Fluctuations of the spectrum of matrix-valued Gaussian processes.

Arturo Jaramillo

Centro de Investigación en Matemáticas (CIMAT)

April 2024

Goal

If $Y^{(n)} = (Y^{(n)}(t);\ t \geq 0)$ are centered Gaussian, symmetric matrix-valued processes of dimension n

Goal

If $Y^{(n)} = (Y^{(n)}(t);\ t \geq 0)$ are centered Gaussian, symmetric matrix-valued processes of dimension n and $(\mu_t^{(n)}\ ;\ n \geq 1)$ is the measure that assigns mass $\frac{1}{n}$ to each eigenvalue of $Y^{(n)}(t)$.

Goal

If $Y^{(n)} = (Y^{(n)}(t);\ t \geq 0)$ are centered Gaussian, symmetric matrix-valued processes of dimension n and $(\mu_t^{(n)}\ ;\ n \geq 1)$ is the measure that assigns mass $\frac{1}{n}$ to each eigenvalue of $Y^{(n)}(t)$.

Goal

For $r \in \mathbb{N}$ fixed and a given $F : \mathbb{R} \to \mathbb{R}^r$, what can we say about

$$\left(\int_{\mathbb{R}} F(x)\mu_t^{(n)}(dx) - \mathbb{E}\left[\int_{\mathbb{R}} F(x)\mu_t^{(n)}(dx) \right]\ ;\ t \geq 0 \right)?$$

Goal

If $Y^{(n)} = (Y^{(n)}(t);\ t \geq 0)$ are centered Gaussian, symmetric matrix-valued processes of dimension n and $(\mu_t^{(n)}\ ;\ n \geq 1)$ is the measure that assigns mass $\frac{1}{n}$ to each eigenvalue of $Y^{(n)}(t)$.

Goal

For $r \in \mathbb{N}$ fixed and a given $F : \mathbb{R} \to \mathbb{R}^r$, what can we say about

$$\left(\int_{\mathbb{R}} F(x)\mu_t^{(n)}(dx) - \mathbb{E}\left[\int_{\mathbb{R}} F(x)\mu_t^{(n)}(dx)\right]\ ;\ t \geq 0\right)?$$

My true intentions

Get you to solve some open problems.

Notation

Let $Y^{(n)} = (Y^{(n)}(t);\ t \geq 0)$ be a sequence of $\mathbb{R}^{n\times n}$-valued processes.

Notation

Let $Y^{(n)} = (Y^{(n)}(t);\ t \geq 0)$ be a sequence of $\mathbb{R}^{n\times n}$-valued processes. Assume that $Y^{(n)}(t) = [Y^{(n)}_{i,j}(t)]_{1\leq i,j\leq n}$ is real and symmetric, with

$$Y^{(n)}_{i,j}(t) = \begin{cases} \frac{1}{\sqrt{n}} X_{i,j}(t) & \text{if} \quad i < j, \\ \frac{\sqrt{2}}{\sqrt{n}} X_{i,i}(t) & \text{if} \quad i = j, \end{cases} \tag{1}$$

where $X_{i,j} := (X_{i,j}(t);\ t \geq 0)$ are i.i.d. centered Gaussian with

$$R(s,t) := \mathbb{E}[X_{1,1}(s) X_{1,1}(t)].$$

Notation

We will use the notation

$$\sigma_s := \sqrt{R(s,s)} \qquad \text{y} \qquad \rho_{s,t} := \frac{R(s,t)}{\sigma_s \sigma_t},$$

Notation

We will use the notation

$$\sigma_s := \sqrt{R(s,s)} \qquad \text{y} \qquad \rho_{s,t} := \frac{R(s,t)}{\sigma_s \sigma_t},$$

Assume $\sigma_1 = 1$ and for all $T > 0$ fixed...

Notation

We will use the notation

$$\sigma_s := \sqrt{R(s,s)} \qquad \text{y} \qquad \rho_{s,t} := \frac{R(s,t)}{\sigma_s \sigma_t},$$

Assume $\sigma_1 = 1$ and for all $T > 0$ fixed...

(H1) There exists $\alpha > 1$, such that

$$\sup_{0 \le t \le T} \int_0^T \left| \frac{\partial R}{\partial s}(s,t) \right|^\alpha ds < \infty.$$

Notation

We will use the notation

$$\sigma_s := \sqrt{R(s,s)} \qquad \text{y} \qquad \rho_{s,t} := \frac{R(s,t)}{\sigma_s \sigma_t},$$

Assume $\sigma_1 = 1$ and for all $T > 0$ fixed...

(H1) There exists $\alpha > 1$, such that

$$\sup_{0 \leq t \leq T} \int_0^T \left| \frac{\partial R}{\partial s}(s,t) \right|^\alpha \mathrm{d}s < \infty.$$

(H2) The mapping $s \mapsto \sigma_s^2$ is smooth in $(0, \infty)$ and continuous at zero.

Notation

Examples:

- Brownian motion.

Notation

Examples:

- Brownian motion.
- Fractional Brownian motion with Hurst parameter $H \in (0,1)$.

Notation

Examples:

- Brownian motion.
- Fractional Brownian motion with Hurst parameter $H \in (0, 1)$.
- Ornestein-Uhlenbeck process.

Notation

We will denote by $\lambda_1^{(n)}(t) \geq \cdots \geq \lambda_n^{(n)}(t)$ the ordered eigenvalues of $Y^{(n)}(t)$ and by $\mu_t^{(n)}$ the spectral empirical distribution

$$\mu_t^{(n)}(\mathrm{d}x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i^{(n)}(t)}(\mathrm{d}x).$$

Wigner theorem

Wigner theorem establishes that for all $\varepsilon > 0$ and a test function f,

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\int_{\mathbb{R}} f(x)\mu_1^{(n)}(\mathrm{d}x) - \int_{\mathbb{R}} f(x)\mu_1^{sc}(\mathrm{d}x)\right| > \epsilon\right] = 0, \tag{2}$$

where μ_σ^{sc}, for $\sigma > 0$, denotes the rescaled semicircle distribution

$$\mu_\sigma^{sc}(\mathrm{d}x) := \frac{\mathbb{1}_{[-2\sigma,2\sigma]}(x)}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\mathrm{d}x.$$

Functional Wigner theorem

Many authors (Rogers, Shi, Cépa, Lepingale and Pérez-Abreu) have studied dynamical versions of Wigner's theorem

Theorem (Jaramillo, Pardo and Pérez)

Denote by $\mathcal{C}(\mathbb{R}_+, \mathtt{Pr}(\mathbb{R}))$ the continuous functions with values in probability measures. If $\mu_0^{(n)}$ converges in law to ν, then the random process $\mu_t^{(n)}$ converges functionally to a constant process μ_t, such that

$$\int_{\mathbb{R}} f(x)\mu_t(dx) = \int_{\mathbb{R}} f(x)\nu(dx) + \frac{1}{2}\int_0^t \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \frac{d}{ds}(R(s,s))\mu_s(dx)\mu_s(dy)ds,$$

Fluctuations of Wigner's theorem

The fluctuatinos are known to satisfy

Theorem (Lytova and Pastur)

If we fix a test function f,

$$n \int_{\mathbb{R}} f(x)\mu_1^{(n)}(\mathrm{d}x) - n\mathbb{E}\left[\int_{\mathbb{R}} f(x)\mu_1^{(n)}(\mathrm{d}x) \right] \xrightarrow{d} \mathcal{N}(0, \sigma_f^2), \tag{3}$$

where

$$\sigma_f^2 := \frac{1}{4} \int_{\mathbb{R}^2} \left(\frac{f(x) - f(y)}{x - y} \right)^2 \frac{4 - xy}{(4 - x^2)(4 - y^2)} \mu_1^{sc}(\mathrm{d}x)\mu_1^{sc}(\mathrm{d}y).$$

Functional fluctuations of Wigner's theorem

There are some results on the functional fluctuations of Wigner's theorem in the following particular cases:

Functional fluctuations of Wigner's theorem

There are some results on the functional fluctuations of Wigner's theorem in the following particular cases:

- The entries $X_{i,j}$ are Ornstein-Uhlenbeck processes. This problem was studied by Israelson, Bender and Unterberger. We know that the limit is Gaussian and the limiting covariance function can be explicitly described.

Functional fluctuations of Wigner's theorem

There are some results on the functional fluctuations of Wigner's theorem in the following particular cases:

- The entries $X_{i,j}$ are Ornstein-Uhlenbeck processes. This problem was studied by Israelson, Bender and Unterberger. We know that the limit is Gaussian and the limiting covariance function can be explicitly described.
- The entries $X_{i,j}$ are **complex** Brownian motions and $f : \mathbb{R} \to \mathbb{R}$ is a polynomial. This problem has been studied by Pérez-Abreu and Tudor. It is known that the limit is Gaussian, but the covariance of the limit hasn't been described in an explicit way.

Dynamical fluctuations (notation)

For a test function $F = (f_1, \dots, f_r) \in \mathcal{P}^r$ and $z \in (0,1)$, define the processes

$$Z_F^{(n)}(t) := n \int_{\mathbb{R}} F(x) \mu_t^{(n)}(dx) - n\mathbb{E}\left[\int_{\mathbb{R}} F(x) \mu_t^{(n)}(dx) \right],$$

Dynamical fluctuations (notation)

For a test function $F = (f_1, \dots, f_r) \in \mathcal{P}^r$ and $z \in (0,1)$, define the processes

$$Z_F^{(n)}(t) := n \int_{\mathbb{R}} F(x) \mu_t^{(n)}(dx) - n\mathbb{E}\left[\int_{\mathbb{R}} F(x) \mu_t^{(n)}(dx) \right],$$

and the kernel

$$K_z(x,y) := \frac{1 - z^2}{z^2(x-y)^2 - xyz(1-z)^2 + (1-z^2)^2}.$$

Dynamical fluctuations (variance)

Theorem (Díaz, Jaramillo, Pardo, Pérez)

For all $f, g \in \mathcal{P}$,

$$\lim_{n\to\infty} \mathrm{Cov}\left[Z_f^{(n)}(s), Z_g^{(n)}(t)\right] = 2\int_{\mathbb{R}^2} f'(x)g'(y)\nu_{\sigma_s,\sigma_t}^{\rho_{s,t}}(\mathrm{d}x, \mathrm{d}y),$$

where

$$\nu_{\sigma_s,\sigma_t}^{\rho_{s,t}}(A, B) = \int_0^1 \int_{A\times B} K_{z\rho_{s,t}}(x/\sigma_s, y/\sigma_t)\mu_{\sigma_s}^{sc}(\mathrm{d}x)\mu_{\sigma_t}^{sc}(\mathrm{d}y)\mathrm{d}z.$$

Dynamical fluctuations (CLT)

Theorem (Díaz, Jaramillo, Pardo)

Let $\Lambda_F = ((\Lambda_{f_1}(t), \dots, \Lambda_{f_r}(t)); t \geq 0)$ *be centered Gaussian, independent of* $\{X_{i,j}; j \geq i \geq 1\}$*, with*

$$\mathbb{E}\left[\Lambda_{f_i}(s)\Lambda_{f_j}(t)\right] = \int_{\mathbb{R}^2} f_i'(x) f_j'(y) \nu_{\sigma_s,\sigma_t}^{\rho_{s,t}}(\mathrm{d}x, \mathrm{d}y).$$

Then,

$$(Z_F^{(n)}(t)\ ;\ t \geq 0) \stackrel{Stably}{\longrightarrow} \Lambda_F,$$

uniformly over compact sets.

Dynamical fluctuations (CLT)

Theorem (Díaz, Jaramillo, Pardo)

Let $\Lambda_F = ((\Lambda_{f_1}(t), \dots, \Lambda_{f_r}(t)); t \geq 0)$ *be centered Gaussian, independent of* $\{X_{i,j}; j \geq i \geq 1\}$, *with*

$$\mathbb{E}\left[\Lambda_{f_i}(s)\Lambda_{f_j}(t)\right] = \int_{\mathbb{R}^2} f_i'(x) f_j'(y) \nu_{\sigma_s,\sigma_t}^{\rho_{s,t}}(\mathrm{d}x, \mathrm{d}y).$$

Then,

$$(Z_F^{(n)}(t)\ ;\ t \geq 0) \xrightarrow{Stably} \Lambda_F,$$

uniformly over compact sets. In addition, $d_{TV}(Z_f^{(n)}(t), \Lambda_f(t)) \leq \frac{C}{\sqrt{n}}$.

Ingredient I: relations between D, δ y L

Mehler's formula establishes that $F \in L^2(\Omega)$ and Ψ_F is a measurable mapping from $\mathbb{R}^{\mathfrak{H}^d}$ to $\mathbb{R}$, such that $F = \Psi_F(V)$, then

$$P_\theta F = \widetilde{\mathbb{E}}\left[\Psi_F(e^{-\theta} V + \sqrt{1 - e^{-2\theta}}\widetilde{V})\right],$$

where $\widetilde{V}$ is an independent copy of V and $\widetilde{\mathbb{E}}$.

Ingredient I: relations between D, δ y L

Mehler's formula establishes that $F \in L^2(\Omega)$ and Ψ_F is a measurable mapping from $\mathbb{R}^{\mathfrak{H}^d}$ to $\mathbb{R}$, such that $F = \Psi_F(V)$, then

$$P_\theta F = \widetilde{\mathbb{E}}\left[\Psi_F(e^{-\theta} V + \sqrt{1 - e^{-2\theta}}\widetilde{V})\right],$$

where $\widetilde{V}$ is an independent copy of V and $\widetilde{\mathbb{E}}$. Additionally,

$$LF = -\delta(DF),$$

and if F is centered,

$$-L^{-1}F = \int_{\mathbb{R}_+} P_\theta F \mathrm{d}\theta.$$

Simple application for computation of covariance

See the board, keeping in mind that if $(\theta, \beta) \to A(\theta, \beta)$ is a n-symmetric matrix,

Lemma (Hadamard variational formulas)

$$\frac{\partial \lambda_i}{\partial \theta} = U_i^* \frac{\partial A}{\partial \theta} U_i,$$

$$\frac{\partial^2 \lambda_i}{\partial \theta \partial \beta}(\theta, \beta) = U_i^* \frac{\partial^2 A}{\partial \theta \partial \beta} U_i + 2 \sum_{j=1}^{n} \mathbb{1}_{\{j \neq i\}} \frac{1}{\lambda_i - \lambda_j} (U_j^* \frac{\partial A}{\partial \beta} U_i)(U_i^* \frac{\partial A}{\partial \theta} U_j)$$

CLT via Malliavin calculus

Theorem (Nourdin, Peccati and Réveillac)

Consider centered smooth random vectors $Z_n = (Z_{1,n}, \ldots, Z_{r,n})$. Let C be a covariance such that:

(i) $\mathbb{E}\left[Z_{i,n}Z_{j,n}\right] \to C(i,j)$;

CLT via Malliavin calculus

Theorem (Nourdin, Peccati and Réveillac)

Consider centered smooth random vectors $Z_n = (Z_{1,n}, \dots, Z_{r,n})$*. Let* C *be a covariance such that:*

(i) $\mathbb{E}[Z_{i,n} Z_{j,n}] \to C(i,j)$;

(ii) $\sup_{n \geq 1} \mathbb{E}\left[\|DZ_{i,n}\|_{\mathfrak{H}}^4\right] < \infty$;

CLT via Malliavin calculus

Theorem (Nourdin, Peccati and Réveillac)

Consider centered smooth random vectors $Z_n = (Z_{1,n}, \ldots, Z_{r,n})$*. Let* C *be a covariance such that:*

(i) $\mathbb{E}\left[Z_{i,n}Z_{j,n}\right] \to C(i,j)$;

(ii) $\sup_{n\geq 1} \mathbb{E}\left[\|DZ_{i,n}\|_{\mathfrak{H}}^4\right] < \infty$;

(iii) $\mathbb{E}\left[\|D^2 Z_{i,n} \otimes_1 D^2 Z_{i,n}\|_{(\mathfrak{H}^d)^{\otimes 2}}^2\right] \to 0$ *when* $n \to \infty$.

CLT via Malliavin calculus

Theorem (Nourdin, Peccati and Réveillac)

Consider centered smooth random vectors $Z_n = (Z_{1,n}, \ldots, Z_{r,n})$*. Let* C *be a covariance such that:*

(i) $\mathbb{E}\left[Z_{i,n} Z_{j,n}\right] \to C(i,j)$;

(ii) $\sup_{n \geq 1} \mathbb{E}\left[\|DZ_{i,n}\|_{\mathfrak{H}}^4\right] < \infty$;

(iii) $\mathbb{E}\left[\|D^2 Z_{i,n} \otimes_1 D^2 Z_{i,n}\|_{(\mathfrak{H}^d)^{\otimes 2}}^2\right] \to 0$ *when* $n \to \infty$.

Then $Z_n \overset{Law}{\to} \mathcal{N}(0, C)$ *and*

$$d_{TV}(Z_{1,n}, \mathcal{N}(0, C_{1,1})) \leq C\mathbb{E}\left[\left\|D^2 Z_{i,n} \otimes_1 D^2 Z_{i,n}\right\|_{(\mathfrak{H}^d)^{\otimes 2}}^2\right]^{\frac{1}{4}}$$

Some interesting open problems:

Good news

Very long, but completely tractable computations, allow us to handle

(ii) $\sup_{n\geq 1} \mathbb{E}\left[\|DZ_{i,n}\|_{\mathfrak{H}}^4\right] < \infty$;

(iii) $\mathbb{E}\left[\|D^2 Z_{i,n} \otimes_1 D^2 Z_{i,n}\|_{(\mathfrak{H}^d)^{\otimes 2}}^2\right] \to 0$ when $n \to \infty$.

for matrices with dependent entries for many ensambles!

Some interesting open problems:

Good news

Very long, but completely tractable computations, allow us to handle

(ii) $\sup_{n\geq 1} \mathbb{E}\left[\|DZ_{i,n}\|_{\mathfrak{H}}^4\right] < \infty$;

(iii) $\mathbb{E}\left[\|D^2 Z_{i,n} \otimes_1 D^2 Z_{i,n}\|_{(\mathfrak{H}^d)^{\otimes 2}}^2\right] \to 0$ when $n \to \infty$.

for matrices with dependent entries for many ensambles! Example: entries coming from a fractional Brownian sheet, Wishart shapes, symplectic, Hermitian.

Bad news

$\mathbb{E}\left[Z_{i,n} Z_{j,n}\right]$ has not well understood yet.

Some interesting open problems:

Good news

Very long, but completely tractable computations, allow us to handle

(ii) $\sup_{n\geq 1} \mathbb{E}\left[\|DZ_{i,n}\|_{\mathfrak{H}}^4\right] < \infty$;

(iii) $\mathbb{E}\left[\|D^2 Z_{i,n} \otimes_1 D^2 Z_{i,n}\|_{(\mathfrak{H}^d)^{\otimes 2}}^2\right] \to 0$ when $n \to \infty$.

for matrices with dependent entries for many ensambles! Example: entries coming from a fractional Brownian sheet, Wishart shapes, symplectic, Hermitian.

Bad news

$\mathbb{E}\left[Z_{i,n}Z_{j,n}\right]$ has not well understood yet.

Good news

Asymptotic covariances seems a tractable problem.

Second interesting problem:

Tecall that $Y^{(n)}(t) = [Y_{i,j}^{(n)}(t)]_{1 \leq i,j \leq n}$, with

$$Y_{i,j}^{(n)}(t) = \begin{cases} \frac{1}{\sqrt{n}} X_{i,j}(t) & \text{if} \quad i < j, \\ \frac{\sqrt{2}}{\sqrt{n}} X_{i,i}(t) & \text{if} \quad i = j, \end{cases} \tag{4}$$

Second interesting problem:

Tecall that $Y^{(n)}(t) = [Y_{i,j}^{(n)}(t)]_{1\leq i,j\leq n}$, with

$$Y_{i,j}^{(n)}(t) = \begin{cases} \frac{1}{\sqrt{n}} X_{i,j}(t) & \text{if} \quad i < j, \\ \frac{\sqrt{2}}{\sqrt{n}} X_{i,i}(t) & \text{if} \quad i = j, \end{cases} \tag{4}$$

Why?

Second interesting problem:

Tecall that $Y^{(n)}(t) = [Y_{i,j}^{(n)}(t)]_{1\leq i,j\leq n}$, with

$$Y_{i,j}^{(n)}(t) = \begin{cases} \frac{1}{\sqrt{n}} X_{i,j}(t) & \text{if} \quad i < j, \\ \frac{\sqrt{2}}{\sqrt{n}} X_{i,i}(t) & \text{if} \quad i = j, \end{cases} \tag{4}$$

Why?

- Alternative I: $\tilde{Y}^{(n)}(t) = [Y_{i,j}^{(tn)}(1)]_{1\leq i,j\leq n}$
- Alternative II: $\tilde{Y}^{(n)}(t_1, t_2) = [Y_{i,j}^{(t_2 n)}(t_1)]_{1\leq i,j\leq n}$.

Third interesting problem:

Everything we have said, but for heavy tails.

Third interesting problem:

Everything we have said, but for heavy tails.

There are several results by Guionnet, Ben Arous, et. al.

Fourth interesting problem:

Eigenvalue collision in fixed dimension.

Fourth interesting problem:

Eigenvalue collision in fixed dimension.

Very well understood (see Yimin Xiao, Nualart, Jaramillo): There is collision of eigenvalues for fBm's if and only if $H \geq 1/2$.

Fourth interesting problem:

Eigenvalue collision in fixed dimension.

Very well understood (see Yimin Xiao, Nualart, Jaramillo): There is collision of eigenvalues for fBm's if and only if $H \geq 1/2$. But...

Fourth interesting problem:

Eigenvalue collision in fixed dimension.

Very well understood (see Yimin Xiao, Nualart, Jaramillo): There is collision of eigenvalues for fBm's if and only if $H \geq 1/2$. But...

- In the critical regime $H = 1/2$, how much time you spend near colliding? (comparison with Bessel processes)
- In the regime $H < 1/2$, how much time you spend colliding?

Fifth interesting problem:

Wigner-type chaos for matrices of fixed dimension

Fifth interesting problem:

Wigner-type chaos for matrices of fixed dimension

Wigner chaos is surprisingly well studied (see Biane, Speicher, Kemp, Nourdin, Peccati):

Fifth interesting problem:

Wigner-type chaos for matrices of fixed dimension

Wigner chaos is surprisingly well studied (see Biane, Speicher, Kemp, Nourdin, Peccati): We can make sense of $I_q^W(f)$, for W a free Brownian motion and $I_q^{W^n}(f)$, for W^n a Dyson Brownian motion

Fifth interesting problem:

Wigner-type chaos for matrices of fixed dimension

Wigner chaos is surprisingly well studied (see Biane, Speicher, Kemp, Nourdin, Peccati): We can make sense of $I_q^W(f)$, for W a free Brownian motion and $I_q^{W^n}(f)$, for W^n a Dyson Brownian motion

- There is criteria for asymptotic freeness for $I_q^{W^n}(f)$.

An interesting conversation with Ronan:

Interacting particle system point of view

An interesting conversation with Ronan:

Interacting particle system point of view

Same questions by for interacting particle systems

An interesting conversation with Ronan:

Interacting particle system point of view

Same questions by for interacting particle systems

Bibliography

- Díaz M., Jaramillo A., Pardo J.C., y Pérez J.L. (2018). Functional Central Limit theorem for Matrix-valued Gaussian processes.
- Perez-Abreu V. y Tudor C. (2007). Functional Limit Theorem for Trace processes in a Dyson Brownian motion. *Communications on Stochastic Analysis*. **3** 415-428.
- Jaramillo, A., Pardo, J. y Pérez, J. (2018). Convergence of the empirical spectral distribution of a Gaussian matrix process. *Electronic Journal of Probability*.
- Israelson S. (2001). Asymptotic fluctuations of a particle system with singular interaction. *Stochastic Process and their Applications*. **93** 25-56.

Proving tightness

The main observation is that the random variable $\int f(x)\mu_t^{(n)}(dx)$ satisfies the following stochastic equation

$$\begin{aligned}
&\int f(x)\mu_t^{(n)}(dx) \\
&\quad = f(0) + \frac{1}{n^{\frac{3}{2}}}\sum_{i=1}^{n}\sum_{k\leq h}\int_0^t f'(\Phi_i(Y^{(n)}(s)))\frac{\partial \Phi_i}{\partial y_{k,l}}(Y^{(n)}(s))\delta X_{k,h}(s) \\
&\quad + \frac{1}{2}\int_0^t\int_{\mathbb{R}^2}\mathbb{1}_{\{x\neq y\}}\frac{f'(x)-f'(y)}{x-y}\mu_s^{(n)}(dx)\mu_s^{n}(dy)v_s' ds \\
&\quad + \frac{1}{2n^2}\sum_{i=1}^{n}\int_0^t f''(\Phi_i(Y^{(n)}(s)))v_s' ds,
\end{aligned}$$

where $v_s := \sigma_s^2$.