SELF-INTERSECTION LOCAL TIME OF THE FLUCTUATION LIMIT OF A PARTICLE SYSTEM WITH MUTATION

Ekaterina Todorova
Centro de Investigación en Matemáticas, A.C.

Abstract

We give a necessary and sufficient condition for existence and continuity of self-intersection local time (SILT) of the fluctuation limit of a two-type particle system in R^d, were the particles evolve according to symmetric stable motions with different stability parameters, and switch from type one to type two at random times with given distribution. We show that existence of SILT is determined by the "most mobile" of the stable motions.

1. Introduction

Let $X = (X(t))_{t\in[0,1]}$ be a centered, continuous, Gaussian process with values in the space $\mathcal{S}'(R^d)$ of tempered distributions on R^d. An intuitive definition of self-intersection local time (SILT) of X up to time $t \in [0, 1]$ is given by the formal expression

$$\int_0^t \int_0^t \langle X(s) \otimes X(r), \delta(x-y)\varphi(x)\rangle \, ds \, dr, \tag{1.1}$$

where $\otimes$ denotes the tensor product in $\mathcal{S}'(R^d)$, $\varphi \in \mathcal{S}(R^d)$ (the C^∞ rapidly decreasing functions), δ is the Dirac distribution, and $\langle \cdot, \cdot \rangle$ stands for duality. Since $\delta(x - y)\varphi(x) \notin \mathcal{S}(R^{2d})$, the first question is how to give a rigorous meaning to (1.1). This problem was studied by Adler, Feldman and Lewin (1991), and Adler and Rosen (1993), in case of the Brownian and α-stable density processes. They proved that SILT of X exists if and only if $d < 2\alpha$, and for $\alpha = 2$ the SILT process, when it exists, has cadlag paths. The cadlag result in the Brownian case was obtained by means of the approximating particle system.

Motivated by their work, Bojdecki and Gorostiza (1995,1999) gave a rigorous definition of (1.1) for a wide class of $\mathcal{S}'(R^d)$-valued Gaussian processes and a criterion for existence and continuity of SILT in terms of the covariance of a given process and applied this criterion to various examples of generalized processes. In particular, they proved that SILT of the α-stable density process exists and is continuous if and only if $d < 2\alpha$. Bojdecki and Gorostiza (1995) noted that the definition of SILT for $\mathcal{S}'(R^d)$-processes is not an extension of the definition in the finite-dimensional case. Indeed, they showed that a finite-dimensional Wiener process can be embedded in $\mathcal{S}'(R^d)$ in two ways, such that the SILT exists for one of them but not for the other.

In order to understand better the dependence of SILT on the spacial structure of the process X and to give a "particle picture" interpretation, Gorostiza and Todorova (1999) obtained an existence and continuity result for SILT of a more general density process that corresponds to a system of particles of two types, where the particles of type i, $i = 1, 2$, move according to a symmetric α_i-stable process, and each particle switches back and forth between the two types with respective exponential waiting times with parameters V_i. Gorostiza and Todorova (1999) proved that SILT exists and is continuous if and only if $d < 2\min\{\alpha_1, \alpha_2\}$, thus generalizing the above results. The intuitive interpretation of this result is that the existence of SILT is determined by the "most mobile" of the stable motions.

With a view toward increasing our understanding of the meaning of SILT for Gaussian $\mathcal{S}'(R^d)$-processes which have associated particle picture, we study in this paper the existence and continuity of SILT for a density process where the particles begin with type 1 (performing symmetric α_1-stable motion) and switch to type 2 (corresponding to a symmetric α_2-stable motion) in random time τ, where τ is a random variable concentrated in a interval $[T_1, T_2], 0 < T_1 < T_2 < 1$. We show (Theorem 3.2) that the SILT exists and is continuous if and only if $d < 2\min\{\alpha_1, \alpha_2\}$; that is, again the " most mobile" motion determines existence of SILT. Note that the result does not depend on τ. In the case $\alpha_1 = \alpha_2$, this result reduces to the previous studied by Adler, Feldman and Lewin(1991), Adler and Rosen (1993) and Bojdecki and Gorostiza (1995).

The organization of this paper is as follows. In Section 2 we recall the definition of SILT of generalized Gaussian processes given by Bojdecki and Gorostiza (1995), as well as their criterion for existence and continuity of SILT. In Section 3 we introduce the particle system which we consider here, and state our main result on the existence of SILT (Theorem 3.2) which is proved in Section 4.

2. A general criterion on existence and continuity of SILT for $\mathcal{S}'(R^d)$-valued processes

We will condense the main result from Bojdecki and Gorostiza (1995). We refer the reader to that paper for additional information and details.

Let $\mathcal{F}$ denote the class of non-negative symmetric C^∞ functions f on R^d with bounded support, and such that $f(0) > 0$ and $\int f(x)\,dx = 1$. For $f \in \mathcal{F}$, $\varepsilon > 0$ and $\varphi \in \mathcal{S}(R^d)$ we define

$$f_\varepsilon(x) = \varepsilon^{-d} f\left(\frac{x}{\varepsilon}\right), \quad x \in R^d,$$

and

$$\Phi^f_{\varepsilon,\varphi}(x,y) = f_\varepsilon(x-y)\varphi(x), \quad x,\, y \in R^d.$$

Note that $\Phi^f_{\varepsilon,\varphi} \in \mathcal{S}(R^{2d})$ and $\Phi^f_{\varepsilon,\varphi}$ approximates $\delta(x-y)\varphi(x)$ as $\varepsilon \to 0$. In order to give a meaning to (1.1), the idea is to replace $\delta(x-y)\varphi(x)$ by $\Phi^f_{\varepsilon,\varphi}$, so that it makes sense, and take the limit as $\varepsilon \to 0$. For existence of a limit it is also necessary to replace $X(s) \otimes X(r)$ by the Wick product $: X(s) \otimes X(r) :$, which is an $\mathcal{S}'(R^{2d})$-valued random field such that

$$\langle : X(s) \otimes X(r) :, \varphi \otimes \psi \rangle = \langle X(s), \varphi\rangle \langle X(r), \psi\rangle - E(\langle X(s), \varphi\rangle\langle X(r), \psi\rangle).$$

This leads to defining an approximate SILT $L^f_\varepsilon(t)$ by

$$\langle L^f_\varepsilon(t), \varphi\rangle = \int_0^t \int_0^t \langle : X(s) \otimes X(r) :, \Phi^f_{\varepsilon,\varphi}\rangle \, ds\, dr, \quad t \in [0,1], \quad \varphi \in \mathcal{S}(R^d),$$

which is a continuous $\mathcal{S}'(R^d)$-process.

We can now give a precise meaning to (1.1).

Definition 2.1. For a given continuous centered Gaussian $\mathcal{S}'(R^d)$-valued process $X = (X(t))_{t\in[0,1]}$, if there exists an $\mathcal{S}'(R^d)$-valued process $L = (L(t))_{t\in[0,1]}$ such that for any $t \in [0,1], \varphi \in \mathcal{S}(R^d)$ and $f \in \mathcal{F}$,

$$\left\langle L^f_\varepsilon(t), \varphi \right\rangle \to \langle L(t), \varphi\rangle$$

in L^2 as $\varepsilon \to 0$, then L is called the *self-intersection local time* (SILT) of X.

Let K denote the covariance functional of X :

$$K(s,\varphi;t,\psi) = E\left(\langle X(s),\varphi\rangle \langle X(t),\psi\rangle\right), \quad \varphi,\psi \in \mathcal{S}(R^d), \quad s,t \in [0,1].$$

For test functions $\Phi^{(1)}, \Phi^{(2)} \in \mathcal{S}(R^d) \otimes \mathcal{S}(R^d)$ of the form

$$\Phi^{(1)} = \sum_{i=1}^{n} \varphi_i^{(1)} \otimes \psi_i^{(1)}, \quad \Phi^{(2)} = \sum_{j=1}^{m} \varphi_j^{(2)} \otimes \psi_j^{(2)}, \;\; \varphi_i^{(1)}, \psi_i^{(1)}, \varphi_j^{(2)}, \psi_j^{(2)} \in \mathcal{S}(R^d), \tag{2.1}$$

and $s,r,u,v \in [0,1]$, we consider the functional

$$\begin{aligned} & J_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)}) \\ = & \sum_{i=1}^{n}\sum_{j=1}^{m} \Big(K(s,\varphi_i^{(1)};u,\varphi_j^{(2)})K(r,\psi_i^{(1)};v,\psi_j^{(2)}) + K(s,\varphi_i^{(1)};v,\psi_j^{2})K(r,\psi_i^{(1)};u,\varphi_j^{(2)}) \Big), \end{aligned} \tag{2.2}$$

which is the covariance functional of the random field $: X(s) \otimes X(r) :$ for test functions of the given form.

We now state the existence and continuity criterion for SILT:

Theorem 2.2. *Given a continuous centered Gaussian $\mathcal{S}'(R^d)$-process X, assume that $J_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})$ has a well defined extension on $\mathcal{S}(R^{2d}) \times \mathcal{S}(R^{2d})$ such that*

a) The functional

$$(\Phi^{(1)},\Phi^{(2)}) \mapsto \int_{[0,t]^4} J_{s,r,u,v}(\Phi^{(1)},\Phi^{(2)})\, ds\, dr\, du\, dv$$

is continuous on $\mathcal{S}(R^{2d}) \times \mathcal{S}(R^{2d})$ for each $t \in [0,1]$.

b) $J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^g_{\delta,\varphi})$ converges to a finite limit as $\varepsilon, \delta \to 0$, for each $f, g \in \mathcal{F}$, $\varphi \in \mathcal{S}(R^d)$, $s,r,u,v \in [0,1]$, and this limit does not depend on f, g.

c)

$$\left| J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^g_{\delta,\varphi}) \right| \leq G_\varphi(s,r,u,v)$$

for some measurable function G_φ on $[0,1]^4$ which depends on φ but is independent of $\varepsilon, \delta, f, g$, and such that

$$\int_{[0,1]^4} G_\varphi(s,r,u,v)\, ds\, dr\, du\, dv < \infty \tag{2.3}$$

for each $\varphi \in \mathcal{S}(R^d)$.

Then the SILT L of the process X exists.

Assume in addition that

d) There exists a non-decreasing continuous function F on $[0,1]$ and a number $\gamma > 0$ such that for all $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $\varphi \in \mathcal{S}(R^d)$,

$$\begin{aligned}\int\limits_{[0,1]^4} &\left(1_{[0,t_2]^2}(s,r) - 1_{[0,t_1]^2}(s,r)\right)\left(1_{[0,t_2]^2}(u,v) - 1_{[0,t_1]^2}(u,v)\right) G_\varphi(s,r,u,v)\, ds\, dr\, du\, dv \\ &\le C(\varphi)\left(F(t_2) - F(t_1)\right)^{1+\gamma}, \end{aligned} \tag{2.4}$$

where $C(\varphi)$ is a positive constant depending only on φ.

Then the SILT L is a continuous $\mathcal{S}'(R^d)$-process, and moreover L^f_ε converges weakly to L in $C([0,1], \mathcal{S}'(R^d))$ as $\varepsilon \to 0$.

Theorem 2.3. *a) Suppose that $J_{s,r,u,v}$ satisfies condition (i) but*

$$\lim_{\varepsilon \to 0} \int_{[0,1]^4} J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^f_{\varepsilon,\varphi})\, ds\, dr\, du\, dv = \infty$$

for some $\varphi \in \mathcal{S}(R^d)$ and $f \in \mathcal{F}$. Then X does not have SILT.

b) Conditions (2.3) and (2.4)are satisfied by the function

$$G_\varphi(s,r,u,v) = C(\varphi)((|s-u| + |r-v|)^{-\beta} + (|s-v| + |r-u|)^{-\beta}), \beta \in (0,2), \tag{2.5}$$

where $C(\varphi)$ is some positive constant depending only on φ.

We denote by $p^{(\alpha)}_t$ the transition density of the symmetric α-stable motion in R^d.

We will use also the following result from Gorostiza and Todorova (1999).

Theorem 2.4. *For $\alpha_1, \alpha_2 \in (0,2]$ given, there is a positive constant $C = C(d, \alpha_1, \alpha_2)$ such that*

$$\sup_{x,y \in R^d} \int p_r^{(\alpha_1)}(x,z) p_s^{(\alpha_2)}(z,y)\, dz \le C(r+s)^{-d/\min\{\alpha_1,\alpha_2\}}$$

for all $s, r \in (0,1]$.

3. The particle model and the main result

We consider a particle system of two types in R^d, that correspond to a symmetric α_i-stable motions, $\alpha_i \in (0,2]$, $i = 1,2$, and $t \in [0,1]$. The particles are initially distributed according to a Poisson random field with intensity measure $n\Lambda$, where $n \in N$, and Λ denotes the Lebesgue measure on R^d, and change of type in a random time τ, when it switches to a type 2 and moves according to a symmetric α_2-stable motion for the rest of time. The random variable τ has a given distribution μ_τ concentrated in the time interval $[T_1, T_2]$, where $0 < T_1 < T_2 < 1$.

Let

$$N(t) = \sum_i \delta_{\xi_i(t)},$$

where $\{\xi_i(t)\}$ are the positions of the particles at time t, and consider the normalized fluctuation process X^n defined by

$$X^n(t) = n^{-1/2}(N^n(t) - EN^n(t)),\, t \in [0,1].$$

The following result is well-known.

Theorem 3.1. *When $n \to \infty$, X^n converges weakly in $D([0,1], \mathcal{S}'(R^d))$ to a centered, continuous, Gaussian process X with covariance functional $K(s,\phi;t,\psi) = E(\langle X(s),\phi\rangle\langle X(t),\psi\rangle)$ given by*

$$K(s,\phi;t,\psi) = \int E_x[\phi(\xi_s)\psi(\xi_t)]dx.$$

The process X is the high-density fluctuation limit of the given branching particle system.

We obtain the following main result.

Theorem 3.2. *If $d < 2\min\{\alpha_1, \alpha_2\}$ then the process X has SILT, and SILT is a continuous $\mathcal{S}'(R^d)$-process. if $d \geq 2\min\{\alpha_1, \alpha_2\}$, then X does not have SILT.*

This result reveals that it does not matter when and how the particles change of type- the most "mobile" type has a determining property for the existence of SILT.

4. Proof of Theorem 3.2

We will prove that the conditions of Theorem 2.2 and Theorem 2.3 are satisfied using the methods of Gorostiza-Todorova (1999).

First we obtain an expression for the covariance functional of X in terms of the transition densities $p_t^{(i)}$ of the symmetric α_i-stable motion, $i = 1, 2$. The covariance functional depends on the ordering of T_1 on $[0, 1]$ with respect to s, t.

We will omit writing R^d in the integrals on R^d.

Lemma 4.1. *For $s \leq t$ the covariance functional $K(s, \phi; t, \psi)$ of the process X is given by the following expressions:*

a) If $s \leq t \leq T_1$, then $K(s, \phi; t, \psi) = \int \varphi(y)\psi(z)p_{t-s}^{(1)}(y - z)\, dy\, dz$.

b) If $s \leq T_1 < t$, then $K(s, \phi; t, \psi) = \int \int_{T_1}^{t\wedge T_2} \varphi(y)\psi(w)p_{r-s}^{(1)}(y-z)p_{t-r}^{(2)}(z-w)\, dw\, dz\, dy\, \mu_\tau(dr)$.

c) If $T_1 < s \leq t$, then

$$\begin{aligned} K(s, \phi; t, \psi) &= \mu_\tau[T_1, s \wedge T_2] \int \varphi(z)\psi(w)p_{t-s}^{(2)}(z - w)\, dz\, dw && (4.1) \\ &+ \int \int_{s\wedge T_2}^{t\wedge T_2} \varphi(y)\psi(w)p_{t-r}^{(2)}(z - w)p_{r-s}^{(1)}(y - z)\, dy\, dz\, dw \mu_\tau(dr). && (4.2) \end{aligned}$$

Remark. When $T_2 \leq s \leq t$ we obtain in case c) the same formula like in a) but with stability parameter α_2.

Proof. Let $T_t^{\alpha_i}, i = 1, 2$, denote the semigroup of the α_i-stable symmetric motion and $F_s = \sigma\{X_t, t \leq s\}$. Using the Markov property and the definition of $T_t^{\alpha_i}$, we obtain in case a)

$$E_x(\phi(\xi_s)\psi(\xi_t)) = E_x(\phi(\xi_s)E_x(\psi(\xi_t)|F_s))$$

$$
\begin{aligned}
&= E_x(\phi(\xi_s)T^{\alpha_1}_{t-s}\psi(\xi_s) \\
&= T^{\alpha_1}_s(\phi T^{\alpha_1}_{t-s}\psi)(x). \qquad (4.3)
\end{aligned}
$$

Now using the definition of the semigroup $T^{\alpha_i}_t, i = 1, 2$, we have

$$
\begin{aligned}
T^{\alpha_1}_{t-s}\psi(y) &= \int \psi(z)p^{(1)}_{t-s}(y-z)\,dz \\
T^{\alpha_1}_s(\phi T^{\alpha_1}_{t-s}\psi)(x) &= \int \phi(y)\psi(z)p^{(1)}_{t-s}(y-z)p^{(1)}_s(x-y)\,dy\,dz \\
K(s,\phi;t,\psi) &= \int \phi(y)\psi(z)p^{(1)}_{t-s}(y-z)p^{(1)}_s(x-y)\,dx\,dy\,dz,
\end{aligned}
$$

and using that $p^{(1)}_s$ is a density, we obtain the result in this case. In the second case, conditioning on τ we have

$$
\begin{aligned}
E_x(\phi(\xi_s)\psi(\xi_t)) &= \int_{T_1}^{t\wedge T_2} E_x(E_x(\phi(\xi_s)\psi(\xi_t)|\tau = r)|F_r)\mu_\tau(dr) \\
&= \int_{T_1}^{t\wedge T_2} E_x(\phi(\xi_s)T^{\alpha_2}_{t-r}\psi(\xi_r))\mu_\tau(dr) \\
&= \int_{T_1}^{t\wedge T_2} E_x(E_x(\phi(\xi_s)T^{\alpha_2}_{t-r}\psi(\xi_r)|F_s))\mu_\tau(dr) \\
&= \int_{T_1}^{t\wedge T_2} E_x(\phi(\xi_s)T^{\alpha_1}_{r-s}T^{\alpha_2}_{t-r}\psi(\xi_s))\mu_\tau(dr) \\
&= \int_{T_1}^{t\wedge T_2} T^{\alpha_1}_s(\phi T^{\alpha_1}_{r-s}T^{\alpha_2}_{t-r}\psi)(x)\mu_\tau(dr). \qquad (4.4)
\end{aligned}
$$

Using the definition of $T^{\alpha_i}_t, i = 1, 2$, we finish the proof like in the previous case.

In case c) we have

$$
\begin{aligned}
E_x(\phi(\xi_s)\psi(\xi_t)) &= \int_{T_1}^{t\wedge T_2} E_x(E_x(\phi(\xi_s)\psi(\xi_t)|\tau = r))\mu_\tau(dr) \\
&= \int_{T_1}^{s\wedge T_2} E_x(E_x(\phi(\xi_s)\psi(\xi_t)|\tau = r))\mu_\tau(dr) \\
&\quad + \int_{s\wedge T_2}^{t\wedge T_2} E_x(E_x(\phi(\xi_s)\psi(\xi_t)|\tau = r))\mu_\tau(dr) \\
&= I_1 + I_2, \qquad (4.5)
\end{aligned}
$$

where I_1 and I_2 are the corresponding integrals in (4.5). We have for I_1:

$$\begin{aligned} I_1 &= \int_{T_1}^{s\wedge T_2} E_x(E_x(\phi(\xi_s)\psi(\xi_t)|F_s)|\tau = r)\mu_\tau(dr) \\ &= \int_{T_1}^{s\wedge T_2} E_x(E_x(\phi(\xi_s)T_{t-s}^{\alpha_2}\psi(\xi_s)|\tau = r)\mu_\tau(dr) \\ &= \int_{T_1}^{s\wedge T_2} E_x(T_{s-r}^{\alpha_2}(\phi(\xi_r)T_{t-s}^{\alpha_2}\psi(\xi_r))\mu_\tau(dr) \\ &= \int_{T_1}^{s\wedge T_2} T_r^{\alpha_1}T_{s-r}^{\alpha_2}(\phi T_{t-s}^{\alpha_2}\psi)(x)\mu_\tau(dr). \end{aligned}$$

The calculation of I_2 is similar: for $T_2 \leq s \leq t$ we have $I_2 = 0$; for $s \leq T_2$ we obtain

$$\begin{aligned} I_2 &= \int_{s}^{t\wedge T_2} E_x(E_x(\phi(\xi_s)\psi(\xi_t)|F_r)|\tau = r)\mu_\tau(dr) \\ &= \int_{s}^{t\wedge T_2} E_x(\phi(\xi_s)T_{t-r}^{\alpha_2}\psi(\xi_r)|\tau = r)\mu_\tau(dr) \\ &= \int_{s}^{t\wedge T_2} E_x(E_x(\phi(\xi_s)T_{t-r}^{\alpha_2}(\xi_r)|F_s)|\tau = r)\mu_\tau(dr) \\ &= \int_{s}^{t\wedge T_2} E_x(T_{s-r}^{\alpha_2}(\phi(\xi_r)T_{t-s}^{\alpha_2}(\xi_r)|F_s)|\tau = r)\mu_\tau(dr) \\ &= \int_{s}^{t\wedge T_2} E_x(\phi(\xi_s)T_{r-s}^{\alpha_1}(T_{t-r}^{\alpha_2}\psi(\xi_s))\mu_\tau(dr) \\ &= \int_{s}^{t\wedge T_2} T_s^{\alpha_1}(\phi T_{r-s}^{\alpha_1}(T_{t-r}^{\alpha_2}\psi))(x)\mu_\tau(dr). \end{aligned}$$

Hence

$$I_2 = \int_{s\wedge T_2}^{t\wedge T_2} T_s^{\alpha_1}(\phi T_{r-s}^{\alpha_1}(T_{t-r}^{\alpha_2}\psi))(x)\mu_\tau(dr).$$

Now we finish the proof using the definition of the semigroup $T_t^{\alpha_i}, i = 1, 2$.

Substituting $K(s, \varphi; t, \psi)$ in (2.2), first for Φ, Ψ of the form $\Phi = \varphi \otimes \psi, \Psi = \overline{\varphi} \otimes \overline{\psi}$, and then for Φ, Ψ of the general form, and using the linearity of the tensor product, we obtain analogously to Lemma 5.3 in Gorostiza and Todorova (1998) the functional $J_{s,l,u,v}(\Phi, \overline{\Phi})$, $s \leq l \leq u \leq v$ for all cases of positions of T_1 on $[0, 1]$ with respect to the points s, l, u, v. For example, for $s \leq l \leq u \leq T_1 < v$ we have the following result.

Lemma 4.2. *For $s \le l \le u \le T_1 < v$ we have*

$$J_{s,l,u,v}(\Phi, \overline{\Phi}) = J^{(1)}_{s,l,u,v}(\Phi, \overline{\Phi}) + J^{(2)}_{s,l,u,v}(\Phi, \overline{\Phi}),$$

where

$$\begin{aligned}
&J^{(1)}_{s,l,u,v}(\Phi, \overline{\Phi}) \\
&= \int_{T_1}^{v} \int \Phi(y, y')\overline{\Phi}(z, w') p^{(1)}_{u-s}(y-z) p^{(1)}_{r-l}(y'-z') p^{(2)}_{v-r}(z'-w')\, dy\, dz\, dy'\, dz'\, dw'\, \mu_\tau(dr), \\
&J^{(2)}_{s,l,u,v}(\Phi, \overline{\Phi}) \\
&= \int_{T_1}^{v} \int \Phi(y, y')\overline{\Phi}(w', z) p^{(1)}_{v-l}(y-z) p^{(1)}_{r-s}(y'-z') p^{(2)}_{u-r}(z'-w')\, dy\, dz\, dy'\, dz'\, dw'\, \mu_\tau(dr).
\end{aligned}$$

Having the expression for the function $J_{s,r,u,v}$, condition a) of Theorem 2.2 is proved analogously to the Corollary 5.7 of Gorostiza and Todorova (1998).

We can now prove condition b) of Theorem 2.2 for each one of the orderings of T_1 with respect to s, l, u, v analogously to the Proposition 5.10 of Gorostiza and Todorova (1998), thus obtaining the following result.

Lemma 4.3. *The limit $\lim_{\varepsilon,\delta\to 0} J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^g_{\delta,\varphi})$ exists for all $f, g \in \mathcal{F}$, $\varphi \in \mathcal{S}(R^d)$, $s, r, u, v \in [0, 1]$, and this limit does not depend on f, g. In the case $s \le l \le u \le T_1 < v$ we have*

$$\begin{aligned}
&\lim_{\varepsilon,\delta\to 0} J^{(1)}_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^g_{\delta,\varphi}) \\
&\quad = \int \int_{T_1}^{v} \varphi(y)\varphi(z) p^{(1)}_{u-s}(y-z) p^{(1)}_{r-l}(y-z') p^{(2)}_{v-r}(z'-z)\, dy\, dz\, dz' \mu_\tau(dr)
\end{aligned}$$

$$\begin{aligned}
&\lim_{\varepsilon,\delta\to 0} J^{(2)}_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^g_{\delta,\varphi}) \\
&\quad = \int \int_{T_1}^{v} \varphi(y)\varphi(z) p^{(1)}_{v-l}(y-z) p^{(1)}_{r-s}(y-z') p^{(2)}_{u-r}(z'-z)\, dy\, dz\, dz' \mu_\tau(dr).
\end{aligned}$$

With these preliminary results we now proceed to prove Theorem 3.2.

Proof of Theorem 3.2.

(a) Case $d \geq 2\min\{\alpha_1, \alpha_2\}$.

Let assume first $\alpha_1 \leq \alpha_2$. For $\varphi > 0$, using the Fatou lemma and Lemma 4.3 we obtain

$$\liminf_{\varepsilon\to 0} \int_{[0,1]^4} J_{s,l,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^f_{\varepsilon,\varphi})\, ds\, dr\, du\, dv$$

$$\geq \int_{s\leq l\leq u\leq v\leq T_1} \int p^{(1)}_{u-s}(x,y) p^{(1)}_{v-l}(x,y)\varphi(x)\varphi(y)\, dx\, dy\, ds\, dl\, du\, dv$$

$$+ \int_{s\leq l\leq u\leq v\leq T_1} \int p^{(1)}_{v-s}(x,y) p^{(1)}_{u-l}(x,y)\varphi(x)\varphi(y)\, dx\, dy\, ds\, dl\, du\, dv.$$

But by Theorem 2.3 the last integral is ∞.

If $\alpha_2 \leq \alpha_1$, we proceed analogously using the term that coresponds to $T_2 \leq s \leq l \leq u \leq v$ and the Remark after Lemma 4.1.

By Theorem 2.3 we conclude that in this case the SILT does not exist.

(b) Case $d < 2\min\{\alpha_1, \alpha_2\}$.

We suppose $\alpha_1 \leq \alpha_2$ (the case $\alpha_2 \leq \alpha_1$ is similar). We will verify conditions c) and d) of Theorem 2.2.

From Lemma 4.2 we have for $i = 1, 2$

$$\left| J^{(i)}_{s,l,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^g_{\delta,\varphi}) \right| \leq J^{(i)}_{s,l,u,v}(|\Phi^f_{\varepsilon,\varphi}|, |\Phi^g_{\delta,\varphi}|) \leq \overline{J}^{(i)}_{s,l,u,v}(\Phi^f_{\varepsilon,|\varphi|}, \Phi^g_{\delta,|\varphi|}).$$

Therefore it suffices to bound the terms of $\overline{J}^{(i)}_{s,l,u,v}$, $i = 1, 2$, for all posible cases of orderings for T_1 on $[0,1]$ with respect to s, l, u, v. This can be done by changing variables and using Theorem 2.4.

For example, let consider the term $J \equiv J^{(1)}_{s,l,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^g_{\delta,\varphi})$ corresponding to the case $s \leq u \leq l \leq T_1 < v$. We have

$$|J| \leq \int_{T_1}^{v\wedge T_2} \int |\varphi(y)| |f_\varepsilon(y'-y)| |\varphi(z)| g_\delta(z-w') p^{(1)}_{u-s}(y-z) p^{(1)}_{r-l}(y'-z') p^{(2)}_{v-r}(z'-w')$$
$$dy\, dz\, dw'\, dz'\, dy'\, \mu_\tau(dr)$$

Changing variables $y \to y - y'$, and using Theorem 2.4, we obtain

$$|J| \leq \sup_y |\varphi(y)| \int_{T_1}^{v\wedge T_2} \int |\varphi(z)| f_\varepsilon(y') g_\delta(z-w') p^{(1)}_{|u-s|}(y-z) p^{(1)}_{r-l}(y-y'-z') p^{(2)}_{v-r}(z'-w')$$

$$dy\,dz\,dy'\,dz'\,\mu_\tau(dr)$$

$$\leq \sup_y |\varphi(y)| \int_{T_1}^{v\wedge T_2} \int (u-s+r-l)^{-d/\alpha_1} |\varphi(z)| f_\varepsilon(y') g_\delta(z-w') p^{(2)}_{v-r}(z'-w')$$

$$dz\,dw'\,dz'\,dy'\mu_\tau(dr)$$

$$\leq C(\varphi)(u-s)^{-d/2\alpha_1} \int_{T_1}^{v\wedge T_2} (r-l)^{-d/2\alpha_1} \mu_\tau(dr)$$

$$\leq C(\varphi)(u-s)^{-d/2\alpha_1}(T_1-l)^{-d/2\alpha_1}$$

$$\leq C(\varphi)H(s,l,u,v),$$

where $H(s,l,u,v) = (u-s)^{-d/2\alpha_1}(T_1-l)^{-d/2\alpha_1}$. H is integrable over $s \leq u \leq l \leq T_1 < v$ for $d < 2\alpha_1$.

We proceed analogously with the other terms of $|J|$, obtaining bounds H of the form (2.5).

Therefore conditions c) and d) of Theorem 2.2 are verified and the proof of Theorem 3.2 is complete.

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