must be only bounded, not holomorphic right?

# Moral of the story

For $h$ holomorphic and bounded by one over $[-5, 5]$, there is a constant only depending on $\mathcal{R}$, such that

$$|\langle \mathbf{s}, h \rangle - \langle \nu_n, h \rangle| \leq C \sup_{z \in \mathcal{R}} |\tau[S_n g_z(S_n)] - \langle \mathcal{S}_{\boxplus}^*[\nu_n], \partial g_z \rangle|.$$

By an approximation argument,

$$d_{TV}(\langle \mathbf{s}, h \rangle - \langle \nu_n, h \rangle) \leq C \sup_{z \in \mathcal{R}} |\tau[S_n g_z(S_n)] - \langle \mathcal{S}_{\boxplus}^*[\nu_n], \partial g_z \rangle|.$$

# Non-commutative Lindeberg trick?

Define $S_n^{(k)}$ as the part of $S_n$ that does not involve $\xi_{k,n}$. Observe that

$$\mathbb{E}[S_n g_z(S_n)] = \sum_{k=1}^{n} \mathbb{E}[\xi_{k,n} g_z(S_n)] = \sum_{k=1}^{n} \mathbb{E}[\xi_{k,n}(g_z(S_n) - g_z(S_n^k))]$$

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Do I need a commutative Taylor?

# The only non-commutative calculus we need

**Lemma**

*For non-commutative variables* $a, r$*, define*

$$\Delta\left(a, r\right) := 2\mathfrak{s}[(z-a)r] - r^2,$$

*where* $\mathfrak{s}$ *denotes the symmetrization operator. Then, for all* $q \geq 1$*,*

$$g\left(a+r\right) = g\left(a+r\right)\left(\Delta\left(a, r\right) g(a)\right)^q + \sum_{j=0}^{q-1} g\left(a\right)\left(\Delta\left(a, r\right) g\left(a\right)\right)^j,$$

# Consequence

We have the following expansion for all $q \geq 1$

$$\tau[S_n g_z(S_n)] = \sum_{k=1}^{n} \tau[\xi_{k,n} g\left(S_n\right) \left(\Delta\left(S_n^k, \xi_{k,n}\right) g(S_n^k)\right)^q] + \sum_{j=1}^{q-1} \tau[g\left(S_n^k\right) \left(\Delta\left(S_n^k, \xi_{k,n}\right) g\left(S_n^k\right)\right)^j].$$

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For $q = 2$, the boundedness of the $g$ then yields

$$\begin{aligned}\tau[S_n g_z(S_n)] &= O(\sum_{k=1}^{n} \tau[|\xi_{k,n}|^3]) + \sum_{k=1}^{n} \tau[g\left(S_n^k\right) \Delta\left(S_n^k, \xi_{k,n}\right) g\left(S_n^k\right)] \\ &= O(\sum_{k=1}^{n} \tau[|\xi_{k,n}|^3]) + 2\sum_{k=1}^{n} \tau[g\left(S_n^k\right) \mathfrak{s}\left[(z - S_n^k)\xi_{k,n}\right] g\left(S_n^k\right)].\end{aligned}$$

# Consequence

Finally, using freeness,

$$\begin{aligned}\tau[S_n g_z(S_n)] &= O(\sum_{k=1}^{n} \tau[|\xi_{k,n}|^3]) + 2\sum_{k=1}^{n} \tau[|\xi_{k,n}|^2]\tau[g\left(S_n^k\right)[(z-S_n^k)]\tau[g\left(S_n^k\right)] \\ &= O(\sum_{k=1}^{n} \tau[|\xi_{k,n}|^3]) + 2\sum_{k=1}^{n} \tau[|\xi_{k,n}|^2]\tau[g\left(S_n\right)[(z-S_n)]\tau[g\left(S_n\right)].\end{aligned}$$

The unit variance condition then implies

$$\sup_{z\in\mathcal{R}} |\tau[S_n g_z(S_n)] - \langle \mathcal{S}^*_{\boxplus}[\nu_n], \partial g_z\rangle| \leq C\sum_{k=1}^{n} \tau[|\xi_{k,n}|^3].$$

# Wrapping things up

**Theorem (Diaz-Jaramillo)**
*Under the above considerations,*

$$d_{TV}(\mathbf{s}, \nu_n) \leq C \sum_{k=1}^{n} \int_{\mathbb{R}} |x|^3 \mu_{k,n}(dx).$$

Some improvements:

1. Neighborhoods of dependency.
2. Uniform convergence of the density.
3. The Mauricio Salazar trick holds.

Some unsolved improvements:

1. Uniform convergence of the derivatives of the density.

## Some questions that I thought could be interesting

- Free law of rare events
- For Boolean or monotone convolutions, can we still say something in Wasserstein distance?
- Can we change $\boxplus$ by $\boxplus_m$ and still say something?
- Extended Mauricio Salazar trick or Edgeworth expansions
- Implementations in large matrix problems
- Multidimensional versions
- Free stable limits

Thanks!

# References


- Diaz M., Jaramillo A. Non-commutative Stein's method: Applications to free probability and sums of non-commutative variables.
- G. P. Chistyakov and F. Gotze. Limit theorems in free probability theory.