# On the Large Deviations for Stochatic Differential Equations Driven by a Fractional Brownian Motion

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**Abstract**

In this paper we prove large deviations for stochastic differential equations driven by a fractiona Brownian motion of any Hurst parameter $H \in (0, 1)$.

**Key Words:** fractional brownian motion, stochastic differential equations, large deviations
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## 1. Introduction

In recent years many phenomena in telecommunication network, mathematical finance, filtering theory, biology, etc., are modeled by fractional Brownian motion (fBm for short). It is well known that the fBm is not Markovian, nor a semimartingale, but has a long-range dependence and it is self-similar.

There are several attempts to construct a stochastic calculus with respect to fBm, such as: sample paths theory, semimartingale approach and the Skorohod integral. SDE with respect to fBm are considered in [6], [3]. Such stochastic equations are of Volterra type with a weakly regular kernel and are not covered by standard stochastic Volterra equations as treated for example in Berger and Mizel [4], Protter [18].

Large deviations for stochastic equations driven by semimartingales are by now well known (see Azencot [1], Baldi and Chaleyat-Maurel [2], Cutland [7], Dozzi [10], Freidlin and Wentzell [11], Liptser and Pukhalskii [14], Nualart and Rovira [15], Pérez-Abreu and Tudor [16], Priouret [17], Rovira and Sanz-Solé [19]).

In the present paper we consider Itô equations driven by a fBm and prove large deviations for them.

Xiong [22] (see also Kallianpur and Xiong [13]) obtained a large deviations criterium for Banach space valued random variables which come from random transformations of the Wiener process and for which, since the transformation is not pointwise, the classical contraction principle does not work

In the paper we use such an alternative for SDEs driven by fBm, together with an exponential inequality for a type of stochastic integrals that are not martingales.

## 2. Preliminaries about large deviations

Let $(S,d)$ be a Polish space, $\{\mu_\varepsilon\}_{\varepsilon>0}$ a sequence of probability measures on the Borel $\sigma$-algebra of $S$ and $\Lambda : S \to [0,+\infty]$ be a function.

Recall that the sequence $\{\mu_\varepsilon\}_{\varepsilon>0}$ is said to satisfy the large deviations principle with the rate function $\Lambda$ if :

*(a)* $\Lambda$ is lower semicontinuous.

*(b) (Lower estimate).* For every nonempty open set $G \subset S$

$$-\Lambda(G) \leq \underline{\lim}_{\varepsilon\to 0} \varepsilon \log\ \mu_\varepsilon(G). \tag{2.1}$$

*(c) (Upper estimate).* For every closed set $F \subset S$

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log\ \mu_\varepsilon(F) \leq -\Lambda(F), \tag{2.2}$$

where, for a set $A \subset S$, $\Lambda(A) = \inf_{x\in A} \Lambda(x)$.

The rate function $\Lambda$ is said to be *good* if the level set $\Psi_\Lambda(r) = \{x \in S : \Lambda(x) \leq r\}$ is compact for each $r \geq 0$.
A useful criterion for proving large deviations is due to Xiong [22] (see also Kallianpur and Xiong [13]). We state it here part of it which is enough for our purpose.

**Theorem 2.1** *Let $B$ be a real separable Banach space, $\mathcal{A}_B$ be a linear space of $B$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$.*
*Let $\Lambda : B \to [0, \infty]$ be a good rate function with the skeleton $K_\Lambda = \bigcup_{r>0} \Psi_\Lambda(r)$.*
*Let $A : B \to B$, $G : \mathcal{A}_B \to \mathcal{A}_B$, $F : B \times K_\Lambda \to B$ and $\{X^\varepsilon\}_{\varepsilon>0}$ be a family of $B$-valued random variables. Assume that:*
*(a) $B \subset \mathcal{A}_B$.*
*(b) $F(x, .) \mid_{\Psi_\Lambda(r)}$ is continuous for all $x \in B, r > 0$ and $A, F$ are Lipschitz continuous, i.e., there are positive constants $K_1$, $\{K(r)\}_{r>0}$ such that*

$$\|A(x) - A(y)\| \leq K_1 \|x - y\|, \quad \forall x, y \in B,$$

$$\|F(x, \varphi) - F(y, \varphi)\| \leq K(r) \|x - y\|, \quad \forall x, y \in B, \varphi \in \Psi_\Lambda(r).$$

*(c) $X^\varepsilon \in \mathcal{A}_B$ and*

$$X^\varepsilon = A(X^\varepsilon) + \varepsilon^{\frac{1}{2}} G(X^\varepsilon), \ a.s. in \ B.$$

*(d) $G$ is exponentially continuous in the following sense: for any $M > 0$ there exist $\delta > 0$ such that for any $\{X_1^\varepsilon\}$, $\{X_2^\varepsilon\} \subset \mathcal{A}_B$ we have*

$$\overline{\lim_{\varepsilon \to 0}} \varepsilon \log \ P(\varepsilon^{\frac{1}{2}} \|G(X_1^\varepsilon) - G(X_2^\varepsilon)\| > \sqrt{\delta}, \|X_1^\varepsilon - X_2^\varepsilon\| < \delta) \leq -M.$$

*(e) $\{X^\varepsilon\}_{\varepsilon>0}$ is exponentially tight, i.e., for any $M > 0$ there exist a compact $C_M \subset B$ such that*

$$\overline{\lim_{\varepsilon \to 0}} \varepsilon \log \ P(X^\varepsilon \notin C_M) \leq -M.$$

*(f) For any $x \in B$,the family $B$-valued random variables*

$$X^{\varepsilon, x} = A(x) + \varepsilon^{\frac{1}{2}} G(x), \ \varepsilon > 0,$$

*satisfies the LDP ion $B$ with the good rate function*

$$I^x(y) = \inf \{\Lambda(\varphi) : \varphi \in K_\Lambda, F(x, \varphi) = y\}, \ y \in B. \tag{2.3}$$

*Then the function $I : B \to [0,\infty]$ defined by*

$$I(x) = \inf \{\Lambda(\varphi) : \varphi \in K_\Lambda, F(x,\varphi) = x\}\,,\ x \in B, \tag{2.4}$$

*is a rate function (not necessarily good!) and the upper large deviations bound hold for the sequence $\mu_\varepsilon = P \circ (X^\varepsilon)^{-1}$.*

We shall need the following theorem due to Kallianpur and Odaira (see [12], [13] or [21]).

**Theorem 2.2** *Let $\mu$ be a centered Gaussian measure on a separable Banach space $E$. Let $S$ be a continuous linear map from a separable Hilbert space $H$ to $E$ such that*

$$\|S'l\|_H^2 = \int_E l(x)^2 d\mu(x),\ \forall l \in E', \tag{2.5}$$

*where $S' : E' \to H' = H$ is the dual of $S$.*
*For $\varepsilon > 0, l$ let $\mu_\varepsilon$ be the probability measure on $E$ defined by*

$$\mu_\varepsilon(C) = \mu(x \in E : \varepsilon^{\frac{1}{2}} x \in C), \forall C \in \mathcal{B}(E).$$

*Then $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies the LDP in $E$ with the good rate function*

$$I_\mu(x) = \inf \left\{ \frac{1}{2} \|h\|_H^2 : h \in H, S(h) = x \right\},\ x \in E. \tag{2.6}$$

# 3. Stochastic Differential Equations Driven by Fractional Brownian Motion

We consider $T = [0,1]$ and $W = \{x \in C(T,R) : x(0) = 0\}$ equipped with the supremum norm. We denote by $W'$ the strong topological dual of $W$. For $H \in (0,1)$ we denote by $P_H$ the unique probability measure on $W$ such that the canonical process $\{W_t\}_{t\in T}$ is a fBm with Hurst parameter $H$. Recall that the covariance $R_H$ of $W$ is given by

$$R_H(s,t) = \frac{C_H}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \tag{3.1}$$

$$C_H = \frac{\Gamma(2-2H)\cos\pi H}{\pi H(1-2H)}. \tag{3.2}$$

In order to represent the fBm in terms of standard Brownian motion (obtained for $H=\frac{1}{2}$) we need hypergeometric functions.
We consider the Gauss hypergeometric function $F(\alpha,\beta,\gamma,z)$ which is the analytic continuation on $C\times C\times C\setminus\{0,-1,-2,....\}\times\{z\in C:\operatorname{Arg}(1-z)<\pi\}$ of the power series

$$F(\alpha,\beta,\gamma,z)=\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\beta)_k}{(\gamma)_k\,k!}z^k, \tag{3.3}$$

$(a)_k=a(a+1)....(a+k-1)$.
Define

$$K_H(t,r)=\frac{(t-r)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}F(\frac{1}{2}-H,H-\frac{1}{2},H+\frac{1}{2},1-\frac{t}{r})1_{[0,t)}(r). \tag{3.4}$$

It is known that (see [9])

$$R_H(t,s)=\int_0^{t\wedge s}K_H(t,r)K_H(s,r)dr. \tag{3.5}$$

Next we shall denote by $\nabla_H$ the Gross-Sobolev derivative operator, $\delta_H$ its dual (divergence operator) and we define the stochastic integral with respect the fBm by

$$\int_0^1 u_s\delta_H W_s=\delta_H(K_H u), \tag{3.6}$$

for processes $u$ for which $K_H u$ is in the domain of $\delta_H$.

**Remark 3.1** It is known that (see [9])
(i) $\left\{\int_0^t K_H(t,s)\delta_H W_s\right\}_{t\in T}=\{W_t\}_{t\in T}$.
(ii) $\left\{\int_0^t 1_{[0,t]}\delta_H W_s\right\}_{t\in T}:=\{B_t\}_{t\in T}$ is a standard Brownian motion on $(W,\mathcal{B}(W),P_H)$.
(iii) For every $u\in L^2(W\times T)$, $u$ adapted,

$$\int_0^t u_s\delta_H W_s=\int_0^t u_s dB_s, t\in T,$$

(the last integral is the usual Itô integral).
(iv) We have the following representation for the fBm

$$W_t=\int_0^t K_H(t,s)dB_s,\ t\in T.$$

By a *stochastic differential equation driven by a fBm* we mean an equation of the form

$$X_t = \xi + \int_0^t K_H(t,s)b(s,X_s)ds + \int_0^t K_H(t,s)\sigma(s,X_s)dB_s,\ t \in T, \tag{3.7}$$

where $\xi \in R$ and $\sigma(t,x), b(t,x) : T \times R \to R$ are measurable.
Next we denote

$$H_0 = \left| H - \frac{1}{2} \right|,\ \Delta_H = \left[0, \frac{1}{H_0}\right),$$

and for $p \in \Delta_H$ we put

$$\alpha_p = (1 - pH_0)^{-1},\ L_+^{\alpha_p} = \cup_{\alpha > \alpha_p} L^\alpha(T).$$

**Definition 3.2** By a solution of (3.7) we mean an nonanticipating process $\{X_t\}_{t \in T}$ such that $t \longrightarrow E\left[|X_t|^2\right] \in L_+^{\alpha_2}$ and (3.7) is satisfied $dP_H \otimes dt$ a.s..
The following result is due tu Coutin and Decreusefond [6]
**Theorem 3.3** *If $b, \sigma$ are Lipschitz in the second variable, uniformly with respect to the first variable, and there exist $x_0, y_0 \in R$, $\alpha_b > \alpha_1, \alpha_\sigma > 2\alpha_2$ such that*

$$b(.,x_0) \in L^{\alpha_b}(T),\ \sigma(.,x_0) \in L^{\alpha_\sigma}(T),$$

*then (3.7) has a unique sulution $\{X_t\}_{t \in T}$.*
*Moreover for every $p \in \Delta_H \cap [2, \infty)$*

$$\sup_{t \in T} E\left[|X_t|^p\right] < \infty. \tag{3.8}$$

*If $\sigma$ is bounded then $X$ is continuous.*

## 4. Main Result

$$X_t^\varepsilon = \xi + \int_0^t K_H(t,s)b(s,X_s^\varepsilon)ds + \varepsilon^{\frac{1}{2}} \int_0^t K_H(t,s)\sigma(s,X_s^\varepsilon)dB_s,\ t \in T, \tag{4.1}$$

Let $H_{\frac{1}{2}}$ be the Cameron-Martin space associated to the standard Brownian motion, that is

$$H_{\frac{1}{2}} = \left\{ h \in L^2(T) : h(t) = \int_0^t \dot{h}(s)ds,\ \dot{h} \in L^2(T) \right\}.$$

We have the following main result.

**Theorem 4.1** *Assume that $\sigma, b$ are bounded and Lipschitz in the second variable, uniformly with respect to the first variable. Let $X^\varepsilon$ be the unique continuous solution of (4.1).*

*Then the family $\{X^\varepsilon\}_{\varepsilon>0}$ satisfies a LDP in $W$ with the good rate function*

$$I(f) = \inf\left\{\frac{1}{2}\left\|\dot{h}\right\|^2_{L^2(T)} : h \in H_{\frac{1}{2}}, f = S(h)\right\}, \tag{4.2}$$

*where $S(h)$ is the unique continuous solution of*

$$S(h)_t = \xi + \int_0^t K_H(t,s)b(s,S(h)_s)ds + \int_0^t K_H(t,s)\sigma(s,S(h)_s)\dot{h}(s)ds,\ t \in T. \tag{4.3}$$

For the proof of Theorem 4.1 we need the following lemmas.

Next a process $\{U_t\}$ and $\alpha > 0$ we define

$$[U]_\alpha = \sup_{0\le s<t\le T} \frac{|U_t - U_s|}{|t-s|^\alpha}.$$

We have the following exponential estimate for stochastic Volterra integrals with singular kernel(see [15] for the case of regular coefficients).

**Lemma 4.2** *Let $\{f(t)\}_{t\in T}$ be a bounded nonanticipating real process and $0 < \alpha < H$. Then, for any $a > 0$,*

$$P(\left[\int_0^\cdot K_H(.,s)f(s)dB_s\right]_\alpha > a) \le C_1 \exp\left\{-(aC_2 - 1)^2\right\}, \tag{4.4}$$

*where $0 < C_i = C_i(\alpha, H, \|f\|_\infty) < \infty$.*

*In particular*

$$P(\sup_{0\le t\le 1}\left|\int_0^t K_H(t,s)f(s)dB_s\right| > a) \le C_1 \exp\left\{-(aC_2 - 1)^2\right\}. \tag{4.5}$$

**Proof.** By the Kolmogorov criterium the process $t \longrightarrow I(t) := \int_0^t K_H(t,s)f(s)dB_s$ is $\alpha$-Hölder. Fix $0 \le s < t \le 1$ and define

$$g_{t,s}(r) = 1_{s\le r\le t}K_H(t,r) + 1_{r\le s}\left[K_H(t,r) - K_H(s,r)\right],\ r \in T.$$

Then we have

$$I(t) - I(s) = \int_0^1 g_{t,s}(r) f(r) dB_r. \tag{4.6}$$

Define the martingales

$$M_u = C_H^{-\frac{1}{2}} (t-s)^{-\alpha} \int_0^u g_{t,s}(r) f(r) dB_r,$$

$$\tilde{M}_u = C_H^{-\frac{1}{2}} (t-s)^{-\alpha} \int_0^u \frac{M_r g_{t,s}(r) f(r)}{(1+M_r^2)^{\frac{1}{2}}} dB_r, \ u \in T.$$

By Itô's formula we have

$$d\left(1+M_u^2\right)^{\frac{1}{2}} = \frac{1}{2}\left(1+M_u^2\right)^{-\frac{1}{2}} g_{t,s}^2(r) f^2(r) dr -$$

$$\frac{1}{4}\left(1+M_u^2\right)^{-\frac{3}{2}} g_{t,s}^2(r) f^2(r) dr + \tilde{M}_u^2. \tag{4.7}$$

By the time change in stochastic integral we choose a Brownian motion $\left\{\tilde{W}_u\right\}_{u\in T}$ such that

$$\tilde{M}_u = \tilde{W}_{\tau(u)}, \ \ \tau(u) = C_H^{-1} (t-s)^{-2\alpha} \int_0^u \frac{M_r^2 g_{t,s}^2(r) f^2(r)}{1+M_r^2} dr.$$

Next we have

$$\tau(u) \leq C_H^{-1} (t-s)^{-2\alpha} \|f\|_\infty^2 \int_0^u g_{t,s}^2(r) dr =$$

$$C_H^{-1} (t-s)^{-2\alpha} \|f\|_\infty^2 \left[\int_s^t K_H^2(t,r) dr + \int_0^s \left(K_H(t,r) - K_H(s,r)\right)^2 dr\right]$$

$$\leq (t-s)^{2(H-\alpha)} \leq 1,$$

where from by using this in (4.7) we obtain

$$\left(1+M_1^2\right)^{\frac{1}{2}} \leq 1 + \sup_{t\leq 1}\left|\tilde{W}_t\right| + C_1,$$

and thus

$$M_1^2 \leq C_1 \left(1 + \sup_{t\leq 1}\left|\tilde{W}_t\right|^2\right)^2.$$

Now by Fernique's inequality we obtain that

$$E\left[\exp\left\{C_2 M_1^2\right\}\right] = C_3 < \infty. \tag{4.8}$$

If we denote

$$\Psi(x) = e^{C_2 x^2} - 1, p(x) = x^\alpha,$$

$$\eta = \int_0^1 \int_0^1 \exp\left\{C_2 \frac{|I(t) - I(s)|^2}{|t-s|^{2\alpha}}\right\} dtds,$$

then (4.6) and (4.8) imply

$$E(\eta) \leq \int_0^1 \int_0^1 E\left[\exp(C_2 M_1^2)\right] dtds \leq C_3. \tag{4.9}$$

Also we have

$$\eta - 1 = \int_0^1 \int_0^1 \Psi\left(\frac{|I(t) - I(s)|}{p(|t-s|)}\right) dtds$$

From Kallianpur-Xiong [13]-pp. 313-314 and (4.9) it follows that

$$\left|I_1^t(f) - I_1^s(f)\right| \leq C_4 \left(\sqrt{|\log \eta|} + 1\right) |t-s|^\alpha, \tag{4.10}$$

and then

$$P\left([I(.)]_\alpha > a\right) \leq P\left[\eta > \exp\left\{\left(aC_4^{-1} - 1\right)^2\right\}\right] \leq$$

$$\leq C_3 \exp\left\{\left(aC_4^{-1} - 1\right)^2\right\},$$

i.e., the estimate (4.4). ■

**Lemma 4.3** *Under the assumptions of Theorem 4.1 the equation (4.3) has a unique continuous solution.*

**Proof.** If $f, g$ are continuous solutions of (4.3) then it follows

$$|f_t - g_t| \leq C \int_0^t |K_H(t,s)| \varphi_s ds, \ \forall t \in T, \tag{4.11}$$

where

$$\varphi_t = |f_t - g_t| \left(1 + \left|\dot{h}_t\right|\right) \in L^2(T).$$

Of course

$$\int_0^t |K_H(t,s)| \varphi_s ds \leq \left[\int_0^t |K_H(t,s)|^2 ds\right]^{\frac{1}{2}} \left[\int_0^1 \varphi_s^2 ds\right]^{\frac{1}{2}} \leq D < \infty. \tag{4.12}$$

If we define for $r \geq 1$,

$$K_n^{(r)}(t) = \int_0^t |K_H(t,s_1)|^r ds_1 \int_0^{s_1} |K_H(s_1,s_2)|^r ds_2 ... \int_0^{s_{n-1}} |K_H(s_{n-1},s_n)|^r ds_n,$$

then iterating (4.11) and using Lemma 3.3 of [6] we obtain

$$|f_t - g_t| \leq DC^n K_{n-1}^{(1)}(t) \longrightarrow 0.$$

For existence we use the successive approximations. Define

$$f_t^0 = \xi,\ f_t^n = \xi + \int_0^t K_H(t,s) b(s, f_s^{n-1}) ds + \int_0^t K_H(t,s)\sigma(s, f_s^{n-1})\dot{h}(s) ds,\ t \in T.$$

The the successive approximations $f_t^n$ are continuous since $K_H(L^2(T)) \subset W$ (see [20]-pag. 187). If we denote

$$\alpha_n(t) = \left| f_t^{n+1} - f_t^n, \right|$$

then we have

$$\alpha_0(t) \leq C \int_0^t |K_H(t,s)| \left(1 + \left|\dot{h}_s\right|\right) ds \leq D,$$

$$\alpha_n(t) \leq C \int_0^t |K_H(t,s)| \alpha_{n-1}(s) \left(1 + \left|\dot{h}_s\right|\right) ds,$$

and by iteration

$$\alpha_n(t) \leq C^n D K_n^{(1)}(t),$$

and by Lemma 3.3 of [6], we deduce that $\alpha_n(t) \longrightarrow \alpha(t)$ uniformly in $t$ and $\alpha$ is a solution of (4.3). ■

**Lemma 4.4** *For $f \in W$ consider the gaussian process*

$$Y_t^\varepsilon = \xi + \int_0^t K_H(t,s) b(s, f_s) ds + \varepsilon^{\frac{1}{2}} \int_0^t K_H(t,s)\sigma(s, f_s) dB_s,\ t \in T. \tag{4.13}$$

*Then the family* $\{Y^{\varepsilon}\}_{\varepsilon>0}$ *satisfies a LDP in* $W$ *with the good rate function*

$$I^f(h) = \inf\left\{\frac{1}{2}\|\dot{\varphi}\|^2_{L^2(T)} : \varphi \in H_{\frac{1}{2}}, h = S_f(\varphi)\right\}, \tag{4.14}$$

*where*

$$S_f(\varphi)_t = \xi + \int_0^t K_H(t,s)b(s,f_s)ds + \int_0^t K_H(t,s)\sigma(s,f_s)\dot{\varphi}(s)ds,\ t \in T. \tag{4.15}$$

From the above lemma and Theorem 3.3 of [9] we obtain in particular the standard result (see for example [5]-Corollary 4.9.3 or [21]-Theorems 3.45 and 3.48)

**Corollary 4.5** *The the family* $\left\{\varepsilon^{\frac{1}{2}}W\right\}_{\varepsilon>0}$ *satisfies a LDP in* $W$ *with the good rate function*

$$I(h) = \inf\left\{\frac{1}{2}\left\|\dot{h}\right\|^2_{L^2(T)} : \dot{h} \in L^2(T), h(.) = \int_0^{\cdot} K_H(.,s)\dot{h}(s)ds\right\}. \tag{4.16}$$

**Proof of Lemma 4.4** Without loss of generality we may assume that $\xi = b = 0$. We shall apply Theorem 2.2 to the operator $S_f : H_{\frac{1}{2}} \longrightarrow W$.
Consider the Gaussian measure $\mu$ on $W$

$$\mu = P_H \circ \left(\int_0^{\cdot} K_H(.,r)\sigma(r,f_r)dB_r\right)^{-1}.$$

For $\eta \in W'$ we have

$$\int \eta^2(w)d\mu(w) = \int_0^1\int_0^1 E\left\{\left[\int_0^t K_H(t,r)\sigma(r,f_r)dB_r\right]\times\right.$$

$$\left.\times\left[\int_0^s K_H(s,r)\sigma(r,f_r)dB_r\right]\right\}d\eta(t)d\eta(s) =$$

$$\int_0^1\int_0^1\int_0^1 K_H(t,r)K_H(s,r)\sigma^2(r,f_r)dr\,d\eta(t)d\eta(s) =$$

$$\int_0^1\left[\int_0^1 K_H(t,r)d\eta(t)\right]^2\sigma^2(r,f_r)dr.$$

Next for each $\varphi \in H_{\frac{1}{2}}$ we have

$$\left\langle S_f' \eta, \varphi \right\rangle_{H_{\frac{1}{2}}} = \int_0^1 S_f \varphi(t) d\eta(t) =$$

$$\int_0^1 \left[ \int_0^1 K_H(t,r) d\eta(t) \right] \sigma(r, f_r) \dot{\varphi}(r) dr,$$

which implies that

$$\left\| S_f' \eta \right\|_{H_{\frac{1}{2}}}^2 = \int_0^1 \left[ \int_0^1 K_H(t,r) d\eta(t) \right]^2 \sigma^2(r, f_r) dr. \quad \blacksquare$$

**Proof of Theorem 4.1** Next we shall denote:
$B = C(T)$, $\mathcal{A}_B$ be the family of all real continuous adapted processes $\{f_t\}_{t \in T}$,

$$\Lambda : B \to [0, \infty], \ \Lambda(\varphi) = \begin{cases} \frac{1}{2} \|\dot{\varphi}\|_{L^2(T)}^2, & \varphi \in H_{\frac{1}{2}} \\ \infty, & \varphi \notin H_{\frac{1}{2}}, \end{cases}$$

$$A : B \to B, \ A(f)_t = \xi + \int_0^t K_H(t,s) b(s, f_s) ds,$$

$$G : \mathcal{A}_B \to \mathcal{A}_B, \ G(X)_t = \int_0^t K_H(t,s) \sigma(s, X_s) dB_s,$$

$$F : B \times H_{\frac{1}{2}} \to B, \ F(f, \varphi)_t = S_f(\varphi).$$

We check the hypotheses of Theorem 2.1 in the above setting.
The conditions (a), (b) are easily seen. In order to see (d), we define

$$f(t) = \delta^{-1} \left[ \sigma(s, X_\varepsilon^1(s)) - \sigma(s, X_\varepsilon^2(s)) \right] 1_{\{|X_\varepsilon^1(t) - X_\varepsilon^2(t)| < \delta\}}, \ t \in T.$$

Then by Lemma 4.2 we have

$$\varepsilon \log P_H \left( \varepsilon^{\frac{1}{2}} \sup_{0 \le t \le 1} \left| G(X_\varepsilon^1)_t - G(X_\varepsilon^2)_t \right| > \sqrt{\delta}, \sup_{0 \le t \le 1} \left| X_\varepsilon^1(t) - X_\varepsilon^2(t) \right| < \delta \right)$$

$$\varepsilon P_H(\left[\int_0^{\cdot} K_H(.,s)f(s)dB_s\right]_\alpha > \frac{1}{(\varepsilon\delta)^{\frac{1}{2}}}) \to -\infty \text{ as } \varepsilon \to 0.$$

For the condition (e), define the compact set

$$C_M = \{x \in C(T) : \ [x]_\alpha \leq M\}, \ M > 0, \ 0 < \alpha < H.$$

It is easily seen that for $s \leq t$,

$$\left| \int_0^t K_H(t,r)b(r,X_r^\varepsilon)dr - \int_0^s K_H(s,r)b(r,X_r^\varepsilon)dr \right| \leq D_1 (t-s)^H. \tag{4.17}$$

Then by Lemma 4.2 and (4.17),

$$\lim_{M\to\infty} \overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log\ P_H(X^\varepsilon \notin C_M) \leq$$

$$\lim_{M\to\infty} \max\left\{ \overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log\ P_H(\left[\int_0^{\cdot} K_H(.,r)b(r,X_r^\varepsilon)dr\right]_\alpha > \frac{M}{2}), \right.$$
$$\left. \overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log\ P_H(\left[\int_0^{\cdot} K_H(.,r)\sigma(r,X_r^\varepsilon)dr\right]_\alpha > \frac{M}{2}) \right\} = -\infty.$$

Next if $f \in C(T)$ then the family

$$X^{\varepsilon,f} = \xi + \varepsilon^{\frac{1}{2}} B(f),$$

satisfies, by Lemma 4.4, the LDP in $C(T)$ with the good rate function

$$I^f(h) = \inf\left\{ \frac{1}{2} \|\dot{\varphi}\|^2_{L^2(T)} : \varphi \in H_{\frac{1}{2}}, h = F(f,\varphi) \right\},$$

which is the assertion in the condition (f).
Therefore by Theorem 2.1 the upper estimate of large deviations hold.
For the lower estimate all we need to show is that for every $f \in C(T)$ with $I(f) < \infty$ and for all $\delta, \eta > 0$ we have

$$\underline{\lim}_{\varepsilon\to 0}\, \varepsilon \log\ P(\sup_{0\leq t\leq 1} |X_t^\varepsilon - f_t| < \delta) \geq -I(f) - \eta.$$

Choose $\varphi \in H_{\frac{1}{2}}$ such that $f = S_f(\varphi)$ and $\frac{1}{2}\|\dot{\varphi}\|^2_{L^2(T)} \leq I(f) + \eta$.
By Girsanov's theorem the process

$$\tilde{B}_t = B_t - \frac{1}{\sqrt{\varepsilon}}\varphi_t, t \in T,$$

is a Brownian Motion with respect to the probability measure

$$\tilde{P} = \exp\left\{\frac{1}{\sqrt{\varepsilon}}\int_0^1 \dot{\varphi}_t dB_t - \frac{1}{2\varepsilon}\int_0^1 |\dot{\varphi}_t|^2 dt\right\}.P.$$

Consider the process $\tilde{X}^\varepsilon_t = X^\varepsilon_t - f_t$, $t \in T$. It is clear that $\tilde{X}^\varepsilon_t$ satisfies under $\tilde{P}$ the stochastic equation

$$\begin{aligned}\tilde{X}^\varepsilon_t = \varepsilon^{\frac{1}{2}}\int_0^t K_H(t,s)\sigma(s,\tilde{X}^\varepsilon_s + f_s)d\tilde{B}_s + \\ \int_0^t K_H(t,s)\left[b(s,\tilde{X}^\varepsilon_s + f_s) - b(s,f_s)\right] ds + \\ \int_0^t K_H(t,s)\left[\sigma(s,\tilde{X}^\varepsilon_s + f_s) - \sigma(s,f_s)\right]\dot{\varphi} ds.\end{aligned} \tag{4.18}$$

If we denote

$$A^\varepsilon = \left\{\sup_{0\leq t\leq 1}\left|\tilde{X}^\varepsilon_t\right| < \delta\right\},$$

then we obtain

$$\begin{aligned}P(A^\varepsilon) = \int_{A^\varepsilon}\exp\left\{-\frac{1}{\sqrt{\varepsilon}}\int_0^1 \dot{\varphi}_t dB_t + \frac{1}{2\varepsilon}\int_0^1 |\dot{\varphi}_t|^2 dt\right\} d\tilde{P} = \\ \exp\left\{-\frac{1}{2\varepsilon}\int_0^1 |\dot{\varphi}_t|^2 dt\right\}\int_{A^\varepsilon}\exp\left\{-\frac{1}{\sqrt{\varepsilon}}\int_0^1 \dot{\varphi}_t d_{\tilde{P}}\tilde{B}_t\right\} d\tilde{P},\end{aligned}$$

and from Hölder and Jensen's inequalities,

$$\varepsilon \log P(A^\varepsilon) \geq -I(f) - \eta + \varepsilon \log \tilde{P}(A^\varepsilon) - \sqrt{\varepsilon}E_{\tilde{P}_{|A^\varepsilon}}\left(\int_0^1 \dot{\varphi}_t d_{\tilde{P}}\tilde{B}_t\right),$$

which shows that it is enough to prove that

$$\log \tilde{P}(A^\varepsilon) \text{ and } E_{\tilde{P}_{|A^\varepsilon}}\left(\int_0^1 \dot{\varphi}_t d_{\tilde{P}}\tilde{B}_t\right),$$

are bounded for $\varepsilon \leq \varepsilon_0$.
Since

$$\left| E_{\tilde{P}_{|A^\varepsilon}} \left( \int_0^1 \dot{\varphi}_t d_{\tilde{P}} \tilde{B}_t \right) \right| = \frac{1}{\tilde{P}(A^\varepsilon)} \int_{A^\varepsilon} \left( \int_0^1 \dot{\varphi}_t d_{\tilde{P}} \tilde{B}_t \right) d\tilde{P}$$

$$\leq \frac{1}{\tilde{P}(A^\varepsilon)} \left[ E_{\tilde{P}} \left( \left| \int_0^1 \dot{\varphi}_t d_{\tilde{P}} \tilde{B}_t \right|^2 \right) \right]^{\frac{1}{2}} \tilde{P}(A^\varepsilon)^{\frac{1}{2}} = \frac{\left[ E_{\tilde{P}} \left( \int_0^1 |\dot{\varphi}_t|^2 \, dt \right) \right]^{\frac{1}{2}}}{\tilde{P}(A^\varepsilon)^{\frac{1}{2}}},$$

the assertion follows if we prove that $\tilde{P}(A^\varepsilon) \to 1$ as $\varepsilon \to 0$.
In fact we show more, that is $E_{\tilde{P}} \left[ \sup_{0 \leq t \leq 1} \left| \tilde{X}_t^\varepsilon \right|^r \right] \to 0$ as $\varepsilon \to 0$ for $r$ large enough.
From (4.18), by using a standard procedure, we have for $r$ large enough

$$E_{\tilde{P}} \left[ \left| \tilde{X}_t^\varepsilon \right|^r \right] \leq C_1 \left\{ \varepsilon^{\frac{r}{2}} + \int_0^t K_H^r(t,s) E_{\tilde{P}} \left[ \left| \tilde{X}_s^\varepsilon \right|^r \right] ds \right\},$$

where from

$$E_{\tilde{P}} \left[ \left| \tilde{X}_t^\varepsilon \right|^r \right] \leq C_2 \sum_{l=1}^{\infty} \varepsilon^{\frac{lr}{2}} \left( K_l^{(r)} 1 \right)(t),$$

for $\varepsilon \leq \varepsilon_0$, with $C_1$ independent of $\varepsilon$. Therefore

$$\sup_{t \leq 1} E_{\tilde{P}} \left[ \left| \tilde{X}_t^\varepsilon \right|^r \right] \leq C_3 \varepsilon^{\frac{r}{2}} \to 0 \text{ as } \varepsilon \to 0. \tag{4.19}$$

From Lemma 3.2 of [6] we have

$$\int_0^s K_H^r(s,s_1) \left| \tilde{X}_{s_1}^\varepsilon \right|^r ds_1 \leq \sup_{t \leq 1} \| K_H^r(t,.) \|_{L^{r_1}} \int_0^1 \left| \tilde{X}_{s_1}^\varepsilon \right|^{r_2} ds_1. \tag{4.20}$$

Now from Remark 4.1 of [8],(4.19) and (4.20) it follows that

$$E_{\tilde{P}} \left[ \sup_{s \leq t} \left| \tilde{X}_s^\varepsilon \right|^r \right] \leq C_4 \left\{ \varepsilon^{\frac{r}{2}} + E \left[ \sup_{s \leq t} \int_0^s K_H^r(s,s_1) \left| \tilde{X}_{s_1}^\varepsilon \right|^r ds_1 \right] \right\} \leq$$

$$C_5 \left\{ \varepsilon^{\frac{r}{2}} + \sup_{s \leq 1} E_{\tilde{P}} \left( \left| \tilde{X}_s^\varepsilon \right|^r \right) ds \right\} \to 0 \text{ as } \varepsilon \to 0.$$

Finally the exponentially tightness and the lower estimate imply that the rate function is good.

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