Abelian varieties with group action

by

H. Lange [1] and S. Recillas [2]

Introduction. Let G be a finite group acting on a smooth projective curve X. This induces an action of G on the Jacobian JX of X and thus a decomposition of JX up to isogeny. The most prominent example of such a situation is the case of the group $G \simeq \mathbb{Z}/2\mathbb{Z}$ of two elements. Let $\pi : X \to Y = X/G$ denote the canonical quotient map. The first to notice that JX is isogenous to the product $JY \times P(X/Y)$ was probably Wirtinger (see [W]). The abelian variety $P(X/Y)$ was later called by Mumford the Prym variety of the covering π, because of its relation to certain Prym differentials.

The group action of some other special groups were studied by Recillas and Ries (who in [Re] and [Ri] decomposed the Jacobian of the Galois covering of a simple trigonal cover), Recillas-Rodríguez (S_3 in [RR1] and S_4 in [RR2]), Donagi (S_3, S_4 and WD_4 in [DM]), Carocca-Recillas-Rodríguez (the dihedral groups in [CRR]), and Sánchez-Argáez (A_5 in [SA2]). There are also some general results: First of all there is the classical paper of Chevalley-Weil (see [W]) which computes how many times an irreducible complex representation appears in the tangent space $T_0 JX$ in terms of the fixed points under elements of G and its stabilizer subgroups. Finally Donagi in [D] and Mérindol in [M] found the decomposition of a complex abelian variety A in the case of a finite group G all of whose irreducible $\mathbb{Q}$-representations are absolutely irreducible.

It is the aim of the present paper to prove some general results on abelian varieties with action by a finite group G and deduce from them a method to compute the Jacobian JX of a curve X with action by G which will be applied for several examples. The contents of the paper is as follows: In Section 1 we prove an analogue of Poincaré's reducibility theorem for polarized abelian varieties with G-action. This is however only an existence result and does not help if one actually wants to decompose a given abelian variety with G-action. Our point of view for doing this is the following:

Let G be a finite group acting on an abelian variety A. The action induces an algebra homomorphism $\rho : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(A)$. We define $\mathrm{Im}\,(\alpha) := \mathrm{Im}\,(\rho(m\alpha)) \subseteq A$ for any $\alpha \in \mathbb{Q}[G]$ where m is some positive integer such that $m\alpha \in \mathbb{Z}[G]$. Let

$$1 = e_1 + \ldots + e_r$$

denote the decomposition of $1 \in \mathbb{Q}[G]$ into the sum of orthogonal idempotents e_i of $\mathbb{Q}[G]$ corresponding to the simple $\mathbb{Q}$-algebras associated to $\mathbb{Q}[G]$. If $A_i = \mathrm{Im}\,(e_i)$, then it is obvious that the A_i are G-stable abelian subvarieties of A with $\mathrm{Hom}_G(A_i, A_j) = 0$ for $i \neq j$ such that the addition map induces an isogeny

$$A_1 \times \ldots \times A_r \to A.$$

called the *isotypical decomposition* of A. The essential point here is that one can explicitly describe the elements e_i in terms of the complex characters of G, which allows to actually

[1]Supported by CONACYT 27962-E
[2]Supported by CONACYT 27962-E and DAAD

compute the isotypical decomposition in many cases. In Sections 3 and 4 we decompose the G-abelian subvarieties A_i further. As a simple $\mathbb{Q}$-algebra $\mathrm{End}_{\mathbb{Q}}(A_i)$ decomposes unto a direct sum of minimal left ideals. Correspondingly the idempotent e_i decomposes into a sum of orthogonal idempotents

$$e_i = f_{i_1} + \ldots + f_{i_{n_i}}$$

which gives rise to an isogeny decomposition

$$B_{i_1} \times \ldots \times B_{i_{n_i}} \to A_i$$

with $B_{i_j} = \mathrm{Im}\,(f_{i_j})$. Again the essential point is that one can describe the f_{i_j} explicitly in terms of the irreducible $\mathbb{Q}$-representations of G. Using them one can show that $B_{i_i}, \ldots, B_{i_{n_i}}$ are isogenous to each other, so that finally we obtain an isogeny decomposition

$$A \sim B_1^{n_1} \times \ldots \times B_r^{n_r} \tag{1}$$

Here the abelian variety B_i corresponds to the different irreducible $\mathbb{Q}$-representations of G. This generalizes a result of Mérindol [M] who deduced the same decomposition in the special case that all irreducible $\mathbb{Q}$-representations of G are absolutely irreducible.

In Section 5 we apply these results to the case of a Jacobian variety with G-action. Let X be a smooth projective curve of genus ≥ 2 over the field of complex numbers and let G be a finite group acting on X with quotient $\pi : X \to Y := X/G$. Let $W_1, \ldots, W_r$ denote the irreducible $\mathbb{Q}$-representations of G, where we assume that W_1 is the trivial representation. Then if m_i denotes the Schur invariant of W_i and $n_i := \dim_{D_i} W_i, D_i = \mathrm{End}^G(W_i)$, then the isogeny decomposition (1) of the Jacobian JX reads

$$JX \sim \pi^* JY \times B_2^{m_2 n_2} \times \ldots \times B_r^{m_r n_r}. \tag{2}$$

The first problem is of course, which of the abelian varieties B_i are nonzero. This certainly depends on the curve X and not only on the group G. However, we show in Proposition 5.2 that if the genus of Y is ≥ 2, then $B_i \neq 0$ for all i,
The second problem is to describe the abelian variety B_i in terms of the curve X and its quotients, that is in terms of the Prym varieties of the intermediate coverings of $\pi : X \to Y$. Recall that if $M < N < G$ are subgroups with intermediate covering $\varphi : X_M \to X_N$ where $X_M = X/M$ and $X_N = X/N$, then the *Prym variety* of φ is defined as the complement of $\varphi^* JX_N$ in JX_M with respect to the canonical polarization and we have

$$JX_M \sim \varphi^* JX_N \times P(X_M/X_N).$$

It is by no means clear that the isogeny decomposition of $P(X_M/X_N)$ underlies the isogeny decomposition (2), since the abelian varieties B_i are not simple in general. The main result of this section Theorem 5.7 says that this is the case. In fact we show using the explicit description of the projectors f_{i_j} of Section 3 that there are non negative integers s_i such that

$$P(X_M/X_N) \sim B_2^{s_2} \times \ldots \times B_r^{s_r}.$$

This can be used to compute the isogeny decomposition of JX in many cases. In Section 6 we work out the examples $G = S_4, A_5$ the dihedral group D_n and the quaternion group $\mathbb{Q}_8$.

Acknowlegements: The second author wants to thank C. Procesi and A. Sánchez-Argáez for helpful conversations. This work was carried out during a visit of the first author to Morelia and of the second author to Erlangen under the DAAD-UNAM exchange program.

1. Poincaré's reducibility theorem with G-action:

Let G be a finite group acting on an abelian variety A defined over an algebraically closed field k. A is called *G-simple*, if there is no G-equivariant isogeny $A_1 \times A_2 \to A$ with nontrivial abelian varieties A_1 and A_2 with G-action. In this section we will show that any abelian variety A with G-action admits a G equivariant isogeny $\varphi : A_1 \times \ldots \times A_r \to A$ with G-simple abelian varieties A_i. Since the image of A_i under such an isogeny φ is a G-simple abelian subvariety of A, it suffices to decompose A into a sum of G-simple abelian subvarieties. The main result of this section is the following theorem which might be called Poincaré's reducibility theorem with G-action.

Theorem 1.1: *Let A be an abelian variety with G-action.*

(a) *There are G-simple abelian subvarieties $A_1, \ldots, A_r$ of A such that the addition map gives a G-equivariant isogeny*

$$\mu : A_1 \times \ldots \times A_r \to A$$

(b) *This decomposition is unique in the following sense: Let $B_1, \ldots, B_s$ be another set of G-simple abelian subvarieties such that the addition map induces a G-equivariant isogeny $B_1 \times \ldots \times B_s \to A$. Then $r = s$ and there is a permutation σ of degree r such that for $i = 1, \ldots, r$ there is a G-isogeny $A_i \to B_{\sigma(i)}$.*

For the proof we need some preliminaries.

Remark 1.2: If L denotes a polarization on A, that is the first Chern class of an ample line bundle L on A, we say that G *acts on the polarized abelian variety* (A, L), and L is called a *G-equivariant polarization*, if $g^*L \approx L$ (algebraically equivalent) for any $g \in G$. Note that for any abelian variety A with G-action there exists a polarization L on A, such that G acts on (A, L).

To see this let $G = \{g_1, \ldots, g_n\}$ and let M denote an arbitrary ample line bundle on A. Then $L = g_1^*M \otimes \ldots \otimes g_n^*M$ defines a polarization on A with $g_\nu^*L \approx L$ for all $\nu = 1, \ldots, n$.

Remark 1.3: If $\varphi : A \to B$ is a G-isogeny of abelian varieties with G-action, it is easy to see that there is a G-isogeny $\psi : B \to A$ such that $\psi\varphi = n_A$ and $\varphi\psi = n_B$ for some positive integer n. Notice however, that if $\varphi : (A, L) \to (B, M)$ is a G-isogeny of polarized abelian varieties, there is in general no G-isogeny $\psi : (B, M) \to (A, L)$.

Now fix a G-equivariant polarization L on A and let $\phi_L : A \to \hat{A}$ denote the associated isogeny and "$'$" the corresponding Rosati-involution of $\mathrm{End}_{\mathbb{Q}}(A)$, i.e. $\varphi' = \phi_L^{-1}\hat{\varphi}\phi_L$ for any endomorphism φ of A.

For an abelian subvariety B of A let $\iota_B : B \to A$ denote the canonical embedding. If $e(B)$ denotes the exponent of the induced polarization ι^*L, then $\psi_{\iota^*L} := e(B)\phi_{\iota^*L}^{-1} : \hat{B} \to B$ is an isogeny. Define the norm endomorphism $N_B \in \mathrm{End}(A)$ of B to be the composition

$$N_B := \iota_B \psi_{\iota^*L} \hat{\iota}_B \phi_L.$$

Then

$$\varepsilon_B = \frac{1}{e(B)} N_B$$

is a symmetric idempotent of $\mathrm{End}_{\mathbb{Q}}(A)$, i.e. $\varepsilon_B^2 = \varepsilon_B = \varepsilon'_B$. So to every abelian subvariety B of A we associated a symmetric idempotent ε_B of $\mathrm{End}_{\mathbb{Q}}(A)$. Conversely if $\varepsilon \in \mathrm{End}_{\mathbb{Q}}(A)$ is a symmetric idempotent, then $n\varepsilon \in \mathrm{End}(A)$ for some $n > 0$ and $\mathrm{Im}\ (n\varepsilon)$ is an abelian subvariety. Thus the polarization L induces bijection between the sets of abelian subvarieties of A and symmetric idempotents of $\mathrm{End}_{\mathbb{Q}}(A)$ (see [LB]). For the proof of Theorem 1.1 we need 2 observations.

(i) For an automorphism g of A we have $g^*L \approx L$ if and only if $\phi_{g^*L} = \phi_L$. Since $\phi_{g^*L} = \hat{g}\phi_L g$ this is equivalent to

$$\hat{g}\phi_L = \phi_L g^{-1} \tag{1}$$

(ii) Let $\iota_B : B \to A$ be the canonical embedding of the abelian subvariety B. Then gB is an abelian subvariety with canonical embedding

$$\iota_{gB} = g\iota_B g^{-1} \tag{2}$$

Lemma 1.4.: $\varepsilon_{gB} = g\varepsilon_B g^{-1}$

Proof: Since $g^*L \approx L$ we have $e(gB) = e(B)$. Hence the assertion follows from

$$\begin{aligned}
N_{gB} &= e(B) g\iota_B g^{-1} \phi^{-1}_{(g\iota_B g^{-1})^*L} \hat{g}^{-1} \hat{\iota}_B \hat{g} \phi_L && \text{(using (2))} \\
&= e(B) g\iota_B g^{-1} g \phi^{-1}_{\iota_B^* L} \hat{g} \hat{g}^{-1} \hat{\iota}_B \hat{g} \phi_L && (\text{since } \phi_{(g\iota_B g^{-1})^*L} = \hat{g}^{-1}\phi_{\iota_B^* L} g^{-1}) \\
&= e(B) g\iota_B \phi^{-1}_{\iota_B^* L} \hat{\iota}_B \phi_L g^{-1} && \text{(by (1))} \\
&= g N_B g^{-1}
\end{aligned}$$

□

Lemma 1.5: *If $\varepsilon \in End_{\mathbb{Q}}(A)$ is a symmetric idempotent, so is $g\varepsilon g^{-1}$.*

Proof:

$$\begin{aligned}
(g\varepsilon g^{-1})' &= \phi_L^{-1}(g\varepsilon g^{-1})^\wedge \phi_L \\
&= (\phi_L g^{-1})^{-1} \hat{\varepsilon} (\phi_L g^{-1}) && \text{(by (1))} \\
&= g\phi_L^{-1} \hat{\varepsilon} \phi_L g^{-1} \\
&= g\varepsilon g^{-1}
\end{aligned}$$

□

Hence the action of G on the set of abelian subvarieties corresponds to conjugation on the set of symmetric idempotents of $\mathrm{End}_{\mathbb{Q}}(A)$. In particular we have with the notation as above

Corollary 1.6: $gB = B$ *if and only if* $g\varepsilon_B g^{-1} = \varepsilon_B$.

Theorem 1.1 (a) follows by induction from the subsequent lemma.

Lemma 1.7.: *Let B be a G-stable abelian subvariety of A. There exists a G-stable abelian subvariety C of A, such that the addition map induces an isogeny $\mu : B \times C \to A$.*

Proof: Since B is G-stable, ε_B is G-invariant by Corollary 1.6. Hence $1 - \varepsilon_B$ is a G-invariant symmetric idempotent. Hence again by Corollary 1.6 the abelian subvariety C

corresponding to $1-\varepsilon_B$ is G-stable. Moreover $\varepsilon_B+\varepsilon_C=1$ implies that $\mu : B\times C\to A$ is an isogeny. $\square$

Proof of Theorem 1.1 (b): Suppose $A_1\times\ldots\times A_r\to A$ and $B_1\times\ldots\times B_s\to A$ are 2 isogenies with G-simple abelian subvarieties A_i, B_i of A. By Remark 1.3 there is a surjective G-homomorphism $A_1\times\ldots\times A_r\to B_1$. Now A_j being G-simple for all j there is an A_i, which without loss of generality we may assume to be A_1, such that the restriction $A_1\to B_1$ is a G-isogeny.
Renumerate the A_i and B_j such that $A_1,\ldots,A_{r_1}$ are and $A_{r_1+1},\ldots,A_r$ are not G-isogenous to A_1 and similarly $B_1,\ldots B_{s_1}$ are and $B_{s_1+1},\ldots,B_s$ are not G-isogenous to A_1. Then the G-isogeny $A_1\times\ldots\times A_r\to B_1\times\ldots\times B_s$ splits into the product of 2 isogenies $f_i : A_1\times\ldots\times A_{r_1}\to B_1\times\ldots\times B_{s_1}$ and $f_2 : A_{r_1+1}\times\ldots\times A_r\to B_{s_1+1}\times\ldots\times B_s$. Since all of $A_1,\ldots,A_{r_1},B_1,\ldots B_{s_1}$ are G-isogenous to each other we get $r_1=s_1$ and the assertion follows by induction. $\square$

The following example shows that the G-simple abelian subvarieties A_i in Theorem 1.1 are not uniquely determined in general.

Example 1.8: Let $E_i=\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ be the elliptic curve with automorphism group $\mathrm{Aut}E_i=<i>\simeq\mathbb{Z}/4\mathbb{Z}$ and $A=E_i\times E_i$ with the product polarization L. Then G acts on (A,L) and

$$\mu : (E_i\times\{0\})\times(\{0\}\times E_i)\to A$$

is a decomposition into G-simple abelian subvarieties. On the other hand the diagonal Δ and the antidiagonal $\Xi=\{(x,-x)\in A\,|\,x\in E_i\}$ are G-simple abelian subvarieties giving a G-decomposition

$$\tilde{\mu} : \Delta\times\Xi\to A.$$

Note that the induced polarizations ι_Δ^*L and ι_Ξ^*L are of degree 2.

Let $\mu : A_1\times\cdots\times A_r\to A$ be as in Theorem 1.1. Renumerating we may assume that there is a sequence of integers $0=r_0<r_1<r_2<\ldots<r_{s-1}<r_s=r$ such that $A_{r_i+1},\ldots,A_{r_{i+1}}$ are G-isogenous to each other but not to the other A_j for $i=0,\ldots,s-1$. Defining B_i to the abelian subvariety of A generated by $A_{r_{i-1}+1},\ldots,A_{r_i}$ for $i=1,\ldots,s$ we obtain a G-decomposition

$$\nu : B_1\times\ldots\times B_s\to A.$$

with $\mathrm{Hom}_G(B_i,B_j)=0$ for $i\neq j$.

Proposition 1.9: *The decomposition $\nu : B_1\times\ldots\times B_s\to A$ is uniquely determined.*

Proof: Suppose $\nu' : B_1'\times\ldots\times B_t'\to A$ is a second such decomposition. According to Remark 1.3 there is a G-isogeny $f : B_1'\times\ldots\times B_t'\to B_1\times\ldots\times B_s$. Now $\mathrm{Hom}_G(B_i,B_j)=0$ for $i\neq j$ implies that $f(B_1')$ is exactly one of the components, say B_1 of $B_1\times\ldots\times B_s$. Similarly $\mathrm{Hom}_G(B_i',B_j')=0$ for $i\neq j$ implies $f(B_1')\cap f(B_i')=0$ for $i\geq 2$. Hence f splits into a product

$$f=f_1\times f_2$$

with G-isogenies $f_1 : B_1'\to B_1$ and $f_2 : B_2'\times\ldots\times B_t'\to B_2\times\ldots\times B_s$. Hence $B_1'\sim B_1\sim A_1^{k_1}$ with a G-simple abelian subvariety A_1. If $B_1\neq B_1'$ as abelian subvarieties of A, we would have

$$B_1'+B_1\sim A_1^{e_1}$$

with $e_1 > k_1$, contradicting the uniqueness statement of Theorem 1.1. Now the assertion follows by induction. □

Proposition 1.10: *For any B_i in Proposition* 1.9 *there is a simple abelian variety E_i uniquely determined up to isogeny and a positive integer n_i such that*

$$B_i \sim E_i^{n_i}.$$

Proof: It suffices to prove the analogous statement for any G-simple abelian subvariety A_i of B_j. Let E_i denote any simple abelian subvariety of A_i. The sum $\sum_{g \in G} gE_i$ is a G-simple abelian subvariety of A_i and hence coincides with A_i. This implies the assertion. □

Note that E_i and E_j in Proposition 1.10 might be isogenous for $i \neq j$.

2. The isotypical decomposition

Let the finite group G act on an abelian variety A. The action induces an algebra homomorphism

$$\rho : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(A)$$

where $\mathbb{Q}[G]$ denotes the group algebra of G over $\mathbb{Q}$. If α is any element in $\mathbb{Q}[G]$, we define

$$\mathrm{Im}\ (\alpha) := \mathrm{Im}\ (\rho(m\alpha)) \subseteq A$$

where m is some positive integer such that $m\alpha \in \mathbb{Z}[G]$. Im (α) is an abelian subvariety of A, which certainly does not depend on the chosen integer m. In order to obtain proper abelian subvarieties we have to choose suitable element α of $\mathbb{Q}[G]$.

For this recall that $\mathbb{Q}[G]$ is a semisimple $\mathbb{Q}$-algebra of finite dimension. As such it admits a unique decomposition

$$\mathbb{Q}[G] = Q_1 \times \ldots \times Q_r \tag{1}$$

with simple $\mathbb{Q}$-algebras $Q_1, \ldots, Q_r$. Let

$$1 = e_1 + \ldots + e_r \tag{2}$$

denote the decomposition of the unit element 1 in $\mathbb{Q}[G]$. The elements e_i, the unit elements of the algebras Q_i, form, when considered as elements of $\mathbb{Q}[G]$, a set of orthogonal idempotents contained in the center of $\mathbb{Q}[G]$.

Proposition 2.1: *Let $A_i = Im\ (e_i)$ for $i = 1, \ldots, r$.*

(a) *A_i is a G-stable abelian subvariety of A with $Hom_G(A_i, A_j) = 0$ for $i \neq j$.*

(b) *The addition map induces an isogeny*

$$\mu : A_1 \times \ldots \times A_r \to A.$$

This decomposition is called the *isotypical decomposition* of A. It is unique up to permutation, since the idempotents e_i are uniquely determined by $\mathbb{Q}[G]$.

Proof: (a) Let $n_i \in \mathbb{Z}$ be such that $n_i e_i \in \mathbb{Z}[G]$. The idempotent e_i being in the center of $\mathbb{Q}[G]$, we obtain

$$gA_i = g(n_i e_i)A = n_i e_i g A = n_i e_i A = A_i.$$

The second assertion follows from the fact that $e_i e_j = 0$ for $i \neq j$. Assertion (b) is an immediate consequence of (2). □

In order to obtain the isotypical decomposition of A, one proceeds as follows: The action of G induces a commutative diagram

$$\begin{tikzcd} & \mathrm{End}(T_0 A) \\ \mathbb{Q}[G] \arrow[ur, "\tilde{\rho}"] \arrow[dr, "\rho"'] & \\ & \mathrm{End}_{\mathbb{Q}}(A) \arrow[uu, "\rho_a"'] \end{tikzcd} \tag{3}$$

where $T_0 A$ denotes the tangent space of A in 0 and ρ_a is the analytic representation of $\mathrm{End}_{\mathbb{Q}}(A)$. So G acts compatibly on the tangent space $T_0 A$ and the representation $\tilde{\rho}$ is defined over $\mathbb{Q}$. Hence we can decompose $T_0 A$ into a direct sum

$$T_0 A = (W_1^{n_1} \otimes \mathbb{C}) \oplus \ldots \oplus (W_r^{n_r} \otimes \mathbb{C})$$

where W_i denotes the irreducible representation of G over $\mathbb{Q}$ corresponding to the Q-algebra Q_i (in fact W_i is a minimal left ideal of Q_i) and the n_i are non negative integers for $i = 1, \ldots, r$. The W_i being pairwise nonisomorphic, we have $\mathrm{Hom}_G(W_i, W_j) = 0$ for $i \neq j$. If Λ denotes the lattice in $T_0 A$ defining A, then the commutativity of diagram (3) implies that $(W_i^{n_i} \otimes \mathbb{C}) \cap \Lambda$ is a lattice in $W_i^{n_i} \otimes \mathbb{C}$ and

$$A_i \simeq (W_i^{n_i} \otimes \mathbb{C})/(W_i^{n_i} \otimes \mathbb{C}) \cap \Lambda.$$

In terms of representations of the group G the idempotents e_i can be explicitly determined as follows: Let $V_1, \ldots, V_s$ denote the irreducible $\mathbb{C}$-representations of the group G with corresponding characters $\chi_1, \ldots, \chi_s$. Then

$$p_j := \frac{\deg \chi_j}{|G|} \sum_{g \in G} \chi_j(g) g \in \mathbb{C}[G]$$

is a projector of $\mathbb{C}[G]$ with image the sum of all sub-representations of $\mathbb{C}[G]$ isomorphic to V_j.
According to a theorem of Brauer (see [S] Théorème 24) the field

$$K_j = \mathbb{Q}(\chi_j(g), g \in G)$$

is cyclotomic of degree say d_j over $\mathbb{C}$. In particular $K_j \,|\, \mathbb{Q}$ is a Galois extension. The conjugate representations of V_j, i.e. the representations associated to the conjugate characters $\sigma\chi_j$ ($\sigma \in \mathrm{Gal}\ (K_j|(\mathbb{Q}))$ are again irreducible over $\mathbb{C}$ and non isomorphic to each other. Hence they are among the V_i's. Let $V_{j_1} = V_j, V_{j_2}, \ldots, V_{j_{d_j}}$ be the different irreducible representations conjugate to V_j. Their projectors p_{j_ν} are pairwise orthogonal and their sum $p_{j_1} + \ldots + p_{j_{d_j}}$ is again a projector. It is a projector of $\mathbb{Q}[G]$, since

$$\begin{aligned} p_{j_1} + \ldots + p_{j_{d_j}} &= \frac{\deg \chi_j}{|G|} \sum_{g \in G} (\chi_{j_1}(g) + \ldots + \chi_{j_{d_j}}(g)) \cdot g \\ &= \frac{\deg \chi_j}{|G|} \sum_{g \in G} tr_{K_j|\mathbb{Q}}(\chi_j(g)) g \in \mathbb{Q}[G] \end{aligned}$$

Here we use the fact that all characters χ_{j_ν} are of the same degree. The image of the projector $p_{j_1} + \ldots + p_{j_{d_j}}$ in $\mathbb{C}[G]$ consists of the sum of all subrepresentations of $\mathbb{C}[G]$ isomorphic to one of the V_{j_ν}. But these are just all the irreducible components of $W_{k(j)} \otimes \mathbb{C}$ where $W_{k(j)}$ denotes the isotypical component of $\mathbb{Q}[G]$ containing V_j. Hence its image in $\mathbb{Q}[G]$ consists just of the isotypical component $W_{k(j)}^{n_{k(j)}}$ of $\mathbb{Q}[G]$. On the other hand this is just the idempotent $e_{k(j)}$ considered as a projector of $\mathbb{Q}[G]$. Hence we have

$$e_{k(j)} = p_{j_1} + \ldots + p_{j_{d_j}} = \frac{\deg \chi_j}{|G|} \sum_{g \in G} tr_{K_j|\mathbb{Q}}(\chi_j(g)) g \tag{4}$$

Now suppose L is a polarization of A invariant under the action of G. Then the isotypical components A_i correspond to symmetric idempotents ε_{A_i} as defined in Section 1. The idempotents e_i and ε_{A_i} of $\mathrm{End}_{\mathbb{Q}}(A)$ are related as follows

Proposition 2.2: $\varepsilon_{A_i} = e_i e_i'$.

Note that here e_i is considered as an element of $\mathrm{End}_{\mathbb{Q}}(A)$. (To be more precise one should write $\rho(e_i)$ instead of e_i). The element e_i' cannot necessarily be realized in $\mathbb{Q}[G]$.

Proof: Certainly e_i is an idempotent in the center of $\mathrm{End}_{\mathbb{Q}}(A)$. In particular $e_i e_i' = e_i' e_i$ which implies that $e_i e_i'$ is an idempotent. It is symmetric since " $'$ " is an anti involution. It is easy to see that Im $(e_i') =$ Im (e_i), which completes the proof. □

3. Decomposition of the projectors e_i over $\mathbb{Q}$

Let Q_i denote one of the $\mathbb{Q}$-algebras of the decomposition (1) of Section 2. As a simple $\mathbb{Q}$-algebra Q_i is a direct sum of minimal left ideals

$$Q_i = W_{i_1} \oplus \ldots \oplus W_{i_{n_i}}. \tag{1}$$

Considered as $\mathbb{Q}$-representations of the group G the W_{i_j} are all isomorphic to an irreducible $\mathbb{Q}$-representation W_i. Let

$$e_i = f_{i_1} + \ldots + f_{i_{n_i}} \tag{2}$$

be the decomposition of the idempotent e_i according to (1). The f_{i_j} are also orthogonal idempotents of $\mathbb{Q}[G]$ and give rise to an isogeny decomposition

$$B_{i_1} \times \ldots \times B_{i_{n_i}} \to A_i$$

of the isotypical component A_i of the G-abelian variety A. In order to say more about the abelian subvarieties B_{i_j} we have to determine the f_{i_j} in terms of the irreducible $\mathbb{Q}$-representation W_i.

In order to avoid indices let W denote an irreducible representation of G over $\mathbb{Q}$. By Schur's Lemma $D := \mathrm{End}^G(W)$ is a skew-field of finite dimension over $\mathbb{Q}$. The simple sub algebra of $\mathbb{Q}[G]$ defined by W (i.e. one of the algebras Q_i of Section 2 equation (1)) is then $\mathrm{End}_D(W)$.
Let $n := \dim_D W$ and denote by K the center of D. Then $[D : K] = m^2$, where m is the *Schur index* of the representation W. Decompose $W \otimes \mathbb{C}$ into a direct sum of irreducible representations and fix one of the irreducible components, say $\overline{V}$ over $\mathbb{C}$. Let L denote a field of definition, i.e. a smallest field such that there is a representation V over L with $\overline{V} = \mathbb{C} \otimes_L V$. According to a theorem of Brauer (see [S] Théorème 24) we may assume that L is a subfield of the cyclotomic field $\mathbb{Q}(\xi_e)$ where e denotes the exponent of G. In particular $L|\mathbb{Q}$ is Galois. Denote its Galois group by

$$\Sigma := \mathrm{gal}(L|\mathbb{Q}).$$

For any $\sigma \in \Sigma$ consider the L-vector space

$$L \otimes_\sigma V := L \otimes_L V$$

where the map $L \to L$ defining the tensor product is just the automorphism σ. In particular we have

$$a \otimes_\sigma (\alpha v) = (\sigma(\alpha)a) \otimes_\sigma v$$

for all $v \in V$, $a, \alpha \in L$. The L-vector space structure of $L \otimes_\sigma V$ is given in the usual way by $\alpha(a \otimes_\sigma v) = (\alpha a) \otimes_\sigma v$. Using these definitions it is easy to see that the map

$$\varphi = \varphi_\sigma : \begin{cases} V & \to & L \otimes_\sigma V \\ v & \mapsto & 1 \otimes_\sigma v \end{cases}$$

is bijective and σ-sesquilinear, i.e. $\varphi(\alpha v) = \sigma(\alpha)\varphi(v)$ for all $\alpha \in L, v \in V$.

Lemma 3.1: (1) *If $G \times V \to V, (g, v) \mapsto gv$ is the action of G on V, the commutative diagram*

$$\begin{array}{ccc} V & \xleftarrow{\varphi^{-1}} & L \otimes_\sigma V \\ g \downarrow & & \downarrow g \\ V & \xrightarrow{\varphi} & L \otimes_\sigma V \end{array}$$

defines an action of G on $L \otimes_\sigma V$. In particular $\varphi \circ g = g \circ \varphi$.
(2): *If ρ (respectively ρ_σ) denotes the matrix representation of G on V (respectively on $L \otimes_\sigma V$) with respect to the basis $\{e_1, \ldots, e_d\}$ (respectively $\{1 \otimes_\sigma e_1, \ldots, 1 \otimes_\sigma e_d\}$) then*

$$\rho_\sigma(g) = \sigma(\rho(g))$$

for all $g \in G$.

Proof: For (1) it suffices to show that $\varphi g \varphi^{-1} : L \otimes_\sigma V \to L \otimes_\sigma V$ is L-linear. But φ^{-1} is σ^{-1}-sesquilinear, g is linear and φ is σ-sesquilinear. This implies the assertion. (2) is a straight-forward computation. □

It is well-known and easy to see that the representation $\bigoplus_{\sigma\in\Sigma}(L \otimes_\sigma V)$ (where we identify $V = L \otimes_{id_L} V$) is defined over $\mathbb{Q}$ and as such is isomorphic to W. By Schur's lemma the isomorphism is uniquely determined up to a nonzero constant. We fix such an isomorphism and identify both, that is we have

$$L \otimes_{\mathbb{Q}} W = \bigoplus_{\sigma \in \Sigma} (L \otimes_\sigma V). \tag{3}$$

For abbreviation we denote the right hand side by W_L. Let χ denote the character of V. The field $K := \mathbb{Q}(\chi(g), g \in G)$ generated by the values $\chi(g)$ over $\mathbb{Q}$ coincides with the center of the algebra $\mathrm{End}^G(W)$, which was also denoted by K. The field K is a subfield of L with

$$[L : K] = m.$$

¿From this we can conclude

Proposition 3.2: $\dim_L V = m \cdot n$

Proof: $\dim_{\mathbb{Q}} W = \dim_D W \cdot [D : K][K : \mathbb{Q}] = nm^2[K : \mathbb{Q}]$. On the other hand $\dim_L L \otimes_{\mathbb{Q}} W = \dim_L V \cdot m \cdot [K : \mathbb{Q}]$. This implies the assertion. □

If $\Delta \subseteq \Sigma$ denotes the subgroup of Σ associated to the subfield K of L, it is well-known that V and $L \otimes_\sigma V$ are isomorphic as representations over L if and only if $\sigma \in \Delta$. Hence we get by Schur's Lemma

$$\mathrm{Hom}_G(L \otimes_\sigma V,\ L \otimes_\tau V) = \begin{cases} L & \sigma\tau^{-1} \in \Delta \\ & \text{if} \\ 0 & \sigma\tau^{-1} \notin \Delta. \end{cases}$$

The Galois group Σ acts on $L \otimes_{\mathbb{Q}} W$ via $\tau(a \otimes v) = (\tau(a)) \otimes v$ and we have

$$(L \otimes_\sigma W)^\Sigma = W.$$

The corresponding action of Σ on the right hand side W_L of (3) is

$$\tau((\alpha_\sigma \otimes_\sigma v_\sigma)_{\sigma \in \Sigma}) = ((\alpha_\sigma \otimes_{\tau\sigma} v_\sigma)_{\sigma \in \Sigma}).$$

Hence the action just permutes the components of elements in W_L. This implies:

Proposition 3.3: $W = (W_L)^\Sigma$.

Since both groups G and Σ act on the vector space W_L and we can speak of G– and Σ–invariant forms.

Lemma 3.4: *Up to a constant in L there is exactly one G– and Σ–invariant sesquilinear form on the vector space W_L. Choosing the constant suitably, this form is a hermitian scalar product.*

Proof: Since V is absolutely irreducible, there is up to a constant exactly one G-invariant sesquilinear from. Choosing the constant suitably, this is a hermitian scalar product, denoted by $<,>_1$ (1 stands for $1 \in \mathrm{gal}(L|\mathbb{Q})$). Since Σ permutes the components of W_L, there is at most one G- and Σ–invariant sesquilinear form on this vector space and it suffices to construct such a form. It is a straightforward computation that the diagram

$$\begin{array}{ccc} V \times V & \overset{\varphi^{-1} \times \varphi^{-1}}{\longleftarrow} & (L \otimes_\sigma V) \times (L \otimes_\sigma V) \\ {\scriptstyle <,>_1} \downarrow & & \downarrow {\scriptstyle <,>_\sigma} \\ L & \overset{\sigma}{\longrightarrow} & L \end{array}$$

defines a G-invariant hermitian scalar product on $L \otimes_\sigma V$. Define $<,>$ on $W_L \times W_L$ by

$$< (\alpha_\sigma \otimes_\sigma v_\sigma)_{\sigma\in\Sigma},\ (\beta_\sigma \otimes_\sigma w_\sigma)_{\sigma\in\Sigma} >:= \sum_{\sigma\in\Sigma} < \alpha_\sigma \otimes_\sigma v_\sigma,\ \beta_\sigma \otimes_\sigma w_\sigma >_\sigma .$$

By construction this is a G- and Σ-invariant hermitian scalar product on W_L. $\square$

Now fix a G- and Σ-invariant hermitian scalar product $<,>$ on W_L. Using Proposition 3.3 the following lemma follows from the fact that $<,>$ is Σ-invariant.

Lemma 3.5: *$<,>$ induces a scalar product on W.*

Now the proceeding is very similar as in $[M]$. Define a function $F : W_L \times W_L \times W_L \times W_L \to L$ by

$$F(x, y, a, b) := \sum_{g\in G} < x, ga >< gb, y >$$

for all $x, y, a, b \in W_L$.

Lemma 3.6: *There is a positive constant $q \in \mathbb{Q}$ such that*

$$F(x, y, a, b) = q < x, y >< b, a >$$

for all $x, y, a, b \in W_L$.

Proof: It is easy to see that F is a G- and Σ-invariant sesquilinear form in (x, y) for fixed (a, b). According to Lemma 3.4 there is a constant $Q(b, a)$ such that $F(x, y, a, b) = Q(b, a) < x, y >$. Now $Q(b, a)$ is a G- and Σ-invariant sesquilinear form in (b, a). Applying Lemma 3.4 again we obtain that there is a constant $q \in L$ such that $F(x, y, a, b) = q < x, y >< b, a >$ for all $x, y, a, b \in W_L$. In order to see that q is a positive rational number choose an orthogonal basis $\{v_1, \ldots, v_t\}$ on W_L and write $gv_1 = \sum_{j=1}^{t} \alpha_j(g)v_j$ with $\alpha_j(g) \in L$. Then $q \cdot |v_1|^4 = F(v_1, v_1, v_1, v_1) = \sum_{g\in G} |\alpha_1(g)|^2|v_1|^4$. Hence $q = \sum_{g\in G} |\alpha_1(g)|^2 > 0$, since $\alpha_1(1_G) = 1 > 0$. $\square$

The constant q does not depend on the choice of the hermitian scalar product. It will be computed in Corollary 3.10 below. Now for any $v \in V\backslash\{0\}$ consider the element

$$\tilde{v} := ((1 \otimes_\sigma v)_{v\in\Sigma}) \in W_L.$$

According to Proposition 3.3 $\tilde{v}$ is an element of the $\mathbb{Q}$-vector space W. Hence

$$h_v := \sum_{g\in G} < \tilde{v}, g\tilde{v} > g$$

is an element of $\mathbb{Q}[G]$, since $< \tilde{v}, g\tilde{v} >\in \mathbb{Q}$ by Lemma 3.5. Denoting by $tr = tr_{L|\mathbb{Q}}$ the trace of the field extension $L|\mathbb{Q}$, we have

Lemma 3.7: *Let $u, v \in V\backslash\{0\}$. Then*

(a) $\quad h_u h_v = q\, tr(< u, v >_1) \sum_{g\in G} < \tilde{u}, g\tilde{v} > g$

(b) $h_u h_v h_u = q^2\, tr^2(< u, v >_1) h_u$.

Proof of (a)

$$\begin{aligned}
h_u h_v &= \sum_{g\in G}\sum_{h\in G} < \tilde{u}, g\tilde{u} >< \tilde{v}, h\tilde{v} > gh \\
&= \sum_{g\in G}\left[\sum_{g\in G} < \tilde{u}, g\tilde{u} >< g\tilde{v}, k\tilde{v} >\right] k \\
&\quad \text{(setting } k = gh \text{ and using the } G\text{-invariance)} \\
&= \sum_{k\in G} q < \tilde{u}, k\tilde{v} >< \tilde{v}, \tilde{u} > k \qquad \text{(by Lemma 3.6)} \\
&= q\, tr(< u, v >_1) \sum_{k\in G} < \tilde{u}, k\tilde{v} > k
\end{aligned}$$

since $< \tilde{v}, \tilde{u} >= \sum_{\sigma\in\Sigma} < 1 \otimes_\sigma v, 1 \otimes_\sigma u >_\sigma = \sum_{\sigma\in\Sigma} \sigma < v, u >_1 = tr(< v, u >_1) = tr(< u, v >_1)$.
The proof of (b) is a similar computation using (a). □

Recall $[L : K] = m, [K : \mathbb{Q}] = d$. Defining

$$p_v := \frac{1}{qdm|v|_1^2}\, h_v,$$

an immediate consequence of Lemma 3.7 (a) is:

Corollary 3.8: (a) $p_v^2 = p_v$
(b) $p_u p_v = 0$ if $< u, v >_1 = 0$.

Recall that $\Delta := gal(L|K)$ and let $\sigma_1 = 1, \sigma_2, \ldots, \sigma_d \in \Sigma$ be a set of representatives of the Galois group $\overline{\Sigma} := \Sigma/\Delta$ of $K|\mathbb{Q}$, where $d = [K : \mathbb{Q}]$. Let χ_i denote the character of the representation $L \otimes_{\sigma_i} V$. So χ_1 is the character of V and $\chi_i = \sigma_i \chi_i$ for $i = 2, \ldots, m$. In Section 2 we considered the projectors $p_i = \dfrac{\deg \chi_i}{|G|} \sum_{g\in G} \chi_i(g) g \in \mathbb{C}[G]$ and showed that the image of the projector

$$e_W = p_1 + \ldots + p_d = \frac{mn}{|G|} \sum_{g\in G} tr(X_1(g))\, g \in \mathbb{Q}[G]$$

is just the isotypical component of W in $\mathbb{Q}[G]$. the following theorem is the main result of this section.

Theorem 3.9: *If* $v_1, \ldots, v_s$ $(s = mn)$ *is an orthogonal basis of* V*, then*

$$e_W = p_{v_1} + \ldots + p_{v_s}.$$

$$
\begin{aligned}
\textit{Proof:}\quad \sum_{i=1}^{s} p_{v_i} &= \frac{1}{qdm}\sum_{i=1}^{s}\frac{1}{|v_i|_1^2}\sum_{g\in G} < \widetilde{v_i}, g\widetilde{v_i} > g, \quad (\text{since } g\widetilde{v_i} = \widetilde{gv_i} \text{ by Lemma 3.1})\\
&= \frac{1}{qdm}\sum_{i=1}^{s}\frac{1}{|v_i|^2}\sum_{g\in G}\sum_{\sigma\in\Sigma}\sigma(< v_i, \sum_{j=1}^{s} g_{ji}v_j >_1)g\\
&\quad (\text{where } (g_{ij}) \text{ is the matrix of } g \text{ with respect to } \{v_1,\dots,v_s\})\\
&= \frac{1}{qdm}\sum_{g\in G}\sum_{\sigma\in\Sigma}\sum_{i=1}^{s}\sigma(g_{ii})g\\
&= \frac{1}{qd}\sum_{g\in G}\sum_{j=1}^{d}\chi_j(g)g\\
&= \frac{1}{qd}\sum_{g\in G} tr(\chi_1(g))g = \frac{|G|}{qdmn}e_W.
\end{aligned}
$$

But $\sum_{i=1}^{s} p_{v_i}$ and e_W are idempotents of $\mathbb{Q}[G]$. Hence $\frac{|G|}{qdmn} = 1$ □

An immediate consequence of the last line of the proof is

Corollary 3.10: $q = \frac{|G|}{dmn} = \frac{|G|}{\dim_{\mathbb{Q}} V}$.

4. Decomposition of isotypical components of an abelian variety.

Let G be a finite group acting on an abelian variety A. Let $\mu : A_1 \times \ldots \times A_r \to A$ denote the isotypical decomposition of Proposition 2.1. Let now A° denote one of the isotypical components A_i of A. Suppose W is the irreducible $\mathbb{Q}$-representation of G defining A° with Schur index m and invariant n $(= \dim_D W, D = \mathrm{End}^G(W))$. In this section we will use the projectors of the last section to show that if $mn > 1$ and $A^\circ \neq 0$, then A° decomposes into a product of mn mutually isogenous abelian subvarieties.

Let V be an irreducible $\mathbb{C}$-representation of G contained in $W \otimes_{\mathbb{Q}} \mathbb{C}$. If W_L and the metric $<,>$ on W_L are as in the last section, for any $v \in V \backslash \{0\}$ there is an idempotent p_v of $\mathbb{Q}[G]$.
Define an abelian subvariety B_v of A° by

$$B_v := \mathrm{Im}\ (p_v) \qquad (= \mathrm{Im}\ (n \cdot p_v) \quad \text{for suitable} \quad n \in \mathbb{Z}).$$

Lemma 4.1: *For any $v_1, v_2 \in V - \{0\}$ the abelian subvarieties B_{v_1} and B_{v_2} are isogenous.*

Proof: Assume first that $< v_1, v_2 >_1 \neq 0$ and let $n_1, n_2 \in \mathbb{Z}$ such that $f_i = n_i p_{v_i} \in \mathrm{End}(A^\circ)$ and hence $B_{v_i} = f_i(A^\circ)$ for $i = 1, 2$. Lemma 3.7. (b) implies

$$f_1 f_2(B_{v_1}) = f_1 f_2 f_1(A^\circ) = f_1(A^\circ) = B_{v_1}.$$

Hence $f_1 f_2 | B_{v_1}$ is an isogeny, and thus $f_1 : B_{v_2} \to B_{v_1}$ is surjective. Similarly one sees that $f_2 : B_{v_1} \to B_{v_2}$ is surjective. So both are isogenies. If $< v_1, v_2 >_1 = 0$, chose a $v_3 \in V$ with $< v_1, v_3 >_1 \neq 0$ and $< v_3, v_2 > \neq 0$. Then B_{v_2} is isogenous to B_{v_3} and B_{v_3} is isogenous to B_{v_2}, which implies the assertion. □

Now let $v_1, \ldots, v_{mn}$ be an orthonormal basis of V and recall the equation $e_W = p_{v_1} + \ldots + p_{v_{mn}}$ of Theorem 3.9. Considered as an equation in $\mathrm{End}_{\mathbb{Q}}(A^\circ)$, this means

$$p_{v_1} + \ldots + p_{v_{mn}} = 1_{A^\circ}. \tag{1}$$

This implies

Lemma 4.2: *The addition map induces an isogeny*

$$\mu : B_{v_1} \times \ldots \times B_{v_{mm}} \to A^\circ.$$

Proof: Equation (1) implies that μ is surjective. Hence it suffices to show that if $< v_i, v_j >_1 = 0$, then the intersection $B_{v_i} \cap B_{v_j}$ is finite. But this is a consequence of $p_{v_i} p_{v_j} = 0$ for $i \neq j$. □

Combining Lemma 4.1 and 4.2 we obtain

Proposition 4.3: *Let $A^\circ \neq 0$ be the isotypical component of an abelian variety A with G-action associated to an irreducible $\mathbb{Q}$-representation W with invariants m and n. If $mn > 1$, A° is isogenous to the (mn)-th fold product of an abelian variety B with itself: $A^\circ \sim B^{mn}$.*

Combining Propositions 2.1 and 4.3 we obtain

Theorem 4.4: *Let G be a finite group acting on an abelian variety A. Let $W_1, \ldots, W_r$ denote the irreducible $\mathbb{Q}$-representations of G with invariants m_i and n_i for $i = 1, \ldots, r$. Then there are abelian varieties $B_1, \ldots, B_r$ and an isogeny*

$$A \sim B_1^{m_1 n_1} \times \ldots \times B_r^{m_r n_r}.$$

We call this decompostition the *isogeny decomposition of A with respect to G*. Note that all the a belian varieties B_i are only determined up to isogeny. Moreover G does not neccessarily act on a component B_i, not even on its image in A. It acts however on the image of $B_i^{m_i n_i}$ in A, which is just the isotypical component A_i of A corresponding to the $\mathbb{Q}$-representation W_i. Note moreover that the abelian varieties B_i might be zero. For example if one starts with an isotypical component A_i, its isogeny decomposition is just $A_i \sim B_i^{m_i n_i}$. So even one gets a proper decomposition if only $m_i n_i > 1$.

5. Jacobians with group action

In this section we apply the previous results to the Jacobians of algebraic curves.

A curve X is assumed to be smooth, irreducible, projective and defined over $\mathbb{C}$ and of genus $g_X \geq 2$. We will denote by $JX = H^0(X, \omega_X)^\vee / H_1(X, \mathbb{Z})$ its Jacobian variety, which is canonically isomorphic to $\mathrm{Pic}^0 X = H^1(X, \mathcal{O}_X) / (H^1(X, \mathbb{Z})$. The intersection product on

$H_1(X,\mathbb{Z})$ gives a canonical principal polarization $\phi_\Theta : JX \overset{\approx}{\to} \widehat{JX}$. By an automorphism $g : JX \to JX$ we will understand a polarization preserving one.

Let A be an abelian variety and $E \in H^2(A,\mathbb{Z}) \cong \Lambda^2 H^1(A,\mathbb{Z})$ a polarization, i.e. the Chern class of a line bundle L on A, denote by $H(v,w) = E(iv,w) + iE(v,w)$ the corresponding positive definite hermitian form in T_0JX $\quad(\Lambda \cong H^1(A,\mathbb{Z})$ and $E(iv,iw) = E(v,w)$ since $E \in H^{1,1})$. Then $g \in \mathrm{Aut}(A)$ preserves polarization if and only if $H(gv,gw) = H(v,w)$ for all $v,w \in T_0JX$.

If a curve X has a non trivial group of automorphisms, then by Torelli's theorem we have:

$$\mathrm{Aut}(X) = \begin{cases} \mathrm{Aut}(JX) & \text{if } X \text{ is hyperelliptic} \\ \mathrm{Aut}(JX)/\{\pm 1\} & \text{if not} \end{cases}$$

One knows that $|\mathrm{Aut}(X)\,| \leq 84(g_X - 1)$ and also that any finite group G is the group of automorphisms of a compact Riemann surface (see [H]). One can easily see this as follows:

1. The action of G on itself represents $G \subset S_n$ as transitive subgroup of the symmetric group ($n = |G|$ and we may assume $n \geq 3$, since for $n = 2$ the hyperelliptic curves give an example).

2. Since $\pi_1(\mathbb{P}^1_{\mathbb{C}} - \{a_1,\ldots,a_s\}) = <\sigma_1,\ldots,\sigma_s \,|\, \sigma_1 \ldots \sigma_s = 1>$, we obtain a surjective homomorphism $f_1 : \pi_1(\mathbb{P}^1_{\mathbb{C}} - \{a_1,\ldots,a_{2n}\}) \to S_n$ as follows: $\quad f_1(\sigma_{2i-1}) = f_1(\sigma_{2i}) = (1, i+1)$ for $i = 1,\ldots,n-1$ and $f_1(\sigma_{2n-1}) = f_1(\sigma_{2n}) = (1,n)$. So we get a cover $f : X \to \mathbb{P}^1_{\mathbb{C}}$ with simple ramification over $a_1,\ldots,a_{2n}$, whose associated Galois extension
$$\begin{tikzcd} \widetilde{X} \arrow[dd, "h"'] \arrow[dr] & \\ & X \arrow[dl] \\ \mathbb{P}^1 & \end{tikzcd}$$
has group $\mathrm{gal}(\widetilde{X} \,|\, \mathbb{P}^1_{\mathbb{C}}) = S_n$. So G appears as a a group of automorphisms of $\widetilde{X}$.

Let us now consider a curve X with group action $G \subset \mathrm{Aut}X$, so G acts on JX and we want to describe the isotypical decomposition

$$JX \sim A_1 \times \ldots \times A_r.$$

That is, according to Section 2, if

$$T_0JX = H^1(X,\mathcal{O}_X) = (W_1^{\alpha_1} \otimes \mathbb{C}) \oplus \ldots \oplus (W_r^{\alpha_r} \otimes \mathbb{C}),$$

where W_i denote the irreducible $\mathbb{Q}$-representations of G and if $\Lambda = H^1(X,\mathbb{Z})$, then from Proposition 2.1 we know that

$$A_i = (W_i^{\alpha_i} \otimes \mathbb{C})/(W_i^{\alpha_i} \otimes \mathbb{C}) \cap \Lambda \quad i = 1,\ldots,r$$

are abelian subvarieties of JX and the addition map

$$\mu : A_1 \times \ldots \times A_r \to JX$$

is an isogeny.

Recalling from Theorem 4.4 that if m_i and n_i are the invariants of the representation W_i, then A_i is isogenous to the $m_i n_i$-fold product of an abelian subvariety B_i with itself, $A_i \sim B_i^{m_i n_i}$. So the isogeny decomposition of JX is:

$$JX \sim B_1^{m_1 n_1} \times \ldots \times B_r^{m_r n_r}. \tag{1}$$

Our aim is to describe the abelian varieties B_i in terms of the curve X, the group action. First we consider the question which of the B_i's are nonzero. We will show that if the genus of the quotient $Y = X/G$ is ≥ 2, then $B_i \neq 0$ for all i. For this we need a result from [SA2]: Let G be a finite group and $V_1, \ldots, V_s$ the irreducible representations of G over $\mathbb{C}$. (In fact, the following theorem is valid over any algebraically closed field k with char $k \not| |G|$.)

Theorem 5.1. [SA2] *Let X be a projective algebraic variety over the field of complex numbers with an action of G. Denote by $Y = X/G$ the quotient variety and by $\pi : X \to Y$ the covering map. Then*

$$\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus (V_1 \otimes_{\mathbb{C}} \mathcal{E}_{V_1}) \oplus \ldots \oplus (V_s \otimes \mathcal{E}_{V_s})$$

where the $\mathcal{E}_{V_i}$ are locally free $\mathcal{O}_Y$-modules such that $rk\mathcal{E}_{V_i} = \dim V_i$.

So once we know that $\pi_* \mathcal{O}_X$ is the regular representation of G, since $H^i(X, \mathcal{O}_X) \cong H^i(Y, \pi_* \mathcal{O}_X)$, $\quad i \geq 0$, then $H^0(X, \mathcal{O}_X) \cong H^0(Y, \pi_* \mathcal{O}_X)$ so $H^0(Y, \mathcal{E}_{V_j}) = \{0\}$ for $j = 1, \ldots, s$ and we also know that

$$H^1(X, \mathcal{O}_X) \cong H^1(Y, \pi_* \mathcal{O}_X) \cong H^1(Y, \mathcal{O}_Y) \oplus \bigoplus_{j=1}^{s} H^1(Y, \mathcal{E}_{V_j})^{\dim V_j}.$$

Now observe that $W_i \otimes \mathbb{C}$ is the direct sum

$$W_i \otimes \mathbb{C} = V_{j_1} \oplus \ldots \oplus V_{j_{s_i}}$$

of some of the V_j's (to each j there is a unique i), so if each $h^1(Y, \mathcal{E}_{V_j}) > 0$, then all representations W_i will appear in the isotypical decomposition of $T_0 JX$.

Proposition 5.2 *Given a Galois cover of curves $\pi : X \to Y$, with group G, all irreducible $\mathbb{Q}$-representations of G appear in the isotypical decomposition of JX if $g_Y \geq 2$.*

Proof. From the structure theorem we mentioned before, the result follows if $h^1(Y, \mathcal{E}_{V_j}) > 0$ for $j = 1, \ldots, s$, where $V_1, \ldots, V_s$ are the irreducible $\mathbb{C}$-representations of G. By Grothendieck duality (see [K]) one knows that $(\pi_* \mathcal{O}_X)^\vee \cong \pi_* \omega_{X/Y}$, where $\omega_{X/Y}$ denotes the relative dualizing sheaf, and since π is a finite morphism, we know that $\pi_* \omega_{X/Y}$ is nef (see [V]), hence all quotients of $(\pi_* \mathcal{O}_X)^\vee$ have non negative degree.

Since

$$(\pi_* \mathcal{O}_X)^\vee \cong \mathcal{O}_Y \oplus \bigoplus_{j=1}^{s} (V_j \otimes_{\mathbb{C}} \mathcal{E}_{V_j})^\vee$$

it follows that $\deg(\mathcal{E}_{V_j}^\vee) \geq 0$. So from Riemann-Roch we obtain

$$h^1(Y, \mathcal{E}_{V_j}) = -\deg \mathcal{E}_{V_j} + \dim V_j (g_Y - 1) > 0$$

if $g_Y > 1$. $\square$

We may assume that each B_i is an abelian subvariety of JX. B_i may be considered as the image of an endomorphism of JX. We give now correspondences on X which give such endomorphisms.

So, given our Galois cover of curves $\pi : X \to Y$ with Galois group G, for each $g \in G$ we consider its graph:

$$\Gamma_g = \{(x, gx);\ x \in X\} \subset X \times X.$$

Given any element $p = \sum_{g \in G} n_g g \in \mathbb{Z}[G]$ we consider the divisor

$$D := \sum_{g \in G} n_g \Gamma_g \in \mathrm{Div}(X \times X)$$

which gives a correspondence of X:

$$X \to \mathrm{Div}^{\sum n_g}(X), \quad x \mapsto \sum_{g \in G} n_g g(x)$$

hence

$$X \to \mathrm{Pic}^{\sum n_g}(X).$$

If we choose an isomorphism $\mathrm{Pic}^{\sum_{g \in g} n_g}(X) \cong JX$, by universality we get an endomorphism:

$$\gamma_D : JX \to JX.$$

An example of this is when $n_g = 1$ for all $g \in G$. In this case we have that $\sum_{g \in G} gx = \pi^*(\pi x)$, so it follows that $\mathrm{Im}\ \gamma_D = \pi^* JY$ for $D = \sum_{g \in G} \Gamma_g$,

¿From now on we will use the above notation and the one in Section 3, that is: $D_i = \mathrm{End}^G(W_i), n_i = \dim_{D_i} W_i, K_i = \text{Center } D_i, m_i^2 = [K_i : \mathbb{Q}], V_i$ an irreducible complex representation belonging to W_i with field of definition $L_i, \mathbb{Q} \subset K_i \subset L_i \subset D_i$, $[D_i : L_i] = m_i$, $[K_i : \mathbb{Q}] = d_i$ and $\Sigma_i = gal(L_i|\mathbb{Q})$.

Given $v_i \in V_i - \{0\}$ we defined:

$$h_{v_i} = \sum_{g \in G} < \tilde{v}_i, g\tilde{v}_i >^i g \in \mathbb{Q}[G]$$

where $<,>^i = \text{diag} <,>^i_\sigma, \sigma \in \Sigma_i$ and $<,>^i_1$ the G-invariant hermitian scalar product on V_i and the corresponding idempotent

$$p_{v_i} = \frac{1}{q_i \cdot d_i \cdot m_i \cdot |v_i|^2} h_{v_i}.$$

In the isogeny decomosition (1) of JX we may assume $B_i = \mathrm{Im}\ (p_{v_i}) \subset JX$ for all i. Moreover we always assume $W_1 = U$, the trivial representation of G, so $m_1 = n_1 = 1$.

We now describe the subvarieties $B_i = \mathrm{Im}\ (p_{v_i})$ in terms of the curve X:

Proposition 5.3 *Let $b_i \in \mathbb{N}^+$ be such that* $n_g^i := \dfrac{b_i}{q_i \cdot d_i \cdot m_i \cdot |v_i|^2} < \tilde{v}_i, g\tilde{v}_i > \in \mathbb{Z}$ *for all* $g \in G$. *Then* $\mathrm{Im}\ \gamma_{\Sigma n_g^i \Gamma_g} = \mathrm{Im}\ (p_{v_i})$.

Proof. The morphism $X \to \mathrm{Pic}^{\Sigma n_g^i}(X), \quad x \mapsto \mathcal{O}_X(\Sigma n_g^i g(x))$ clearly induces $mp_{v_i} : JX \to JX$ for some integer $m > 0$. □

Let us consider now a morphism of curves $f : X \to Y$ of degree d. We define the *complement* of f^*JY on JX with respect to the principal polarization of JX as

$$P := P(X/Y) := \ker(\widehat{f^*} \circ \phi_\Theta)^\circ = \ker(Nmf)^\circ.$$

In many papers (see e.g. [M]) P is also called the *Prym variety* of f. Denote by L_P the induced polarization on P and observe that the induced polarization on JY is $L_{\Theta_Y}^{\otimes d}$ (see [LB]). The map $\sigma : (JY, L_Y^d) \times (P(X|Y), L_P) \to (JX, L_{\Theta_X}) \quad (a,b) \to f^*a + b$ is an isogeny of polarized abelian varieties. This leads to the following

Observation 5.4. *Given the tower of curves*

$$X \overset{f_1}{\to} X_1 \overset{f_2}{\to} X_2 \to \ldots \overset{f_r}{\to} X_r$$

if $g_i = f_i \circ \ldots \circ f_1$, then the decomposition

$$T_0JX = T_0P(X/X_1) \oplus (dg_1^*)_0 T_0P(X_1/X_2) \oplus \ldots \oplus (dg_{r-1}^*)_0 T_0P(X_{r-1}/X_r) \oplus (dg_r^*)_0 T_0JX_r$$

is H-orthogonal, where H is the hermitian from in T_0JX corresponding to Θ_X.

In the spirit of Proposition 5.3 we can say more about the role of the trivial representation U of G. It is easy to see that the isotypical decomposition

$$T_0JX \cong U^{\alpha_U} \oplus \bigoplus_{i=2}^{r} (W_i^{\alpha_i} \otimes_{\mathbb{Q}} \mathbb{C})$$

is H-orthogognal with respect to the hermitian product H on T_0JX corresponding to Θ_X. On the other hand if $\pi : X \to Y = X/G$ as above, by Observation 5.4 the decomposition

$$T_0JX \cong (d\pi^*)_0(T_0JY) \oplus T_0P(X/Y)$$

is H-orthogonal. Since we have seen above that

$$(d\pi^*)_0(T_0JY) \cong U^{\alpha_U}$$

it follows that

$$T_0P(X/Y) \cong \bigoplus_{i=2}^{r} (W_i^{\alpha_i} \otimes_{\mathbb{Q}} \mathbb{C}).$$

We call this observation *orthogonal cancellation.*

¿From this it follows that $\pi^*JY = A_1$ and $P(X/Y) \sim A_2 \times \ldots \times A_r$ with $A_1 = B_1 = \mathrm{Im} \sum_{g \in G} g$ and $\alpha_U =$ genus Y.

One can also get information about the abelian subvarieties $B_i \subset JX$ for the non trivial representations of G. For this let $N < G$ be a subgroup, $N = X/N$ and $\varphi : X \to N$ the quotient map.

For $g \in G$ we consider the divisor

$$\overline{\Gamma}_g := (\varphi \times \varphi)_* \Gamma_g \in \text{ Div}(N \times N).$$

Since for $(x, gx), (y, gy) \in \Gamma_g$, $\varphi(x) = \varphi(y) \Longleftrightarrow \exists\, h \in N$ such that $y = hx$, $\overline{\Gamma}_g$ induces the map:

$$g : Z \longrightarrow \text{Div}^{|H|}(Z) \qquad\qquad z \mapsto g(z) = \sum_{h \in H} \varphi(ghx)$$

where $x \in X$ is such that $\varphi(x) = z$.

Given a divisor $D = \sum_{g \in G} n_g \Gamma_g \in \text{Div}(N \times N)$, consider the divisor $\overline{D} = \sum_{g \in G} n_g \overline{\Gamma}_g \in \text{Div}(N \times N)$.

Denote by $\gamma_D \in \text{End}(JX)$ and $\gamma_{\overline{D}} \in \text{End} JN$ the induced endomorphisms.

Proposition 5.5. *If for each $g \in G$ one has that $n_g = n_{gh} = n_{hg}$ for all $h \in H$, then the following diagram is commutative:*

$$\begin{array}{ccc}
JX & \xrightarrow{\gamma_D} & JX \\
{\scriptstyle Nm\varphi} \downarrow & & \downarrow {\scriptstyle Nm\varphi} \\
JN & \xrightarrow{\frac{1}{|H|}\gamma_{\bar{D}}} & JN \\
{\scriptstyle \varphi^*} \downarrow & & \downarrow {\scriptstyle \varphi^*} \\
JX & \xrightarrow{\gamma_D} & JX
\end{array}$$

In particular, Im $\gamma_D = \varphi^*$Im $\gamma_{\overline{D}} \subset \varphi^* JN$.

Proof. Given $x \in X$, let $z = \varphi(x)$. The commutativity of the upper square follows from:

$$\sum_{g \in G} n_g g(z) = \sum_{g \in G} n_g \sum_{h \in N} \varphi(ghx) = \sum_{h \in N} \sum_{g \in G} n_g \varphi(ghx) = |N| \cdot \sum_{g \in G} n_g g(x)$$

the last equality follows from the conditions satisfied by the coefficients. A similar computation shows that the lower square is commutative. $\square$

Corollary 5.6 *Let W_i be a nontrivial irreducible representation of G defined over $\mathbb{Q}$ and V_i an irreducible complex representation belonging o W_i. Assume that for $v_i \in V_i - \{0\}, hv_i = v_i$ for all $h \in H$. With the notation of Proposition* 5.3, *let $D = \sum_{g \in G} n_g^i \Gamma_g$. Then*

$$B_i = \text{Im } \gamma_D \subset \varphi^* P(X_N/Y).$$

Proof. By definition Im γ_D = Im p_{v_i}. Since $p_{v_i}\left(\sum_{g \in G} g\right) = 0$ in $\mathbb{Q}[G]$, it follows that Im $p_{v_i} \subset A_2 \times \ldots \times A_r$, hence by orthogonal cancellation Im $\gamma_D \subset P(X/Y)$.

By Proposition 5.5 Im $\gamma_D \subset \varphi^* JZ$, hence Im $\gamma_D \subset (\varphi^* JZ \cap P(X/Y))^0 = \varphi^* P(Z/Y)$. $\square$

Observe that we have $C_i \sim$ Im $\gamma_{\bar{D}} \subset P(Z/Y)$. So a natural question arises here:

Given the decomposititon (1)

$$JX \sim JY \times B_2^{m_2 n_2} \times \ldots \times B_r^{m_r n_r}.$$

Is there a similar decomposition for $P(X_N/Y)$? Observe that this is obvious if the abelian varieties B_i are simple. However this need not be the case. The following theorem gives an answer to this question in the slightly more general case of an intermediate cover.

Theorem 5.7. *Given a Galois cover $\pi : X \to Y = X/G$ and subgroups $M < N < G$, let $\psi : X_M \to X_N$ be the associated cover where $X_N = X/N$ and $X_M = X/M$. Let $JX \sim JY \times B_2^{m_2 n_2} \times \ldots \times B_r^{m_r n_r}$ be the decomposition of above. Then there exist non negative integers $s_i, i = 2, \ldots, r$ such that*

$$P(X_M/X_N) \approx B_2^{s_2} \times \ldots \times B_r^{s_r}.$$

Before proving the theorem, let us compute the "difference" between the fixed subspaces $(T_0JX)^N \subset (T_0JX)^M$ in terms of the isotypical decomposition

$$T_0JX = \bigoplus_{i=1}^{r} (W_i^{\alpha_i} \otimes_{\mathbb{Q}} \mathbb{C}).$$

So as before for each $i \in \{2, \ldots, r\}$ let V_i be an irreducible complex representation belonging to $W_i, v_i \in V_i - \{0\}$ and

$$h_{v_i} = \sum_{g \in G} < \tilde{v}_i, g\tilde{v}_i >^i g \in \mathbb{Q}[G].$$

Lemma 5.8. $h_{v_i}(W_i \otimes_{\mathbb{Q}} \mathbb{C}) = \bigoplus_{\sigma \in \Sigma_i} < 1 \otimes_\sigma v_i > \subset \bigoplus_{\sigma \in \Sigma_i} (L_i \otimes_\sigma V_i)$

Proof. Since G acts diagonally on $\bigoplus_{\sigma \in \Sigma_i} (L_i \otimes_\sigma V_i)$, it suffices to prove that for any $\tau \in \Sigma_i$

$$h_{v_i}(L_i \otimes_\tau V_i) =< 1 \otimes_\tau v_i >\subset L_i \otimes_\tau V_i.$$

So let $\overline{w} = (0, \ldots, 0, w, 0, \ldots 0) \in L_i \otimes_\tau V_i$ then

$$\begin{aligned} h_{v_i}(\overline{w}) &= \sum_{g \in G} < \tilde{v}_i, g\tilde{v}_i >^i g\overline{w} \\ &= \sum_{g \in G} ((< 1 \otimes_\sigma v_i, g(1 \otimes_\sigma v_i) >_\sigma^i g\overline{w}_\sigma)_{\sigma \in \Sigma_i}) \\ &= \sum_{g \in G} (0, \ldots, 0, < 1 \otimes_\tau v_i, g(1 \otimes_\tau v_i) >_\tau^i gw, 0, \ldots, 0) \\ &= (0, \ldots, 0, \sum_{g \in G} < 1 \otimes_\tau v_i, g(1 \otimes_\tau v_i) >_\tau^i gw, 0, \ldots, 0) \end{aligned}$$

So $h_{v_i}(\overline{w}) \in L_i \otimes_\tau V_i$. For arbitrary $z \in L_i \otimes_\tau V_i$, we have using Lemma 3.6:

$$\begin{aligned} < h_{v_i}(\overline{w})_\tau, z >_\tau^i &= < h_{v_i}(\overline{w}), \bar{z} >^i \\ &= \sum_{g \in G} < \tilde{v}_i, g\tilde{v}_i >^i < g\overline{w}, \bar{z} >^i \\ &= q_i < \tilde{v}_i, \bar{z} >^i < \overline{w}, \tilde{v}_i >^i \\ &= q_i < 1 \otimes_\tau v_i, z >_\tau^i < w, 1 \otimes_\tau v_i >_\tau^i \end{aligned}$$

So $$h_{v_i}(\overline{w})_\tau = q_i < w, 1 \otimes_\tau v_i >_\tau^i \cdot 1 \otimes_\tau v_i.$$ □

Observe now that if we use a hermitian metric H to decompose T_0JX into a sum of irreducible complex representations, we may assume that H is diagonal on $T_0JX = \bigoplus_{i=1}^{r}(W_i^{\alpha_i} \otimes_{\mathbb{Q}} \mathbb{C})$, H restricted to $W_i \otimes_{\mathbb{Q}} \mathbb{C}$ is diagonal on $W_i \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma\in\Sigma_i}(L_i \otimes_\sigma V_i)$ and that $H|_{V_i\times V_i} =<,>_1^i$. So there are constants $c_\sigma^i \in \mathbb{R}_{>0}$ such that $H|_{L_i\otimes_\sigma V_i \times L_i\otimes_\sigma V_i} = c_\sigma^i \cdot <,>_\sigma^i, \sigma \in \Sigma_i$.

Let us consider now the subspaces:

$$V_i^N \subset V_i^M \subset V_i$$

and choose an orthogonal basis $v_i^1, \dots, v_i^{l_i}, v_i^{l_i+1}, \dots, v_i^{l_i+s_i}, v_i^{l_i+s_i+1}, \dots, v_i^{m_i n_i}$ for V_i in such a way that:

$$< v_i^1, \dots, v_i^{l_i} >= V_i^N$$

and

$$< v_i^1, \dots, v_i^{l_i}, v_i^{l_i+1}, \dots, v_i^{l_i+s_i} >= V_i^M.$$

¿From the way the action of G on $L_i \otimes_\tau V_i$ is defined, it follows that we get an orthogonal basis:

$$< 1 \otimes_\sigma v_i^1, \dots, 1 \otimes_\sigma v_i^{l_i} >= L_i \otimes_\sigma V_i^N = (L_i \otimes_\sigma V_i)^N$$

and

$$< 1 \otimes_\sigma v_i^1, \dots, 1 \otimes_\sigma v_i^{l_i+s_i} >= L_i \otimes_\sigma V_i^M = (L_i \otimes_\sigma V_i)^M.$$

That is for each $\sigma \in \Sigma_i$

$$(L_i \otimes_\sigma V_i)^M = (L_i \otimes_\sigma V_i)^N \oplus \left(\bigoplus_{k=1}^{s_i} < 1 \otimes_\sigma v_i^{l_i+k} > \right)$$

and the sum is orthogonal with respect to $<,>_\sigma^i$, hence also with respect to $c_\sigma^i \cdot <,>_\sigma^i$.

So we get:

$$(W_i \otimes_{\mathbb{Q}} \mathbb{C})^M = (W_i \otimes_{\mathbb{Q}} \mathbb{C})^N \oplus \left(\bigoplus_{\sigma\in\Sigma_i} \left(\bigoplus_{k=1}^{s_i} < 1 \otimes_\sigma v_i^{l_i+k} > \right) \right)$$

the sum being orthogonal with respect to $<,>^i = \mathrm{diag}_{\sigma\in\Sigma_i}(<,>_\sigma^i)$, hence with respect to the restriction of the metric H, which is $\mathrm{diag}_{\sigma\in\Sigma_i}(c_\sigma^i \cdot <,>_\sigma^i)$. Here we use the following fact: Let H and H' be Hermitian scalar products on the complex vector spaces V and W, respecively. Then for subvector spaces V_i of V and W_i of W for $i = 1, 2$ $V_1 \oplus W_1$ is orthogonal to $V_2 \oplus W_2$ with respect to $H + \lambda H'$ for any $\lambda \in \mathbb{R}_{>0}$ if and only if $V_1 \perp V_2$ and $W_1 \perp W_2$.

Using Lemma 5.8 we can write:

$$(W_i \otimes_{\mathbb{Q}} \mathbb{C})^M = (W_i \otimes_{\mathbb{Q}} \mathbb{C})^N \oplus \left(\bigoplus_{k=1}^{s_i} p_{v_i^{l_i+k}}(W_i \otimes_{\mathbb{Q}} \mathbb{C}) \right)$$

and thus

$$(W_i^{\alpha_i} \otimes_{\mathbb{Q}} \mathbb{C})^H = (W_i^{\alpha_i} \otimes_{\mathbb{Q}} \mathbb{C})^N \oplus \left(\bigoplus_{k=1}^{s_i} p_{v_i^{l_i+k}}(W_i^{\alpha_i} \otimes_{\mathbb{Q}} \mathbb{C}) \right)$$

the sum being orthogonal with respect to H.

Observe now that if $1 = e_1 + \ldots + e_r$ is the decomposition into idempotents $e_i = \sum_{k=1}^{m_i \cdot n_i} p_i^k$, we have that $e_i \cdot e_j = 0$, since $p_{v_i^k} \cdot e_j = 0$ for $i \neq j$. From this it follows that

$$p_{v_i^k}(T_0 JX) = p_{v_i^k}(W_i^{\alpha_i} \otimes_{\mathbb{Q}} \mathbb{C}) \quad i \in \{2, \ldots, r\}, k \in \{1, \ldots, s_i + l_i\}.$$

So we have:

$$(T_0 JX)^M = (T_0 JX)^N \oplus \left[\bigoplus_{i=2}^{r} \left(\bigoplus_{k=1}^{s_i} p_{v_i^{l_i+k}}(T_0 JX) \right) \right] \tag{3}$$

the sum being H-orthogonal.

Proof of the theorem: Given the subgroups $M < N < G$ we have the tower of curves:

$$\begin{tikzcd}
X \arrow[dr, "\varphi"] \arrow[ddd, "\pi"'] & \\
& X_M \arrow[d, "\psi"] \\
& X_N \arrow[dl, "\eta"] \\
Y &
\end{tikzcd}$$

By definition we know that

$$T_0 JX_M = (d\psi^*)_0 T_0 JX_N \oplus T_0 P(X_M/X_N)$$

hence

$$(d\varphi^*)_0 T_0 JX_M = (d\varphi^*)_0 \circ (d\psi^*)_0 T_0 JX_N \oplus (d\varphi^*)_0 T_0 P(X_M/X_N) \tag{4}$$

the sum being H-orthogonal by observation 5.4.

¿From invariant theory we know that

$$(d\varphi^*)_0 T_0 JX_M = (T_0 JX)^M$$

$$(d\varphi^*)_0 \circ (d\psi^*)_0 T_0 JX_N = (T_0 JX)^N.$$

So by orthogonal cancellation on (3) and (4) we get:

$$(d\varphi^*)_0 T_0 P(X_M/X_N) = \bigoplus_{i=2}^{r} \left(\bigoplus_{k=1}^{s_i} p_{v_i^{l_i+k}}(T_0 JX) \right).$$

Translating to JX this gives

$$\varphi^* P(X_M/X_N) \sim \prod_{i=2}^{r} \prod_{k=1}^{s_i} p_{v_i^{l_i+k}}(JX).$$

Since φ^* is an isogeny and since $\mathrm{Im}\ p_{v_i'} \sim \mathrm{Im}\ p_{v_i}$ for any $v_i, v_i' \in V_i$, recalling that $\mathrm{Im}\ p_{v_i} = B_i$ we have:

$$P(X_M/X_N) \sim \prod_{i=2}^{r} B_i^{s_i}$$

where $s_i = \dim_{\mathbb{C}} V_i^M - \dim_{\mathbb{C}} V_i^N$. $\square$

We will apply only the following corollary, which is an immediate consequence of the proof of Theorem 5.7.

Corollary 5.9. *with the notation of Theorem 5.7 and $2 \leq i \leq r$ the following statements are equivalent:*

(i) $P(X_M/X_N) \sim B_i$

(ii) $s_i = \dim V_i^M - \dim V_i^N = 1$ *and* $s_j = \dim V_j^M - \dim V_j^N = 0$ *for all* $j \neq i, 2 \leq j \leq r$.

A method for computing the decomposition of the Jacobian JX of a curve X with group action was given in [SA1] in the case $G = A_5$. Theorem 5.7 allows us to deduce a generalization of this method which will be useful for computing other examples. The starting point is the following lemma. Here for any subgroup $M < G$ we denote by $\chi_{\mathbb{C}(G/M)}$ the characteer of the induced representation of G of the trivial representation of M.

Lemma 5.10. *Let G be a finite group, $V_1, \dots, V_t$ its irreducible complex representations. Let $M < N < G$ be subgroups. If we have for some $1 \leq l \leq t$ that:*

$$\chi_{\mathbb{C}(G/M)} - \chi_{\mathbb{C}(G/N)} = \sum_{i=1}^{l} \dim_{\mathbb{C}} V_i^M \cdot \chi_{V_i}$$

then

$$V_j^N = \begin{cases} 0 & \text{if } 1 \leq j \leq l \\ V_j^M & \phantom{\text{if }} l+1 \leq j \leq t \end{cases}$$

Proof. By Frobenius reciprocity:

$$\begin{aligned} \dim_{\mathbb{C}} V_j^N &= < \chi_{V_j}, \chi_{\mathbb{C}(G/N)} > \\ &= < \chi_{V_j}, \chi_{\mathbb{C}(G/M)} - \sum_{i=1}^{l} \dim V_i^M \cdot \chi_{V_i} > \\ &= \dim V_j^M - \sum_{i=1}^{l} \dim V_i^M < \chi_{V_j}, \chi_{V_i} > \\ &= \begin{cases} 0 & \text{if } 1 \leq j \leq l \\ \dim V_j^M & \text{if } l < j \leq t \end{cases} \end{aligned}$$

$\square$

Corollary 5.11. *Let X be a curve with G as a group of automorphisms and $M < N < G$ be subgroups. Let $T_0 JX = U^{\alpha_1} \oplus \bigoplus_{i=2}^{r} (W_i^{\alpha_i} \otimes_{\mathbb{Q}} \mathbb{C})$ be the isotypical decomposition as above, and V_i an irreducible complex representation of G belonging to W_i. Assume that for some $i \in \{2, \dots, r\}$ we have that $\dim V_i^M = 1$ and that*

$$\chi_{\mathbb{C}(G/M)} - \chi_{\mathbb{C}(G/N)} = \sum_{k=1}^{d_i} \chi(L_i \otimes_{\sigma_{j_k}} V_i)$$

where as before $\{\sigma_{j_1},\ldots,\sigma_{j_{d_i}}\}$ *is a set of representatives of* $\Sigma_i/\triangle_i$.
Then

$$P(X_H/X_N) = B_i.$$

Proof. Observe first that $\dim_{\mathbb{C}}(L_i\otimes_{\sigma_{i_k}} V_i)^M = \dim V_i^M = 1,\ 1\le k\le d_i$. Hence by the previous observation we have:

$$\dim_{\mathbb{C}}(L_{i'}\otimes_\sigma V_{i'})^N = \dim_{\mathbb{C}}(L_{i'}\otimes_\sigma V_{i'})^M$$

for $i\neq i'\in\{2,\ldots,r\}$ and all $\sigma\in\Sigma_{i'}$

and
$$\dim_{\mathbb{C}}(L_i\otimes_\sigma V_i)^N = 0 \qquad \sigma\in\Sigma_i.$$

Hence $s_i=1$ and $s_{i'}=0$ for $i\neq i'\in\{2,\ldots,r\}$. So by the previous corollary we have that $P(X_H/X_M)\approx C_i$. □

Finally let us remark that it is easy to deduce other consequences in the same spirit of Corollary 5.11 from Theorem 5.7. But for the examples of Section 6 Corollary 5.10 suffices.

6. Examples

Let X be a curve with G-action, $\pi: X\to Y = X/G$ the corresponding Galois cover. Let

$$T_0JX = (W_1^{\alpha_1}\otimes_{\mathbb{Q}}\mathbb{C})\oplus\ldots\oplus(W_r^{\alpha_r}\otimes_{\mathbb{Q}}\mathbb{C})$$

be the isotypical decomposition. We will denote by A_{W_i} the isotypical components of JX and by $U=W_1$ the trivial representation of G. Since we have seen above that

$$A_U = \pi^*JY$$

and we also know that the isotypical decomposition is H-orthogonal, where H denotes the hermitian form on T_0JX given by the canonical principal polarization. Hence

$$T_0P(X/Y) = (W_2^{\alpha_2}\otimes_{\mathbb{Q}}\mathbb{C})\oplus\ldots\oplus(W_r^{\alpha_r}\otimes_{\mathbb{Q}}\mathbb{C}).$$

In fact G acts on $T_0P(X/Y)$ and the above is the isotypical decomposition of this action.

The next examples except the last one are treated elsewhere, S_3 in [RR1], S_4 in [RR2] and D_n in [CRR], but with adhoc methods, i.e. without the use of Corollary 5.11, which was inspired by [SA1], from where we took the case A_5.

6.1. The symmetric group S_4

First an observation. Let U_n, U_n' and V_n be the trivial, the sign and standard representations of the symmetric group S_n. Then if $S_{n-1}\subset S_n$ is the stabilizer of an element and $A_n\subset S_n$ is the alternating group, we have:

$$\chi_{U_n'} = \chi_{\mathbb{C}(S_n/A_n)} - \chi_{U_n} \qquad (1)$$

$$\chi_{V_n} = \chi_{\mathbb{C}(S_n/S_{n-1})} - \chi_{U_n} \qquad (2)$$

Consider now the homomorphism

$$f : S_4 \to S_3 = S_4/K,$$

where

$$K = \{1, (12)(34), (13)(24), (14)(23)\} \subset S_4$$

is the Klein subgroup, denote by $\gamma : X \to T = X/K$ the covering map. Then S_4 acts on T (as S_3 via f) and the isogeny

$$P(X/T) \times JT \to JX; \quad (x, u) \to x + \gamma^* u$$

is S_4-equivariant. Moreover, the pullbacks U, U' and W via f of the irreducible representations U_3, U_3' and V_3 of S_3, are irreducible, then by orthogonal cancellation the isotypical sum map decomposes in two maps:

$$\begin{array}{rcll} A_U \times A_{U'} \times A_W & \to & \gamma^* JT & \qquad (3) \\ A_{V_4} \times A_{V_4 \otimes U_4'} & \to & P(X/T) & \qquad (4) \end{array}$$

The first is just the S_3-action on U (see [RR1]):
Write $S_3 = <\sigma, \tau | \sigma^2 = \tau^3 = (\sigma\tau)^2 = 1>$ and let $R = T/<\sigma> = X/D_4$, where $D_4 = <\sigma, k>$ and $\Delta = T/<\tau> = X/A_4$.

Since $\dim V^{<\sigma>} = 1$ and $\chi_{U'} - \chi_{\mathbb{C}(S_3/<\tau>)} - \chi_U$, applying Corollary 5.11, we get

$$A_V = A_{V_1} \sim P(R/Y)^2$$

On the other hand, $U'^{<\tau>}_1 = U_1'$, together with $\chi_V = \chi_{\mathbb{C}(S_3/<\sigma>)} - \chi_U$ and Corollary 5.10 shows that

$$A_{U'} = A_U = P(\Delta/Y).$$

So we get an S_3-equivariant isogeny:

$$JY \times P(\Delta/Y) \times P(R/Y)^2 \overset{\sim}{\to} JT \tag{7}$$

Let us work now with the standard and antistandard representations of S_4 :
Let $S_3 = <(13), (123)> \subset S_4$, so $\dim V^{S_3} = 1$ and $W^{S_3} = U'^{S_3} = (V \otimes U')^{S_3} = \{0\}$. Hence from (2) and Corollary 5.10 we get:

$$A_V = P(S/Y)^3.$$

As for the antistandard representation, one computes the character table:

	1	(12)	(123)	(1234)	(12)(34)
$\chi_{\mathbb{C}(S_4/<(1234)>)}$	6	0	0	2	2
$\chi_{\mathbb{C}(S_4/D_4)}$	3	1	0	1	3
$\chi_{V \otimes U'}$	3	-1	0	1	-1

where $D_4 =< (1234), (13) >$.
So

$$\chi_{V\otimes U'} = \chi_{\mathbb{C}(S_4/<(1234)>)} - \chi_{\mathbb{C}(S_4/D_4)}.$$

and since $\dim(V\otimes U')^{<(1234)>} = 1$ and $V^{<(1234)>} = W^{(1234)>} = U'^{<(1234)>} = \{0\}$.
Then from Corollary 5.10 we get:

$$A_{V\otimes U'} = P(Z/R)^3$$

where $Z = X/ < (1234) >$.
So the decomposition is:

$$JY \times P(\Delta/Y) \times P(R/Y)^2 \times P(S/Y)^3 \times P(Z/R)^3 \approx JX. \qquad \square$$

6.2. The alternating group A_5.

Let us write the character table of the complex irreducible representations of A_5:

	1	(123)	(12)(34)	(12345)	(13452)
U	1	1	1	1	1
V	4	1	0	-1	-1
W	5	-1	1	0	0
Y	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
Z	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

Observe that Y and Z are not absolutely irreducible, but $Y+Z$ is defined over $\mathbb{Q}$ and irreducible with invariants $m=1$ and $n=3$,
Let $\mathbb{Z}_5 =< (12345) >$, $D_5 =< (12345), (25)(34) >$ and $A_4 =< (23)(45), (24)(35), (345) >$, then a short computation shows that

$$\dim V^{\mathbb{Z}_5} = \dim W^{D_5} = \dim Y^{A_4} = \dim Z^{A_4} = 1$$

and all other relevant invariant subspaces are zero. Moreover:

$$\begin{aligned} \chi_V &= \chi_{\mathbb{C}(A_5/A_4)} - \chi_U \\ \chi_W &= \chi_{\mathbb{C}(A_5/D_5)} - \chi_U \\ \chi_Y + \chi_Z &= \chi_{\mathbb{C}(A_5/\mathbb{Z}_5)} - \chi_{\mathbb{C}(A_5/D_5)}. \end{aligned}$$

So if X is a curve with A_5 as a group of automorphisms and $\pi : X \to Y = X/A_5$ the quotient cover map, from Corollary 5.11 we get:

$$A_V = P(X_{A_4}/Y)^4;\ A_W = P(X_{D_5}/Y)^5 \quad \text{and} \quad A_{Y+Z} = P(X_{\mathbb{Z}_5}/X_{D_5})^3.$$

So the isotypical decomposition is:

$$JY \times P(A_{A_4}/Y)^4 \times P(X_{D_5}/Y)^5 \times P(X_{\mathbb{Z}_5}/X_{D_5})^3 \sim JX$$

6.3. The dihedral group $D_n =< s, r \mid s^2 = r^n = (sr)^2 = 1 >$

Case $n = 2k$. Then D_n has two 1-dimensional representations

	1	r^h	sr^h
U	1	1	1
U'	1	1	-1

and k irreducible complex representations of dimension 2:

$$\nu_i(r^h) = \begin{pmatrix} \omega^{hi} & 0 \\ 0 & \omega^{-hi} \end{pmatrix}, \quad \nu_i(sr^h) = \begin{pmatrix} 0 & \omega^{-hi} \\ \omega^{hi} & 0 \end{pmatrix} \quad , i = 1, \ldots, k, \ \omega = e^{\frac{2\pi i}{n}}.$$

So the sum $\nu_1 + \ldots + \nu_k$ is defined over $\mathbb{Q}$ and has invariants $m = 1$ and $n = 2$. Using Corollary 5.10 as before if X is a curve with D_n as a group of automorphisms we get the decomposition

$$JY \times P(X_r/Y) \times P(X_s/Y)^2 \sim JX$$

where $Y = X/D_n, \quad X_r = X/ < r >$ and $X_s = X/ < s >$.

Case $n = 2k+1$. Then D_n has four 1-dimensional irreducible representations

	1	r^h	sr^h
U	1	1	1
U'	1	1	-1
U_3	1	$(-1)^h$	$(-1)^h$
U_4	1	$(-1)^h$	$(-1)^{h+1}$

and $k-1$ irreducible representations of dimension 2.

$$\nu_i(r^h) = \begin{pmatrix} \omega^{hi} & 0 \\ 0 & \omega^{-hi} \end{pmatrix}; \quad \nu_i(sr^h) = \begin{pmatrix} 0 & \omega^{-ih} \\ \omega^{ih} & 0 \end{pmatrix} \quad i = 1, \ldots, k-1.$$

Observe that the sum $\nu_1 + \ldots + \nu_{k-1}$ is defined over $\mathbb{Q}$ with invariants $m = 1$ and $n = 2$.

So if X is a curve with D_n as a group of automorphisms, and if $Y = X/D_n, X_s = X/ < s >, Z_s = X/ < s, r^2 >, X_r = X/ < r >$ and $Z_{rs} = X/ < rs, r^2 >$ then the isotypical decomposition of JX is:

$$JY \times P(X_r/Y) \times P(Z_s/Y) \times P(Z_{rs}/Y) \times P(X_s/Z_s)^2 \xrightarrow{\sim} JX.$$

6.4. The quaternion group $\Gamma = \{\pm 1, \pm i, \pm j, \pm ij\}, \ i^2 = j^2 = (ij)^2 = -1$. With this group we cannot apply Corollary 5.10, but we can still use orthogonal cancellation. The center of Γ is $< -1 >$ and we observe that $\Gamma / < -1 > \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Γ has four 1-dimensional irreducible complex representations with characters:

	1	-1	$i,-i$	$j,-j$	$ij,-ij$
U	1	1	1	1	1
U_i	1	1	1	-1	-1
U_j	1	1	-1	1	-1
U_{ij}	1	1	-1	-1	1

and a 2-dimensional irreducible complex representation ν:

$$i \to \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad j \to \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The associated irreducible representation over $\mathbb{Q}$ is $\mathbb{H}_{\mathbb{Q}}$, with invariants $m = 2$ and $n = 1$.

Let X be a curve with Γ as a group of automorphisms, and let $Y = X/\Gamma$, $Z = X/<-1>$, $X_i = X/<i>$, $X_j = X/<j>$ and $X_{ij} = X/<ij>$. Then we have a diagram of coverings of curves:

$$\begin{array}{ccccc} & & X & & \\ & & \downarrow & & \\ & & Z & & \\ & \swarrow & \downarrow & \searrow & \\ X_i & & X_j & & X_{ij} \\ & \searrow & \downarrow & \swarrow & \\ & & Y & & \end{array}$$

Now since $<-1> \ \triangleleft \Gamma$ is a normal subgroup, Γ acts on Z (as $\Gamma/<-1>$) hence on $P(X/Z)$ and on JZ, and the 1-dimensional irreducible complex representations of Γ, are the pullback via $\Gamma \to \Gamma/<-1> \cong \ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ of the four irreducible representations of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. So by orthogonal cancellation the isotypical decomposition of JX splits in two:

$$\begin{array}{rcl} A_U \times A_{U_i} \times A_{U_j} \times A_{U_{ij}} & \sim & JZ \\ A_\nu & \sim & P(X/Z). \end{array}$$

In the case of a $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$-action one has that (see [RR2])

$$JY \times P(X_i/Y) \times P(X_j/Y) \times P(X_{ij}/Y) \sim JZ$$

is the isotypical decomposition.
¿From Theorem 4.3 we know that there exists a subabelian variety $B \subset A$ such that:

$$B^2 \cong A_\nu \cong P(X/Z).$$

So the isotypical decomposition of JX is:

$$JY \times P(X_i/Y) \times P(X_j/Y) \times P(X_{ij}/Y) \times B^2 \sim JX.$$

References

[CW] Chevalley, C., Weil, A.: Über das Verhalten der Integrale erster Gattung bei Automorphismen des Funktionenkörpers, Hamb. Abh. 10 (1934), 358 - 361,

[CRR] Carocca, A., Recillas, S., Rodríguez, R.: The action of dihedral groups on Jacobians, Preprint, Fac. Mat. Pont. Univ. Católica de Chile.

[D] Donagi, R.: Decomposition of spectral covers, Astérisque 218 (1993), 145 - 175.

[DM] Donagi, R., Markman, E.: Spectral covers, algebraically completely integrable Hamiltonian systems, and moduli of bundles; in: Integrable Systems and Quantum groups, Springer LNM 1620 (1196), 1 - 119.

[H] Hurwitz, A.: Über algebraische Gebilde mit eindeutigen Transformationen in sich, Math. Ann. 41, (1893), 403-442.

[K] Kleiman, S. L.: Relative duality for quasi-coherent sheaves, Comp. Math. 41 (1980), 39 - 60.

[LB] Lange, H., Birkenhake, Ch.: Complex Abelian Varieties. Grundlehren No. 317, Springer (1992)

[M] Mérindol, J.-Y.: Variétés de Prym d'une revêtement galoisien. J. reine Ang. Math. *461* (1995), 49-61

[Re] Recillas, S.: La Jacobiana de la extension de Galois de una curva trigonal, Aport. Matem. Com. 14 (1994), 159 - 167.

[Ri] Ries, J.: The Prym variety for a cyclic unramified cover of a hyperelliptic curve, J. reine ang. Math. 340 (1983), 59 - 69.

[RR1] Recillas, S., Rodríguez, R.: Jacobians and representations of S_3, Aport. Mat. Invest. 13 (1998), 117 - 140.

[RR2] Recillas, S., Rodríguez, R.: Prym varieties for Galois covers, Preprint, Fac. Mat. Pont. Univ. Católica de Chile.

[S] Serre, J.-P.: Représentations linéaires des groupes finis. Herrmann (1967).

[SA1] Sánchez- Argáez, A: Acciones del grupo A_5 en variedades jacobianas, Aport. Mat. Com. 25, (1999), 99-108.

[SA2] Sánchez-Argáez, A.: Teoremas de estructura para cubrientes Galois de curvas, Ph. D. Thesis, CIMAT, México (2000).

[V] Viehweg, E.: Positivity of direct image sheaves and applications. Talks at ICTP, Trieste, (2000).

[W] Wirtinger, W.: Untersuchungen über Theta Funktionen, Teubner, Berlin (1985).

Herbert Lange
Mathematisches Institut
Bismarckstr. 1 1/2
D-91054 Erlangen
e-mail: lange@mi.uni-erlangen.de

Sevin Recillas
Instituto de Mathemáticas, UNAM
Morelia, Mich.
58190 México
email: sevin@matmor.unam.mx
and
CIMAT
Callejón Jalisco s/n
Valenciana, Gto.
36000 México
email: sevin@cimat.mx