CLASSIFICATION OF INTEGRATION PATTERNS ON $\mathbb{R}^{1|1}$

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ABSTRACT. The indefinite integration problem in $\mathbb{R}^{1|1}$ is approached by looking at those derivations on this graded manifold that admit a right inverse. The integration formulæ that arise this way are classified in terms of the action of the diffeomorphism group of $\mathbb{R}^{1|1}$. The Berezin integral, and other integral formulæ proposed in the literature for graded manifolds, arise as examples of right inverses for specific derivations. A thorough analysis of the Berezin integral is made for $(1,1)$-dimensional smooth and holomorphic graded manifolds. It is proved that a SUSY-curve carries a canonical indefinite integration defined on its Berezinian sheaf which is not given by the Berezin integral. Finally, the variational calculus is applied to the integrals here classified, and their corresponding Euler-Lagrange equations are given.

INTRODUCTION

The main purpose of this paper is to classify the various indefinite integrals that one can define in $\mathbb{R}^{1|1}$. The classification is performed by looking first at the action of the diffeomorphism group G of $\mathbb{R}^{1|1}$ on the graded Lie algebra of derivations $\operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ of the structure (graded) ring $C^\infty(\mathbb{R}^{1|1})$ of $\mathbb{R}^{1|1}$. This action has several orbits, and some of these orbits have a neat representative; namely, one that we have called a constant coefficient derivation (see **1.6** for conciseness). The set of constant coefficient derivations is a graded Lie subalgebra $\mathfrak{gl}_{1|1} \subset \operatorname{Der} C^\infty(\mathbb{R}^{1|1})$, and one can determine the stabilizer subgroup $H \subset G$ of $\mathfrak{gl}_{1|1}$ under the given G-action. It turns out that the H-orbits in which $\mathfrak{gl}_{1|1}$ gets decomposed are simple to describe (Prop. **1.8**), and representatives (*i.e.,* canonical forms) for constant coefficient derivations can easily be found (Cor. **1.9**). The problem of finding a right inverse I_D for a given derivation $D : C^\infty(\mathbb{R}^{1|1}) \to C^\infty(\mathbb{R}^{1|1})$ is first posed for $D \in \mathfrak{gl}_{1|1}$, and it is determined exactly which H-orbits in $\mathfrak{gl}_{1|1}$ have the property that each D in the orbit admits a right inverse (Thm. **2.2**). There are basically two different types of such orbits, and the derivations belonging to them are called right-invertible of type (1), and of type (2), respectively. The explicit formulæ for the right inverses I_D are also obtained. It is shown that the set of right inverses for a given D are parametrized by pairs of initial data (t_0, α), so that a unique right

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inverse $I_{(D;t_0,\alpha)}$ — that we have called the indefinite integral with respect to D with initial condition α at the base point t_0 — can be associated to the triple $(D; t_0, \alpha)$. It is then shown in Prop. **2.4** that the relationship between the indefinite integrals of different right-invertible derivations from $\mathfrak{gl}_{1|1}$ lying in the same H-orbit is natural: It is an equivariant statement with respect to the action of H on $\operatorname{Der} C^\infty(\mathbb{R}^{1|1})$, on $C^\infty(\mathbb{R}^{1|1})$, and on pairs (t_0, α) of initial data. It is furthermore observed that such an H-equivariance statement is really the content of a basic change of variable formula with respect to diffeomorphisms from H. With that result in sight, we extend the notion of indefinite integral to more general derivations than those in $\mathfrak{gl}_{1|1}$; namely, the equivariance property of the integrals is extended from H to the whole diffeomorphism group G of $\mathbb{R}^{1|1}$, and we call a derivation $D \in \operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ right-invertible if it now lies in the G-orbit of some constant coefficient derivation which can be right inverted as in Thm. **2.2**. The general change of variable formula is then an equivariance statement with respect to diffeomorphisms from G (Thm. **2.9**).

Once the classification of indefinite integrals has been made in $\mathbb{R}^{1|1}$ we observe that most of our work can be reproduced in $\mathbb{C}^{1|1}$ when viewed as a holomorphic graded manifold. It also holds for $(1,1)$-dimensional graded submanifolds of $\mathbb{C}^{1|1}$ with simply-connected base manifold $N \subset \mathbb{C}$. We observe that various integration formulæ for graded manifolds that have been introduced in the literature, now emerge as particular examples of our classification. In particular, we analize in §3 the Berezin integral at the light of our developments. We show first that the assumption about compactness of supports of the Berezin forms can be dropped in the $(m,1)$-dimensional case (Prop. **3.3**), thus obtaining an indefinite Berezin integral on $(1,1)$-dimensional (smooth or holomorphic) graded manifolds with simply-connected base. We then prove that for the so called supersymmetric holomorphic curves (SUSY-curves), there is a canonical integration map defined on the Berezinian sheaf of the curve, with values in the structure sheaf (moded out by the additive action of the constants), which gives more information than the Berezin integral (Prop. **3.5**). We finally give in §4 the analysis needed to set up the Euler-Lagrange equations associated to the variational calculus applied to the integral formulæ we have classified.

There are several reasons to consider the integration problem in $\mathbb{R}^{1|1}$, before attempting a development of a general theory of integration on graded manifolds (*eg,* see [22]-[23] for a discussion of the necessity of an appropriate integration theory from the point of view of cohomology). First of all, the relevance of $\mathbb{R}^{1|1}$ in the theory of graded manifolds stems from the fact that the abstract sections from the structure sheaf of a given graded manifold $\mathbb{M}$ can be set into a one-to-one correspondence with graded manifold morphisms $\mathbb{M} \to \mathbb{R}^{1|1}$ (see **1.2**, and [15], or [21]). Here is another result that emphasizes the relevance of $\mathbb{R}^{1|1}$: Vector fields on a graded manifold $\mathbb{M}$ are, by definition, sections belonging to the sheaf of graded derivations of its structure sheaf. It has been shown in [12] that each vector field defined on $\mathbb{M}$ has a unique integral flow satisfying some initial data which, when globally defined, is a graded manifold morphism $\mathbb{R}^{1|1} \times \mathbb{M} \to \mathbb{M}$.

At this point, however, some peculiarities appear. Even though a unique integral flow exists, it is not true in general that it defines a graded Lie group action of $\mathbb{R}^{1|1}$ in $\mathbb{M}$. As a matter of fact, graded Lie algebras of dimension $(1,1)$ over $\mathbb{R}$

can be classified, and it turns out that there are three different such algebras up to isomorphism. These give rise to three different graded Lie group structures on the same graded manifold $\mathbb{R}^{1|1}$. It has been shown in [12] that a graded Lie group action is defined by the integral flow of a given vector field, if and only if the homogeneous components generate a $(1,1)$-dimensional graded Lie algebra. The action of $\mathbb{R}^{1|1}$ corresponds to that of the graded Lie group structure whose graded Lie algebra is isomorphic to the one generated by the field components. As for another peculiarity we mention that integral flows may no longer define foliations by graded $(1,1)$-dimensional submanifolds (see [15]). Even worse: There is an easy example of an involutive distribution in the cotangent bundle of $\mathbb{R}^{1|1}$ which is not integrable (see [16]). In summary, there are reasons to look first at the simplest geometric problems in $\mathbb{R}^{1|1}$.

Integration on graded manifolds has been an issue of great interest in Physics, ever since the concept of supersymmetry was introduced in the seventies. The most popular resource in performing integrals of fermionic fields has been the Berezin integral. The Berezinian sheaf, on which Berezin integral is defined, has been proved to be a highly useful sheaf for doing analysis, and geometry on a graded manifold. There is, however, only a small fraction of information about the graded manifold that one can squeeze out from the Berezin integral. As a matter of fact, as the odd dimension of the graded manifold gets higher, a greater number of different sections from the Berezinian sheaf (called Berezin forms) have the same Berezin integral (see **3.1**(1) for an expression in local coordinates, and to see the point of our assertion). Thus, for the purpose of making finer distinctions between sections of the Berezinian sheaf, the Berezin integral is rather poor. It is precisely this criticism what has given us motivation for addressing the integration problem on graded manifolds with a somewhat refreshed perspective. We finally mention that there is an extensive recent literature concerning the integration problem in Hopf algebras which, when a $\mathbb{Z}_2$-gradation is incorporated into the picture, the Berezin integral can be recovered.

Conventions. Most of the algebraic objects that appear in this work, have a $\mathbb{Z}_2$-grading inherited from an underlying $\mathbb{Z}_2$-graded vector space. Thus, we shall be dealing with $\mathbb{Z}_2$-graded algebras, $\mathbb{Z}_2$-graded Lie algebras, $\mathbb{Z}_2$-graded derivations, etc. We shall simply refer ourselves to all of these objects as *graded* (instead of as $\mathbb{Z}_2$-graded). We shall also use the standard notation $|\cdot|$ to keep track of the $\mathbb{Z}_2$-grading of the various spaces involved: Say, if $C = C_0 \oplus C_1$ is the decomposition of a $\mathbb{Z}_2$-graded vector space, the *parity function* is defined as $|\cdot| : (C_0 - \{0\}) \cup (C_1 - \{0\}) \to \mathbb{Z}_2$. An element $c \in C$ is by definition *homogeneous* if c belongs to the domain of $|\cdot|$. It is furthermore called *even* if $|c| = 0$, and it is called *odd* if $|c| = 1$. We shall refer the reader to [8], and [11] for the basic definitions that we use here. Other general references are [1], [5], [10], and [17].

1. Basic Preliminaries

1.1 The graded manifold $\mathbb{R}^{1|1}$. The graded manifold $\mathbb{R}^{1|1}$ is the ringed space $\big(\mathbb{R}, \Gamma(\,\cdot\,, \Lambda T)\big)$, where $T = \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the rank-1 trivial vector bundle. The *standard even coordinate* in $\mathbb{R}^{1|1}$ is the element $t \in C^\infty(\mathbb{R}) \simeq \Gamma(\mathbb{R}, \Lambda^0 T) \hookrightarrow \Gamma(\mathbb{R}, \Lambda T)$ thought of as the identity map $\mathbb{R} \to \mathbb{R}$. The *standard odd coordinate* τ is the nowhere-vanishing section $\mathbb{R} \ni t \mapsto (t,1) \in \Gamma(\mathbb{R},T) \simeq \Gamma(\mathbb{R},\Lambda^1 T) \hookrightarrow \Gamma(\mathbb{R},\Lambda T)$. We shall denote by $C^\infty(\mathbb{R}^{1|1})$ the space of global sections of the structure sheaf $\Gamma(\,\cdot\,,\Lambda T)$. Having fixed τ, a global section $f \in C^\infty(\mathbb{R}^{1|1})$ can be identified with the pair of smooth functions $(f_0, f_1) \in C^\infty(\mathbb{R})^2$, where $f = f_0 + f_1\tau$. Thus, $C^\infty(\mathbb{R}^{1|1}) \leftrightarrow C^\infty(\mathbb{R})^2$, and we shall use this identification throughout this work. The parity function that keeps track of the grading in $\Gamma(\,\cdot\,,\Lambda^0 T \oplus \Lambda^1 T)$ in terms of this identification is, $|\cdot| : \{(f_0,0) \in C^\infty(\mathbb{R})^2 \mid f_0 \neq 0\} \cup \{(0,f_1) \in C^\infty(\mathbb{R})^2 \mid f_1 \neq 0\} \to \mathbb{Z}_2$, where, $|f| = 0$ if $f = (f_0, 0)$, and $|f| = 1$ if $f = (0, f_1)$.

What is important for the ringed space $\mathbb{R}^{1|1}$, however, is the algebra structure of $C^\infty(\mathbb{R}^{1|1})$:

$$(1) \qquad f + h \leftrightarrow (f_0 + h_0, f_1 + h_1) \qquad \text{and} \qquad f\,h \leftrightarrow (f_0 h_0, f_0 h_1 + f_1 h_0)$$

On the other hand, one may consider the set $\mathrm{Mor}(\mathbb{R}^{1|1}, \mathbb{R}^{1|1})$ of all morphisms $F : \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ from the ringed space $(\mathbb{R}, \Gamma(\,\cdot\,,\Lambda T))$ into itself. These correspond in a one-to-one fashion with the algebra morphisms $F^* : C^\infty(\mathbb{R}^{1|1}) \to C^\infty(\mathbb{R}^{1|1})$ defined on (global) sections, which in turn are uniquely determined by their effect on a given set of (global) coordinates; *eg.,* t, τ (*cf,* [10], and [11]). Thus,

$$(2) \qquad F^* t = f_0 \qquad \text{and} \qquad F^* \tau = f_1 \tau$$

where f_0, and f_1 are smooth functions on $\mathbb{R}$. Therefore, $\mathrm{Mor}(\mathbb{R}^{1|1}, \mathbb{R}^{1|1})$ can also be identified with $C^\infty(\mathbb{R})^2$ via $F \leftrightarrow (f_0, f_1)$ through these equations.

One of the most valuable results of the theory of graded manifolds is this: The abstract global sections of the structure sheaf $\mathcal{A}$ of a graded manifold $(M, \mathcal{A})$ correspond in a one-to-one fashion with morphisms $(M, \mathcal{A}) \to \mathbb{R}^{1|1}$; thereby the importance of $\mathbb{R}^{1|1}$. The precise statement is given in the following lemma, whose proof can be found in several places (*eg,* [15], and [21]):

1.2 Lemma. *There exist two graded manifold morphisms, $s, m : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$, uniquely defined by the condition that, the correspondence*

$$\mathcal{A}(M) \ni f \quad \longleftrightarrow \quad \varphi_f \in \mathrm{Mor}((M,\mathcal{A}), \mathbb{R}^{1|1})$$

defined by the decomposition of f into its $\mathbb{Z}_2$-homogeneous components via the equality $f = \varphi_f^* t + \varphi_f^* \tau$ ($|\varphi_f^* t| = 0$, and $|\varphi_f^* \tau| = 1$), becomes an isomorphism of graded algebras upon setting,

$$f + h \longleftrightarrow s \circ (\varphi_f \times \varphi_h) \qquad \textit{and} \qquad f\,h \longleftrightarrow m \circ (\varphi_f \times \varphi_h)$$

where $\varphi_f \times \varphi_h\colon (M,\mathcal{A}) \to \mathbb{R}^{1|1} \times \mathbb{R}^{1|1}$ is the unique morphism into the product defined by letting $\varphi_f : (M,\mathcal{A}) \to \mathbb{R}^{1|1}$ be the morphism into the first factor, and $\varphi_h : (M,\mathcal{A}) \to \mathbb{R}^{1|1}$ be that into the second.

As a byproduct of this lemma when $(M,\mathcal{A}) = \mathbb{R}^{1|1}$ itself, one may not only sum and multiply morphisms F and F' from $\mathrm{Mor}(\mathbb{R}^{1|1}, \mathbb{R}^{1|1})$ in the stated manner, but one may also compose global sections f and h from $C^\infty(\mathbb{R}^{1|1})$ via $\varphi_f \circ \varphi_h$. Thus, the monoid structure of $\mathrm{Mor}(\mathbb{R}^{1|1},\mathbb{R}^{1|1})$ given by the composition of morphisms comes into play:

$$(1) \qquad \varphi_f\circ\varphi_h \leftrightarrow (\, f_0\circ h_0\,,\,(f_1\circ h_0)\,h_1\,) \quad \text{whenever} \quad \begin{cases}\varphi_f\leftrightarrow(f_0,f_1)\\ \varphi_h\leftrightarrow(h_0,h_1)\end{cases}$$

Its unit element is the identity morphism $id:\mathbb{R}^{1|1}\to\mathbb{R}^{1|1}$; it corresponds to the pair $(t,1)\in C^\infty(\mathbb{R})^2$.

From now on we shall drop the identification arrows in $C^\infty(\mathbb{R}^{1|1})\ni f\leftrightarrow(f_0,f_1)\in C^\infty(\mathbb{R})^2$, and $\mathrm{Mor}(\mathbb{R}^{1|1},\mathbb{R}^{1|1})\ni\varphi_f\leftrightarrow(f_0,f_1)\in C^\infty(\mathbb{R})^2$, and simply write $f=(f_0,f_1)$, and $\varphi_f=(f_0,f_1)$, respectively. The one-to-one correspondence set by Lemma **1.2** between $\mathrm{Mor}(\mathbb{R}^{1|1},\mathbb{R}^{1|1})$ and $C^\infty(\mathbb{R}^{1|1})$ allows us to parametrize these two different objects with the same set: $C^\infty(\mathbb{R})^2$. Thus, the product $f\,h=(f_0h_0,f_0h_1+f_1h_0)$, and the composition $f\circ h=(f_0\circ h_0,(f_1\circ h_0)\,h_1)$ may be both interpreted in $C^\infty(\mathbb{R}^{1|1})$ as well as in $\mathrm{Mor}(\mathbb{R}^{1|1},\mathbb{R}^{1|1})$.

Notation. We shall need to consider the multiplicative inverse of a given $f=(f_0,f_1)$ (when it exists), on the one hand, and its inverse under the composition (when it exists), on the other. In dealing with these two inverses, the corresponding inverses of the homogeneous component $f_0\in C^\infty(\mathbb{R})$ shall need to be distinguished as well. Thus, we shall write $\dfrac{1}{f_0}\in C^\infty(\mathbb{R})$ for the function such that $f_0\dfrac{1}{f_0}=1$ (when it exists), and $f_0{}^{-1}$ for the C^∞ morphism $\mathbb{R}\to\mathbb{R}$ such that $f_0\circ f_0{}^{-1}=id$ (when it exists).

1.3 The diffeomorphism group of $\mathbb{R}^{1|1}$. Let G be the subset of $\mathrm{Mor}(\mathbb{R}^{1|1},\mathbb{R}^{1|1})$ consisting of all the invertible morphisms $\varphi:\mathbb{R}^{1|1}\to\mathbb{R}^{1|1}$ under composition. Thus G is a group; it is called the diffeomorphism group of $\mathbb{R}^{1|1}$. The identification $\mathrm{Mor}(\mathbb{R}^{1|1},\mathbb{R}^{1|1})\simeq C^\infty(\mathbb{R})^2$ and the composition law **1.2**(1) yield the following:

$$(1) \qquad \varphi=(\varphi_0,\varphi_1) \quad\Longrightarrow\quad \varphi^{-1}=\left(\varphi_0^{-1},\frac{1}{\varphi_1\circ\varphi_0^{-1}}\right)$$

In particular, $\varphi=(\varphi_0,\varphi_1)$ belongs to G if and only if φ_0 is a diffeomorphism of the real line, and φ_1 is a nowhere-vanishing smooth function. The group structure of G, $\varphi\circ\psi=(\varphi_0\circ\psi_0,(\varphi_1\circ\psi_0)\,\psi_1)$, is precisely that of the semidirect product of the diffeomorphism group $\mathrm{Diff}(\mathbb{R})$ under composition, with the multiplicative group of $C^\infty(\mathbb{R})^\times$ of nowhere vanishing smooth functions on the real line, defined by the *right action* $C^\infty(\mathbb{R})^\times\times\mathrm{Diff}(\mathbb{R})\ni(\varphi_1,\psi_0)\mapsto\varphi_1\circ\psi_0\in C^\infty(\mathbb{R})^\times$. **Convention:** We shall write $G=\mathrm{Diff}(\mathbb{R})\ltimes C^\infty(\mathbb{R})^\times$ to emphasize this group structure.

Note that G acts on the right of $C^\infty(\mathbb{R}^{1|1})$ via,

$$(2) \qquad \begin{gathered} C^\infty(\mathbb{R}^{1|1})\times G\ni(f,\varphi)\mapsto\varphi^*f=f\circ\varphi\in C^\infty(\mathbb{R}^{1|1})\\ \big((f_0,f_1),(\varphi_0,\varphi_1)\big)\mapsto\big(\,f_0\circ\varphi_0\,,\,(f_1\circ\varphi_0)\,\varphi_1\,\big)\end{gathered}$$

Furthermore, G defines a right action on the space $\mathrm{Der}\,C^\infty(\mathbb{R}^{1|1})$ of $\mathbb{Z}_2$-graded derivations of the algebra $C^\infty(\mathbb{R}^{1|1})$; namely,

$$(3) \qquad \begin{aligned} &\mathrm{Der}\,C^\infty(\mathbb{R}^{1|1})\times G\to\mathrm{Der}\,C^\infty(\mathbb{R}^{1|1})\\ &\qquad (D,\varphi)\mapsto\varphi^*\circ D\circ\varphi^{-*}=:D\cdot\varphi\end{aligned}$$

where φ^{-*} stands for $(\varphi^{-1})^*$. This G-action is in fact by graded Lie algebra automorphisms. It plays such a fundamental role throughout this work, that we shall give a detailed description of it (Prop. **1.5** below), just after including the pertinent definitions, notation, and recalling the basic results for the benefit of the reader (Parragraph **1.4** next).

1.4 $\mathbb{Z}_2$-graded derivations of $C^\infty(\mathbb{R}^{1|1})$. Let $\operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ be the graded vector space of derivations of $C^\infty(\mathbb{R}^{1|1})$. This is defined as the direct sum of the subspaces $(\operatorname{Der} C^\infty(\mathbb{R}^{1|1}))_0$, and $(\operatorname{Der} C^\infty(\mathbb{R}^{1|1}))_1$, where, for $\mu = 0, 1$,

$$(\operatorname{Der} C^\infty(\mathbb{R}^{1|1}))_\mu = \{D \in (\operatorname{End}_{\mathbb{R}} C^\infty(\mathbb{R}^{1|1}))_\mu \mid D(fh) = (Df)\,h + (-1)^{\mu|f|} f\,(Dh)\}$$

where $(\operatorname{End}_{\mathbb{R}} C^\infty(\mathbb{R}^{1|1}))_\mu$ is the subspace of the space of $\mathbb{R}$-linear maps $D : C^\infty(\mathbb{R}^{1|1}) \to C^\infty(\mathbb{R}^{1|1})$ such that $|Df| = |f| + \mu$ for any homogeneous function f (see **A.5** in the appendix). It is a well-known fact (*cf,* [8], [10], or [11]) that $\operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ is a free (left) graded $C^\infty(\mathbb{R}^{1|1})$-module; the module structure being $(h, D) \mapsto hD$, where $(hD)f = h\,Df$. In fact, $\operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ is freely generated over $C^\infty(\mathbb{R}^{1|1})$ by the derivations $\frac{\partial}{\partial t} \in (\operatorname{Der} C^\infty(\mathbb{R}^{1|1}))_0$ and $\frac{\partial}{\partial \tau} \in (\operatorname{Der} C^\infty(\mathbb{R}^{1|1}))_1$, where,

$$\frac{\partial}{\partial t} f = (f_0', f_1') \qquad \text{and} \qquad \frac{\partial}{\partial \tau} f = (f_1, 0) \tag{1}$$

whenever, $f = (f_0, f_1)$. In other words, any $D \in \operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ can be uniquely written in the form

$$D = a\,\frac{\partial}{\partial t} + b\,\frac{\partial}{\partial \tau} \qquad a, b \in C^\infty(\mathbb{R}^{1|1}) \tag{2}$$

where, $a = Dt$ and $b = D\tau$. Therefore, $\operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ gets identified with $C^\infty(\mathbb{R}^{1|1})^2 \simeq C^\infty(\mathbb{R})^2 \times C^\infty(\mathbb{R})^2$ via,

$$a\,\frac{\partial}{\partial t} + b\,\frac{\partial}{\partial \tau} \leftrightarrow (a, b) \leftrightarrow \big((a_0, a_1), (b_0, b_1)\big) \tag{3}$$

We are now ready to describe in more detail the right G-action **1.3**(3):

1.5 Proposition. *Let $D = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial \tau}$ be a graded derivation of $C^\infty(\mathbb{R}^{1|1})$, with a, and b in $C^\infty(\mathbb{R}^{1|1})$ identified with the pairs (a_0, a_1), and (b_0, b_1) in $C^\infty(\mathbb{R})^2$, respectively. Then, $D \cdot \varphi = \tilde{a}\frac{\partial}{\partial t} + \tilde{b}\frac{\partial}{\partial \tau}$, where*

$$\big((\tilde{a}_0, \tilde{a}_1), (\tilde{b}_0, \tilde{b}_1)\big) = \left(\left(\frac{a_0 \circ \varphi_0}{\varphi_0'}, \frac{(a_1 \circ \varphi_0)\,\varphi_1}{\varphi_0'}\right), \left(\frac{b_0 \circ \varphi_0}{\varphi_1}, b_1 \circ \varphi_0 - \frac{(a_0 \circ \varphi_0)\,\varphi_1'}{\varphi_1 \varphi_0'}\right)\right)$$

whenever $(\varphi_0, \varphi_1) = \varphi \in G$.

Proof. The proof is straightforward. One simply notes that $\tilde{a} = \varphi^* \circ D \circ \varphi^{-*}t$, and $\tilde{b} = \varphi^* \circ D \circ \varphi^{-*}\tau$. Use is made of the fact that,

$$\varphi^* t = \varphi_0 \qquad \varphi^{-*} t = \varphi_0^{-1}$$
$$\varphi^* \tau = \varphi_1 \tau \qquad \varphi^{-*} \tau = \frac{\tau}{\varphi_1 \circ \varphi_0^{-1}}$$

and of the definition of the derivations $\frac{\partial}{\partial t}$, and $\frac{\partial}{\partial \tau}$. $\square$

1.6 The Lie structure of $\operatorname{Der} C^\infty(\mathbb{R}^{1|1})$**.** Let $D \in \operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ and write it in the form **1.4**.(2). By spelling out in detail the sections $a = a_0 + a_1\tau$ and $b = b_0 + b_1\tau$, D can be further decomposed into $D = D_0 + D_1$ with $D_\mu \in \big(\operatorname{Der} C^\infty(\mathbb{R}^{1|1})\big)_\mu$ $(\mu = 0, 1)$:

$$(1) \qquad D_0 = a_0 \frac{\partial}{\partial t} + b_1 \tau \frac{\partial}{\partial \tau} \qquad \text{and} \qquad D_1 = a_1 \tau \frac{\partial}{\partial t} + b_0 \frac{\partial}{\partial \tau}$$

Now let $|\cdot|: \big((\operatorname{Der} C^\infty(\mathbb{R}^{1|1}))_0 - \{0\}\big) \cup \big((\operatorname{Der} C^\infty(\mathbb{R}^{1|1}))_1 - \{0\}\big) \to \mathbb{Z}_2$ be the map defined by $D \mapsto |D| = \mu$ if $D \in \operatorname{Der} C^\infty(\mathbb{R}^{1|1})_\mu - \{0\}$. With its aid, a graded Lie algebra structure can be defined on $\operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ by letting the graded Lie bracket of two homogenoeus derivations be given by (*cf*, [4], [8], or [11]),

$$(2) \qquad [D_1, D_2] = D_1 \circ D_2 - (-1)^{|D_1||D_2|} D_2 \circ D_1$$

and extending it $\mathbb{R}$-bilinearly to $\operatorname{Der} C^\infty(\mathbb{R}^{1|1})$. It is then immediate to verify that the right G-action on $\operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ is by automorphisms of this Lie structure; that is,

$$(3) \qquad [D_1 \cdot \varphi, D_2 \cdot \varphi] = [D_1, D_2] \cdot \varphi$$

It is also easy to verify that the derivations $\{\frac{\partial}{\partial t}, \tau\frac{\partial}{\partial \tau}\} \in (\operatorname{Der} C^\infty(\mathbb{R}^{1|1}))_0$, and $\{\tau\frac{\partial}{\partial t}, \frac{\partial}{\partial \tau}\} \in (\operatorname{Der} C^\infty(\mathbb{R}^{1|1}))_1$ satisfy the following commutation relations:

$$(4) \qquad \Big[\tau\frac{\partial}{\partial \tau}, \tau\frac{\partial}{\partial t}\Big] = \tau\frac{\partial}{\partial t} \qquad \Big[\tau\frac{\partial}{\partial \tau}, \frac{\partial}{\partial \tau}\Big] = -\frac{\partial}{\partial \tau} \qquad \Big[\tau\frac{\partial}{\partial t}, \frac{\partial}{\partial \tau}\Big] = \frac{\partial}{\partial t}$$

with all other Lie brackets equal to zero. In particular, the real graded subspace $\mathfrak{gl}_{1|1} \subset \operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ defined by the $\mathbb{R}$-span of $\{\frac{\partial}{\partial t}, \tau\frac{\partial}{\partial \tau}, \tau\frac{\partial}{\partial t}, \frac{\partial}{\partial \tau}\}$ is closed under $[\cdot,\cdot]$, and therefore defines a graded Lie subalgebra of $\operatorname{Der} C^\infty(\mathbb{R}^{1|1})$; namely, the graded Lie subalgebra of *constant coefficient derivations*. (We refer the reader to **A.5**, and **A.6** in the appendix for an alternative description of this graded Lie algebra). By letting a_0, a_1, b_0, and b_1 be constant functions in Prop. **1.5** above, we obtain the following:

1.7 Proposition. *Let* $\mathfrak{gl}_{1|1} \subset \operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ *be the graded Lie subalgebra of constant coefficient derivations. Let* $H \subset G$ *be the subgroup that stabilizes* $\mathfrak{gl}_{1|1}$ *under the right action of* G *on* $\operatorname{Der} C^\infty(\mathbb{R}^{1|1})$*. Then,* $\varphi = (\varphi_0, \varphi_1) \in H$*, if and only if there exist real constants* $c_0, c_1, d_0 \in \mathbb{R}$*, with* $c_0 c_1 \neq 0$*, such that* $\varphi^* t = c_0 t + d_0$*, and* $\varphi^*\tau = c_1\tau$*.*

Remark. In other words (according to the convention introduced in **1.3**) $H \simeq \operatorname{Aff}(\mathbb{R}) \ltimes \mathbb{R}^\times \subset \operatorname{Diff}(\mathbb{R}) \ltimes C^\infty(\mathbb{R})^\times$ where $\operatorname{Aff}(\mathbb{R})$ is the affine group of the real line, and the semidirect product structure restricts itself to the trivial right action of $\operatorname{Aff}(\mathbb{R})$ on $\mathbb{R}^\times$; that is, H is actually a direct product, and we shall write $\varphi = ((c_0, d_0), c_1) \in H$ with $(c_0, d_0) \in \operatorname{Aff}(\mathbb{R})$, and $c_1 \in \mathbb{R}^\times$.

1.8 Proposition. *The right action* $\operatorname{Der} C^\infty(\mathbb{R}^{1|1}) \times G \to \operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ *restricts to the right action* $\mathfrak{gl}_{1|1} \times H \to \mathfrak{gl}_{1|1}$ *given by,*

$$\big(((a_0, a_1), (b_0, b_1)), ((c_0, d_0), c_1) \big) \mapsto \left(\left(\frac{a_0}{c_0}, \frac{a_1 c_1}{c_0} \right), \left(\frac{b_0}{c_1}, b_1 \right) \right)$$

whenever, $((c_0, d_0), c_1) \in H$. *In particular, the orbit structure is given by,* $\big((a_0, a_1), (b_0, b_1)\big) \sim \big((\tilde a_0, \tilde a_1), (\tilde b_0, \tilde b_1)\big)$ *if and only if, either,*

$$b_1 = \tilde b_1 \qquad \textit{and} \qquad a_0 \tilde a_0 \neq 0 \qquad \textit{and} \qquad b_0 \tilde b_0 \neq 0 \qquad \textit{and} \qquad \tilde a_1 = \frac{\tilde a_0 b_0}{a_0 \tilde b_0}\, a_1 \neq 0$$

or, exactly one of the numbers a_0, a_1, b_0 *is zero (in which case, its corresponding number* $\tilde a_0$, $\tilde a_1$, $\tilde b_0$ *is zero); or, exactly two of them are zero (in which case, their corresponding two numbers are zero); or, exactly the three of them are zero (in which case, the same is true for* $\tilde a_0$, $\tilde a_1$, $\tilde b_0$*).*

Proof. The statement follows from the explicit G action on $\operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ of Prop. **1.5** under the hypotheses that a_0, a_1, b_0, and b_1 are constant functions, and $\varphi^* t = c_0 t + d_0$, $\varphi^* \tau = c_1 \tau$ with $c_0 c_1 \neq 0$, as $\varphi \in H$. $\square$

1.9 Corollary. *The following is a complete list of canonical representatives of* $\mathfrak{gl}_{1|1}$ *under the action of the identity component* H_e *of* H*:*

$$\begin{array}{lccc}
\operatorname{sgn}(a_0)\dfrac{\partial}{\partial t} + \operatorname{sgn}(b_0)\dfrac{\partial}{\partial \tau} + \tau\left(a_1 \left|\dfrac{b_0}{a_0}\right| \dfrac{\partial}{\partial t} + b_1 \dfrac{\partial}{\partial \tau}\right) & a_0 \neq 0 & b_0 \neq 0 & a_1 \neq 0 \\[2ex]
\operatorname{sgn}(a_0)\dfrac{\partial}{\partial t} + \operatorname{sgn}(b_0)\dfrac{\partial}{\partial \tau} + b_1\, \tau\, \dfrac{\partial}{\partial \tau} & a_0 \neq 0 & b_0 \neq 0 & a_1 = 0 \\[2ex]
\operatorname{sgn}(a_0)\dfrac{\partial}{\partial t} + \tau\left(\operatorname{sgn}(a_1)\dfrac{\partial}{\partial t} + b_1 \dfrac{\partial}{\partial \tau}\right) & a_0 \neq 0 & b_0 = 0 & a_1 \neq 0 \\[2ex]
\operatorname{sgn}(b_0)\dfrac{\partial}{\partial \tau} + \tau\left(\operatorname{sgn}(a_1)\dfrac{\partial}{\partial t} + b_1 \dfrac{\partial}{\partial \tau}\right) & a_0 = 0 & b_0 \neq 0 & a_1 \neq 0 \\[2ex]
\operatorname{sgn}(a_0)\dfrac{\partial}{\partial t} + b_1 \tau \dfrac{\partial}{\partial \tau} & a_0 \neq 0 & b_0 = 0 & a_1 = 0 \\[2ex]
\operatorname{sgn}(b_0)\dfrac{\partial}{\partial \tau} + \tau\, b_1 \dfrac{\partial}{\partial \tau} & a_0 = 0 & b_0 \neq 0 & a_1 = 0 \\[2ex]
\tau\left(\operatorname{sgn}(a_1)\dfrac{\partial}{\partial t} + b_1 \dfrac{\partial}{\partial \tau}\right) & a_0 = 0 & b_0 = 0 & a_1 \neq 0 \\[2ex]
\tau\, b_1 \dfrac{\partial}{\partial \tau} & a_0 = 0 & b_0 = 0 & a_1 = 0
\end{array}$$

Remark. For those readers who prefer the Lie theoretic structure of $\mathfrak{gl}_{1|1}$ to be given within the framework of linear algebra (*eg,* through classical Lie algebras and its modules) a brief discussion has been included in **A.5**-**A.8** of the appendix.

2. Indefinite integrals on $\mathbb{R}^{1|1}$

2.1 Problem. Let $D \in \mathfrak{gl}_{1|1}$ be a derivation with constant coefficients $\big((a_0, a_1), (b_0, b_1)\big)$ as in **1.4**(3) which, for the moment, shall be kept fixed. We are interested in determining the precise conditions under which D has a well defined *right inverse*, $I_D : C^\infty(\mathbb{R}^{1|1}) \to C^\infty(\mathbb{R}^{1|1})$. Let $f = (f_0, f_1) \in C^\infty(\mathbb{R}^{1|1})$, and write $I_D(f) = (h_0, h_1)$. It is understood that both, h_0, and h_1 depend on D, and f. This dependence will manifest itself after we find what h_0, and h_1 should be upon imposing the condition that $D \circ I_D(f) = f$, *for any* f. This condition readily leads to a system of differential equations to be solved for h_0, and h_1; namely,

$$\begin{pmatrix} a_0 & 0 & b_0 \\ a_1 & a_0 & b_1 \end{pmatrix} \begin{pmatrix} h_0' \\ h_1' \\ h_1 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \qquad \forall\ f_0, f_1 \in C^\infty(\mathbb{R}) \tag{1}$$

If $a_0 \neq 0$, this system yields the following differential equation for h_1:

$$h_1' + \frac{\Delta(a;b)}{(a_0)^2} h_1 = \frac{\Delta(a;f)}{(a_0)^2} \tag{2}$$

where, for any $x = (x_0, x_1)$, and $y = (y_0, y_1)$ in $C^\infty(\mathbb{R})^2$,

$$\Delta(x;y) = \det \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \end{pmatrix} \tag{3}$$

We now choose a base point $t_0 \in \mathbb{R}$, to find the unique solution of (2) whose value at t_0 is α_1; namely,

$$h_1 = \mathrm{e}^{-\frac{\Delta(a;b)}{(a_0)^2}(t-t_0)}\,\alpha_1 + \int_{t_0}^{t} \mathrm{e}^{-\frac{\Delta(a;b)}{(a_0)^2}(t-s)}\, \frac{\Delta(a; f(s)\,)}{(a_0)^2}\ \mathrm{d}s\,, \qquad a_0 \neq 0 \tag{4}$$

Having thus obtained h_1, we may solve the equation for h_0 so as to obtain the unique solution whose value at t_0 is α_0:

$$h_0 = \alpha_0 + \frac{1}{a_0}\int_{t_0}^{t} (f_0 - b_0 h_1)(s)\ \mathrm{d}s \qquad a_0 \neq 0 \tag{5}$$

If, on the other hand, $a_0 = 0$ we have four possibilities to look at (*cf,* Prop. **1.8**): $(a_1 b_0 \neq 0)$, $(a_1 = 0, b_0 \neq 0)$, $(a_1 \neq 0, b_0 = 0)$, and $(a_1 = b_0 = 0)$. We shall first assume that $a_1 b_0 \neq 0$. In this case, (1) gets considerably reduced, and h_1 can be determined uniquely at once:

$$h_1 = \frac{f_0}{b_0}\,, \qquad a_0 = 0 \tag{6}$$

Now, from $a_1 h_0' = f_1 - b_1 h_1$ one obtains,

$$h_0 = \alpha_0 + \frac{1}{a_1 b_0}\int_{t_0}^{t} \Delta(b; f(s)\,)\ \mathrm{d}s\,, \qquad a_0 = 0 \tag{7}$$

as the unique solution to that differential equation whose value at a chosen base point $t_0 \in \mathbb{R}$ is α_0.

Now, the case $(a_0 = 0, a_1 = 0, b_0 \neq 0)$ does not yield $D \circ I_D(f) = f$, for all f; in fact, the system (1) can be solved only for those f's satisfying $\Delta(b; f) = 0$. Similarly, the cases $(a_0 = 0, a_1$ arbitrary $, b_0 = 0)$, yield a solution of (1) only for those f's with $f_0 = 0$.

Notation. Note that when $a_0 \neq 0$, one may choose the values α_0, and α_1 arbitrarily, *regardless of* the actual inputs D, and f. The pair $(\alpha_0, \alpha_1) \in \mathbb{R}^2$ is just $(h_0(t_0), h_1(t_0))$, and we may call it *the initial condition of* $I_D(f)$ *at the base point* t_0. Thus, an unambiguous notation for the unique solution (h_0, h_1) of (1) above, for which $h_0(t_0) = \alpha_0 \in \mathbb{R}$, and $h_1(t_0) = \alpha_1 \in \mathbb{R}$, is $I_{(D;t_0,\alpha)}(f)$. We shall further write,

$$\text{(8)} \qquad \mathrm{ev}_{t_0}\big(\, I_{(D;t_0,\alpha)}(f)\,\big) = \alpha, \qquad \text{with } \alpha = (\alpha_0, \alpha_1), \text{ whenever } a_0 \neq 0$$

On the other hand, when $a_0 = 0$ with $a_1 b_0 \neq 0$, the value $h_1(t_0)$ depends on D and f (see (6) above). The value $h_0(t_0)$, however, can be arbitrarily chosen, independently of D, and f. We shall use the same notation $I_{(D;t_0,\alpha)}(f)$ for the unique solution (h_0, h_1) of (1) above, such that $h_0(t_0) = \alpha_0$, with the understanding that $\alpha = \alpha_0 \in \mathbb{R}$ in this case. We shall also write,

$$\text{(9)} \qquad \mathrm{ev}_{t_0}\big(\, I_{(D;t_0,\alpha)}(f)\,\big) = \alpha, \qquad \text{with } \alpha = \alpha_0, \text{ whenever } a_0 = 0, \text{ and } a_1 b_0 \neq 0$$

Despite of the fact that the notation $I_{(D;t_0,\alpha)}(f)$ is the same and the meanings of α are different, there can be no danger of confusion because Prop. **1.5** above states that these two cases correspond to D's lying in different G-orbits. Having these definitions and observations in mind, we may now summarize our results in the following:

2.2 Theorem. *Let $D \in \mathfrak{gl}_{1|1}$ be a constant coefficient derivation with coefficients $\big((a_0, a_1), (b_0, b_1)\big)$ as in* **1.4**(3).

(1) *Suppose $a_0 \neq 0$. For each choice of $t_0 \in \mathbb{R}$, and $\alpha = (\alpha_0, \alpha_1) \in \mathbb{R}^2$, the derivation D has a unique right inverse $I_{(D;t_0,\alpha)}$ such that $\mathrm{ev}_{t_0}\big(\, I_{(D;t_0,\alpha)}(f)\,\big) = \alpha$ for any $f \in C^\infty(\mathbb{R}^{1|1})$. Explicitly,*

$$I_{(D;t_0,\alpha)}(f) = \Big(\, \alpha_0 + \frac{1}{a_0}\int_{t_0} (f_0 - b_0 h_1)(s)\ \mathrm{d}s\,,\ h_1 \Big)$$

where,

$$h_1(t) = \mathrm{e}^{-\frac{\Delta(a;b)}{(a_0)^2}(t-t_0)}\,\alpha_1 + \int_{t_0}^{t} \mathrm{e}^{-\frac{\Delta(a;b)}{(a_0)^2}(t-s)}\ \frac{\Delta(a; f(s)\,)}{(a_0)^2}\ \mathrm{d}s\,.$$

(2) *Suppose $a_0 = 0$, and $a_1 b_0 \neq 0$. For each choice of $t_0 \in \mathbb{R}$, and $\alpha = \alpha_0 \in \mathbb{R}$, the derivation D has a unique right inverse $I_{(D;t_0,\alpha)}$ such that $\mathrm{ev}_{t_0}\big(\, I_{(D;t_0,\alpha)}(f)\,\big) = \alpha$ for any $f \in C^\infty(\mathbb{R}^{1|1})$. Explicitly,*

$$I_{(D,t_0,\alpha)}(f) = \Big(\, \alpha_0 + \frac{1}{a_1 b_0}\int_{t_0} \Delta(b; f(s))\ \mathrm{d}s\,,\ \frac{f_0}{b_0} \Big)$$

(3) *If D lies in any H-orbit other than that of cases (1) or (2) above, then D can have no right inverse at all.*

Remark. We shall hereby call $I_{(D,t_0,\alpha)}(f) \in C^\infty(\mathbb{R}^{1|1})$ *the integral of f with respect to D, with initial condition α at the base point t_0* (where, $\alpha \in \mathbb{R}^2$ if $a_0 \neq 0$, and

$\alpha \in \mathbb{R}$ if $a_0 = 0$, and $a_1 b_0 \neq 0$). The name is justified, since all the examples of 'integrals' on the graded manifold $\mathbb{R}^{1|1}$ — at least to the authors' knowledge — are actually obtained for appropriate choices of D's, as it is shown in **2.3** below. Note that there are as many integrals with respect to D, as choices are of pairs (t_0, α) of base point and initial condition. For the sake of reference we shall say that a right invertible derivation $D \in \mathfrak{gl}_{1|1}$ is *of type* (1), or (2), depending on whether it lies in the G-orbit to which cases (1), or (2) of the above proposition are referred, respectively.

2.3 Examples of integration in $\mathbb{R}^{1|1}$. An important consequence of Thm. **2.2** now follows. Several integration schemes on graded manifolds have been proposed in the literature. As far as the authors know, there are at least four of them which have been introduced independently and with no apparent relationship amongst themselves. However, when we specialize these schemes to $\mathbb{R}^{1|1}$, Thm. **2.2** let us gather them all and exhibit them as particular instances of I_D's arising from appropriate D's, as shown in the table below. We refer the reader to each of the indicated references for details about the corresponding integration scheme.

Reference	Integrate $f \in C^\infty(\mathbb{R}^{1|1})$	Actual integral	D from which integral arises
[Newton]	(f_0, f_1)	$\left(\int_{t_0}^t f_0(s)\,\mathrm{d}s\,,\, \int_{t_0}^t f_1(s)\,\mathrm{d}s \right)$	$\dfrac{\partial}{\partial t}$
[11]	$(0, f_1)$	$\left(\int_{t_0}^t f_1(s)\,\mathrm{d}s\,,\, 0 \right)$	$\dfrac{\partial}{\partial \tau} + \tau \dfrac{\partial}{\partial t}$
[20]	(f_0, f_1)	$\left(\int_{t_0}^t (f_0 - \int_{t_0}^s f_1)(s)\,\mathrm{d}s\,,\, \int_{t_0}^t f_1(s)\,\mathrm{d}s \right)$	$\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial \tau}$
[18]	(f_0, f_1)	$\left(\int_{t_0}^s f_1(s)\,\mathrm{d}s, f_0 \right)$	$\dfrac{\partial}{\partial \tau} + \tau \dfrac{\partial}{\partial t}$

We mention in passing that Berezin's integration theory has been enlightened by the works [19], and [7], among others. This is still the most popular 'fermionic integration theory' to date, even though the 'volume forms' that have a nonzero Berezin integral are too reduced. Nevertheless, we shall devote §3 below to make a thorough analysis of the Berezin integral at the light of what we have done so far.

We shall now give the relationship between two integrals, I_D, and $I_{\tilde{D}}$, when D and $\tilde{D} = D \cdot \varphi$ belong to the same H-orbit, H being the stabilizer of $\mathfrak{gl}_{1|1}$ as in Prop. **1.7**. Observe that,

$$\begin{aligned}
(D \cdot \varphi)\Big(I_{D\cdot\varphi}(f \circ \varphi) - \big(I_D(f)\big) \circ \varphi \Big) &= f \circ \varphi - (D \cdot \varphi)(I_D(f) \circ \varphi) \\
&= f \circ \varphi - \varphi^* \circ D \circ \varphi^{-*}\big(\varphi^* I_D(f)\big) \\
&= f \circ \varphi - \varphi^* \circ D\, \big(I_D(f)\big) = 0
\end{aligned}$$

It is therefore clear that $I_{D\cdot\varphi}(f \circ \varphi)$ is equal to $\big(I_D(f)\big) \circ \varphi$ plus some 'boundary terms' lying in $\operatorname{Ker} D\cdot\varphi$, and depending on the base points, and the initial conditions used in I_D, and $I_{D\cdot\varphi}$. At any rate, the boundary terms can be obtained by a straightforward computation from the explicit formulæ of Thm. **2.2**. There is a particular choice, however, of base points, and initial conditions, for which these boundary terms cancel out, and one can then make the following statement:

2.4 Proposition. *Let $D \in \mathfrak{gl}_{1|1}$ be of type (1) or (2), according to Thm.* **2.2**. *Suppose $\tilde{D} = D \cdot \varphi$ with $\varphi \in H$. Then,*

$$I_{(D\cdot\varphi;\varphi_0^{-1}(t_0),\alpha\circ\varphi)}(f \circ \varphi) = \big(I_{(D;t_0,\alpha)}(f) \big) \circ \varphi$$

where, for type (1), or (2), we respectively have the following:

(1) *α is regarded as the pair of constant functions (α_0, α_1) on $\mathbb{R}$, and $\alpha \circ \varphi$ is the composition law in* $\mathrm{Mor}(\mathbb{R}^{1|1}, \mathbb{R}^{1|1})$ *as in §1. Equivalently, $\alpha \circ \varphi = (\alpha_0, c_1\alpha_1)$, whenever $\varphi = ((c_0, d_0), c_1) \in H$ as in* **1.7**.

(2) *α is regarded as the pair of constant functions $(\alpha_0, 0)$ on $\mathbb{R}$, and $\alpha \circ \varphi$ is the composition law in* $\mathrm{Mor}(\mathbb{R}^{1|1}, \mathbb{R}^{1|1})$ *as in §1. Equivalently, $\alpha \circ \varphi = \alpha$.*

Proof. Note first that it suffices to prove the statement for the generators of the group H: In fact, if the statement is true for $\varphi \in H$, and $\psi \in H$, then it is easily seen to be true for $\varphi \circ \psi \in H$. Whence, one only has to take $I_{(D;t_0,\alpha)}(f)$ as given in (1) or (2) of Thm. **2.2** above, and let φ be, case by case, each of the following elements of $H = \mathrm{Aff}(\mathbb{R}) \ltimes \mathbb{R}^\times$: $((c_0, 0), 1)$, $((1, 0), c_1)$, and $((1, d_0), 1)$, respectively. All that has to be done then is to perform appropriate operations in the integral formulæ given in Thm. **2.2**, so as to directly verify that,

$$I_{(D\cdot\varphi;t_0,\alpha)}(f \circ \varphi) = I_{(D;c_0t_0,\alpha)}(f) \circ \varphi \qquad \varphi = ((c_0, 0), 1) \in H$$

$$I_{(D\circ\varphi;t_0,\alpha)}(f \circ \varphi) = I_{(D;t_0,\alpha')}(f) \circ \varphi \qquad \begin{matrix} \varphi = ((1, 0), c_1) \in H \\ \alpha_0' = \alpha_0,\ \alpha_1' = \dfrac{\alpha_1}{c_1} \end{matrix}$$

$$I_{(D\cdot\varphi;t_0,\alpha)}(f \circ \varphi) = I_{(D;t_0+d_0,\alpha)}(f) \circ \varphi \qquad \varphi = ((1, d_0), 1) \in H$$

What all these cases have in common is that the base point appearing in the right hand side is $\varphi_0(t_0)$, and the initial value of the integral in the right is $\alpha \circ \varphi^{-1}$. Thus, after performing the corresponding inverse operations on the left hand side, on obtains the statement. $\square$

Remark. The statement of Prop. **2.4** is particularly appealing. On the one hand, it brings to the foreground the H-equivariance of the integral $I_{(D;t_0,\alpha)}$ when $D \in \mathfrak{gl}_{1|1}$. In other words,

$$I_{(D\cdot\varphi;\varphi_0^{-1}(t_0),\alpha\circ\varphi)} = \varphi^* \circ I_{(D;t_0,\alpha)} \circ \varphi^{-*}$$

and the right hand side clearly defines a right H-action, which gets reflected in the data $(D; t_0, \alpha)$, so that,

$$(D; t_0, \alpha) \cdot \varphi = (D \cdot \varphi; \varphi_0^{-1}(t_0), \alpha \circ \varphi)$$

On the other hand, the statement of Prop. **2.4** is nothing but a *change of variable formula* under diffeomorphisms $\varphi \in H$. As a matter of fact, our entire approach to the general change of variable formula is based on the appropriate generalization of this equivariance statement (*cf,* **2.7** below).

2.5 Right-invertible derivations. We now aim to extend the integration formulæ given in Thm. **2.2** to a class of derivations wider than $\mathfrak{gl}_{1|1}$, and to extend the equivariance property given in Prop. **2.4** to diffoemorphisms φ lying in the wider group $G \supset H$.

Suppose $\tilde{D} \in \operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ lies in the G-orbit of some $D \in \mathfrak{gl}_{1|1}$ for which a well defined right inverse I_D exists according to the criteria of Thm. **2.2** above. Say, $\tilde{D} = D \cdot \varphi$ for some $\varphi \in G$. Then, $\varphi^* \circ I_D \circ \varphi^{-*}$ is clearly a right inverse for $D \cdot \varphi = \varphi^* \circ D \circ \varphi^{-*}$. In other words, *those derivations $\tilde{D} \in \operatorname{Der} C^\infty(\mathbb{R}^{1|1})$ lying in the G-orbit of some $D \in \mathfrak{gl}_{1|1}$ of type (1) or (2) as given by Thm.* **2.2**, *have well defined right inverses.* We shall therefore call them *right-invertible derivations* (of type (1) or (2), depending on the orbit they lie into), and their right inverses shall be called *integrals*.

The argument given right before Prop. **2.4** to show that $I_{D\cdot\varphi}(f \circ \varphi) - \big(I_D(f)\big) \circ \varphi \in \operatorname{Ker} D \cdot \varphi$, works well for $\varphi \in G$. So, again, the important data to keep track of are the base points, and the initial conditions. Now, we have seen that for a given $D \in \mathfrak{gl}_{1|1}$, the set of integrals with respect to D is actually parametrized by the set of pairs (t_0, α), so that $\operatorname{ev}_{t_0}\big(I_D(f)\big) = \alpha$, for all f, and the meaning of α depends on the orbit as stated in Thm. **2.2**. Note that the set of integrals (*ie,* right inverses) with respect to D, and the set of integrals with respect to $\tilde{D} = D \cdot \varphi$ for any $\varphi \in G$, are in one to one correspondence: In fact, the map, $I_D \mapsto \varphi^* \circ I_D \circ \varphi^{-*}$, gives a bijection of sets whose inverse is $I_{D\cdot\varphi} \mapsto \varphi^{-*} \circ I_D \circ \varphi$. Therefore,

2.6 Corollary. *Let $D \in \mathfrak{gl}_{1|1}$ be a right-invertible derivation, and let $\varphi \in G$. Then, the set of integrals $I_{D\cdot\varphi}$ is parametrized by the set of pairs (t_0, α), such that $\operatorname{ev}_{t_0}\big(I_{(D;t_0,\alpha)}(f)\big) = \alpha$, for all f, and the meaning of α depends on whether D is of type (1) or (2) as defined in* **2.5**.

2.7 Equivariant extension. Consider the set of triples $(D; t_0, \alpha)$ where D is right-invertible, t_0 is a base point, and α is an initial condition represented by a point in $\mathbb{R}^2$ or $\mathbb{R}$, depending on whether D is of type (1) or (2) as defined in **2.5** above. Define on this set the following:

$$(D; t_0, \alpha) \cdot \varphi := \left(D \cdot \varphi; \varphi_0^{-1}(t_0), \operatorname{ev}_{\varphi_0^{-1}(t_0)}(\alpha \circ \varphi)\,\right), \qquad \varphi = (\varphi_0, \varphi_1) \in G \tag{1}$$

The term $\operatorname{ev}_{\varphi_0^{-1}(t_0)}(\alpha\circ\varphi)$ on the right hand side of this equation must be interpreted in the sense of the following definition: $\operatorname{ev}_{t_0}(f_0, f_1) = (\,f_0(t_0), f_1(t_0)\,)$ for any $f \in C^\infty(\mathbb{R}^{1|1})$. Thus, if D is of type (1), then $\alpha \in \mathbb{R}^2$ may be viewed as the morphism $\alpha : \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ given by the pair of constant functions (α_0, α_1), and $\alpha \circ \varphi : \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ is just the pullback by $\varphi \in G$. This may no longer be given by a pair of constant functions (as it was the case when $\varphi \in H$, and $D \in \mathfrak{gl}_{1|1}$). Thus, to bring it back to a pair of initial conditions in $\mathbb{R}^2$ we apply $\operatorname{ev}_{\varphi_0^{-1}(t_0)}$ to obtain,

$$\operatorname{ev}_{\varphi_0^{-1}(t_0)}(\alpha \circ \varphi) = (\alpha_0, \varphi_1 \circ \varphi_0^{-1}(t_0)\alpha_1) \tag{2}$$

It can now be directly verified that the assignment $(t_0, \alpha) \mapsto (\,\varphi_0^{-1}(t_0)\,, \operatorname{ev}_{\varphi_0^{-1}(t_0)}(\alpha \circ \varphi)\,)$ defines a right G-action on the set of initial data $\mathbb{R} \times \mathbb{R}^2$. Therefore, (1) defines a right G-action when D is of type (1). Note that when $\varphi = ((c_0, d_0), c_1) \in H$,

$\mathrm{ev}_{\varphi_0^{-1}(t_0)}(\alpha\circ\varphi) = (\alpha_0, c_1\alpha_1) = \alpha\circ\varphi$, whence (2) is enterily consistent with what we have done in Prop. **2.4**. On the other hand, when D is of type (2), we first understand α as the morphism $\alpha : \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ given by the pair of constant funtions $(\alpha_0, 0)$, and repeat the above argument so that $\mathrm{ev}_{\varphi_0^{-1}(t_0)}(\alpha\circ\varphi)$ has the same stated meaning. In summary: *Eq. (1) defines a right G-action on the set of arguments for the integrals, regardless of the type of D.*

2.8 Definition. For any right invertible derivation $D\cdot\varphi$ with $D \in \mathfrak{gl}_{1|1}$, and $\varphi \in G$ we define the integral with respect to $D\cdot\varphi$, having base point t_0, and initial condition α (with the meaning of α depending on whether D is of type (1) or (2) as defined in **2.5**), by

$$I_{(D\cdot\varphi;t_0,\alpha)}(f) = \varphi^* I_{(D;\varphi_0(t_0),\mathrm{ev}_{\varphi_0(t_0)}(\alpha\circ\varphi^{-1}))}(\varphi^{-*}f)$$

Remark. Note that when $D \in \mathfrak{gl}_{1|1}$ is right-invertible, the base point t_0, and the initial condition α are related by $\mathrm{ev}_{t_0}\big(I_{(D;t_0,\alpha)}(f)\big) = \alpha$ (see **2.1**(8), and **2.1**(8)). Observe that the same relationship holds true for the base point, and the initial condition introduced by this definition: In fact, note first that

$$\mathrm{ev}_{t_0}\circ\varphi^* = \mathrm{ev}_{t_0}\circ\varphi^*\circ\,\mathrm{ev}_{\varphi_0(t_0)}$$

as maps from $C^\infty(\mathbb{R}^{1|1})$ into $\mathbb{R}^2$. Therefore,

$$\begin{aligned}
\mathrm{ev}_{t_0}\Big(I_{(D\cdot\varphi;t_0,\alpha)}(f)\Big) &= \mathrm{ev}_{t_0}\Big(\varphi^* I_{(D;\varphi(t_0),\mathrm{ev}_{\varphi_0(t_0)}(\alpha\circ\varphi^{-1}))}(\varphi^{-*}f)\Big)\\
&= \mathrm{ev}_{t_0}\Big(\varphi^*\ \mathrm{ev}_{\varphi_0(t_0)}\big(I_{(D;\varphi_0(t_0),\mathrm{ev}_{\varphi_0(t_0)}(\alpha\circ\varphi^{-1}))}(\varphi^{-*}f)\big)\Big)\\
&= \mathrm{ev}_{t_0}\big(\varphi^*\ \mathrm{ev}_{\varphi_0(t_0)}(\alpha\circ\varphi^{-1})\big)\\
&= \mathrm{ev}_{t_0}\big(\varphi^*\,(\alpha\circ\varphi^{-1})\big)\\
&= \mathrm{ev}_{t_0}(\alpha) = \alpha
\end{aligned}$$

The point now is that Def **2.8** leads to the sought G-equivariant statement that generalizes the H-equivariant formula given in Prop. **2.4**.

2.9 Theorem. *Let D be any right-invertible derivation, and let φ be any element from the group G of diffeomorphisms of $\mathbb{R}^{1|1}$. Then,*

$$I_{(D\cdot\varphi;\varphi_0^{-1}(t_0),\mathrm{ev}_{\varphi_0^{-1}(t_0)}(\alpha\circ\varphi))}(f\circ\varphi) = \varphi^*\big(I_{(D;t_0,\alpha)}(f)\big)$$

Proof. Let $\psi \in G$ be such that $(D\cdot\varphi)\cdot\psi \in \mathfrak{gl}_{1|1}$. Then, Def. **2.8** says that the integral $I_{\left(D\cdot\varphi;\varphi_0^{-1}(t_0),\mathrm{ev}_{\varphi_0^{-1}(t_0)}(\alpha\circ\varphi)\right)}(f)$ is equal to,

$$\psi^{-*}\Big(I_{\left((D\cdot\varphi)\cdot\psi;\psi_0^{-1}(\varphi_0^{-1}(t_0)),\mathrm{ev}_{\psi_0^{-1}(\varphi_0^{-1}(t_0))}((\mathrm{ev}_{\varphi_0^{-1}(t_0)}(\alpha\circ\varphi))\circ\psi)\right)}(\psi^* f)\Big)$$

On the other hand, $D\cdot(\varphi\circ\psi) \in \mathfrak{gl}_{1|1}$ also implies through Def. **2.8** that the integral $I_{(D;t_0,\alpha)}(f)$ is equal to,

$$(\varphi\circ\psi)^{-*}\Big(I_{\big(D\cdot(\varphi\circ\psi);((\varphi\circ\psi)^{-1})_0(t_0),\mathrm{ev}_{((\varphi\circ\psi)^{-1})_0(t_0)}(\alpha\circ(\varphi\circ\psi))\big)}\big(\,(\varphi\circ\psi)^* f\,\big)\Big)$$

Since $((\varphi\circ\psi)^{-1})_0(t_0) = \psi_0^{-1}(\varphi_0^{-1}(t_0))$, all that has to be checked is the equality,

$$\mathrm{ev}_{((\varphi\circ\psi)^{-1})_0(t_0)}(\alpha\circ(\varphi\circ\psi)) = \mathrm{ev}_{\psi_0^{-1}(\varphi_0^{-1}(t_0))}((\mathrm{ev}_{\varphi_0^{-1}(t_0)}(\alpha\circ\varphi))\circ\psi)$$

but this is a straightforward verification. $\square$

2.10 The change of variable formula. Just as remarked after Prop. **2.4**, the statement of Thm. **2.9** is a change of variable formula associated to the integral $I_{(D;t_0,\alpha)}$ for a given right-invertible derivation, and for any $\varphi \in G$. It is now worth looking at the change of variable formula from elementary calculus at the light of what we have done so far. First of all, since $C^\infty(\mathbb{R}^{1|1})$ is the space of sections $\Gamma(\,\cdot\,,\Lambda^0 T\oplus\Lambda^1 T)$ (see **1.1**), we actually have a short exact sequence of $C^\infty(\mathbb{R})$-modules

$$0\to\Gamma(\,\cdot\,,\Lambda^1 T)\to C^\infty(\mathbb{R}^{1|1})\to\Gamma(\,\cdot\,,\Lambda^0 T) = C^\infty(\mathbb{R})\to 0 \tag{1}$$

that splits. In particular the splitting $f_0\mapsto(f_0,0)$ defines a canonical inclusion $C^\infty(\mathbb{R})\hookrightarrow C^\infty(\mathbb{R}^{1|1})$. Note that with respect to the projection $C^\infty(\mathbb{R}^{1|1})\to C^\infty(\mathbb{R})$, and its splitting, the derivation $D=\dfrac{\partial}{\partial t}\in\mathrm{Der}\,C^\infty(\mathbb{R}^{1|1})$ induces the usual derivation $f_0\mapsto f_0'$ of $C^\infty(\mathbb{R})$ which we shall also denote by $\dfrac{\partial}{\partial t}$. In particular, Thm. **2.9** can be applied to $f_0\in C^\infty(\mathbb{R})$ (upon identifying it with $(f_0,0)\in C^\infty(\mathbb{R}^{1|1})$). We shall momentarily change the standard notation $\int_{t_0} f(t)\;\mathrm{d}t$ for the integral in $\mathbb{R}$, and write $\int_{t_0} f\,D^*$ instead, so as to remind ourselves that the $\mathrm{d}t$ term really stands for the element D^* which is dual to the derivation $D=\dfrac{\partial}{\partial t}$. Thus, by writing $I_{(D\cdot\varphi;\varphi_0^{-1}(t_0),\alpha\circ\varphi)}(f\circ\varphi) = \big(\,I_{(D;t_0,\alpha)}(f)\,\big)\circ\varphi$ in full detail we obtain,

$$\int_{\varphi_0^{-1}(t_0)}(f_0\circ\varphi_0)\,(D\cdot\varphi_0)^* = \varphi_0^*\int_{t_0} f_0\,D^*$$

Since, $\dfrac{\partial}{\partial t}\cdot\varphi = \dfrac{1}{\varphi_0'}\dfrac{\partial}{\partial t}$, the dual element $(D\cdot\varphi_0)^*$ is equal to $\varphi_0'\,D^*$, and therefore,

$$\int_{\varphi_0^{-1}(t_0)}(f_0\circ\varphi_0)\,(D\cdot\varphi_0)^* = \int_{\varphi_0^{-1}(t_0)}(f_0\circ\varphi_0)\,\varphi_0'\,D^*$$

In other words,

$$I_{(D;\varphi_0^{-1}(t_0),\alpha\circ\varphi)}\big(\,(f_0\circ\varphi_0)\,\varphi_0'\,\big) = \big(\,I_{(D;t_0,\alpha)}(f)\,\big)\circ\varphi_0 \tag{2}$$

The difference between this formula, and the formula proved in Thm. **2.9** is the following: The integrals that appear in **2.9** are to be performed with respect to *different derivations;* namely, $D \cdot \varphi_0$, and D, respectively, whereas the integrals in Eq. (2) here are made with respect to the *same derivation.* This is what is commonly referred to as the change of variable formula. One way to prove it in $C^\infty(\mathbb{R})$ is by equating the results of computing $I = \int_{t_0} D\,\varphi_0^* \int_{t_0} f_0$ in two different ways: (1) From the fact that $\int_{t_0} h_0' = h_0 - h_0(t_0)$ (*i.e.,* Barrow's integral formula, or the fundamental theorem of calculus), one first obtains that $I = \varphi_0^* \int_{t_0} f_0 - \int_{t_0}^{\varphi_0(t_0)} f_0 = \varphi_0^* \int_{\varphi_0(t_0)} f_0$. (2) By actually performing the derivative that appears in the integrand of I one obtains $I = \int_{t_0} \varphi_0^*(f_0)\,\varphi_0'$.

With these observations in sight, it is now clear what one should do in order to bring the change of variable formula of Thm. **2.9** closer in appearance to the classical formula of elementary calculus; namely, if $\varphi \in G$ is any diffoemorphism of $\mathbb{R}^{1|1}$, and $f \in C^\infty(\mathbb{R}^{1|1})$ is arbitrary, the change of variable formula consists of computing

$$I := I_{(D;t_0,\mathrm{ev}_{t_0}\alpha)}\Big\{\, D\,\varphi^*\big(\, I_{(D;\varphi_0(t_0),\mathrm{ev}_{\varphi_0(t_0)}(\alpha\circ\varphi^{-1}))}(f)\,\big)\,\Big\} \tag{3}$$

in two different ways: First, one needs a formula for the integral of a derivative (*i.e.,* Barrow's formula; see **2.11** below) in order to apply it directly to $I_{(D;t_0,\mathrm{ev}_{t_0}\alpha)}(D\Psi)$ where Ψ stands for $I_{(D;\varphi_0(t_0),\mathrm{ev}_{\varphi_0(t_0)}(\alpha\circ\varphi^{-1}))}(f)$. Second, by acting directly with the derivativation D on Ψ, and keeping the result inside $I_{(D;t_0,\mathrm{ev}_{t_0}\alpha)}$. The computations are tedious but straightforward. For right-invertible derivations of type (1) the final result does not seem to us particularly illuminating as it contains several terms with double and triple integrals. It can be proved, however, that it reduces properly to the usual change of variable formula both, under the splitting $f_0 \mapsto (f_0, 0)$, and under the projection $(f_0, f_1) \mapsto f_0$, as the interested reader can verify. On the other hand, for a right-invertible derivation of type (2), the formula is much simpler and it reduces to,

$$\varphi^*\big(\, I_{(D;\varphi_0(t_0),\mathrm{ev}_{\varphi(t_0)}\alpha}(f)\,\big) = \Big(\, \alpha_0 + \frac{1}{a_1 b_0}\int_{t_0} \varphi_0'\varphi_0^*\Delta(b;f)\;,\; \varphi_1 \frac{\varphi_0^* f_0}{b_0}\,\Big) \tag{4}$$

In any case, the statement of Thm. **2.9** is more descriptive in the sense that it manifestly shows an equivariance property of the integral with respect to the action of the diffeomorphism group G. What is really useful, however, is the explicit formula for $I_{(D;t_0,\alpha)}(Df)$ (*ie,* Barrow's formula). Note that this integral is of the form $f + \operatorname{Ker} D$, since $I_{(D;t_0,\alpha)}(Df) - f \in \operatorname{Ker} D$. We are interested in giving a detailed description of $\operatorname{Ker} D$, and in computing the integral $I_{(D;t_0,\alpha)}(Df)$.

2.11 Theorem (Barrow's integral formula). *Let D be a right-invertible derivation.*

1. *Let $\mathcal{R}$ be either $\mathbb{R}\times\mathbb{R}^2$, or $\mathbb{R}\times\mathbb{R}$, depending on whether D is of type (1), or (2), respectively. Then,*
$$\operatorname{Ker} D = \{ I_{(D;t_0,\alpha)}(0) \mid (t_0,\alpha) \in \mathcal{R}\}$$
2. *If furthermore $D \in \mathfrak{gl}_{1|1}$, the expression $I_{(D;\tilde{t}_0,\tilde{\alpha})}(0)$, where $\tilde{\alpha} = \mathrm{ev}_{\tilde{t}_0}(I_{(D;t_0,\alpha)}(Df) - f)$ is independent of $\tilde{t}_0$, and its value is $I_{(D;t_0,\mathrm{ev}_{t_0}(\alpha - f))}(0)$.*
3. *For any right-invertible derivation D,*
$$I_{(D;t_0,\alpha)}(Df) = f + I_{(\,D;t_0,\mathrm{ev}_{t_0}(\alpha - f)\,)}(0)$$

Proof. Statement (1) follows from the fact that $Dh = 0$ is equivalent to the system **2.1**(1) with $f = 0$. Statement (2) requires the following:

2.12 Lemma. *Let $D \in \mathfrak{gl}_{1|1}$ be a right-invertible derivation.*

(1) *If D is of type (1), then* $I_{(D;t_0,\alpha)}(Df) = f - \mathrm{ev}_{t_0}(f) + \alpha + \mathrm{ev}_{t_0}\big((\alpha - f)_1\big)\,\Theta_{(D;t_0)}$, *where,*

$$\Theta_{(D;t_0)} = -\begin{cases} \dfrac{a_0^2}{\Delta(a;b)}\big(1 - \mathrm{e}^{-\frac{\Delta(a;b)}{a_0^2}(t-t_0)}\big)\Big(\dfrac{b_0}{a_0}, \dfrac{\Delta(a;b)}{a_0^2}\Big) & \text{if} \quad \Delta(a;b) \neq 0 \\ (t - t_0)\Big(\dfrac{b_0}{a_0}, \dfrac{\Delta(a;b)}{a_0^2}\Big) & \text{if} \quad \Delta(a;b) = 0 \end{cases}$$

(2) *If D is of type (2), then $I_{(D;t_0,\alpha)}(Df) = f - \mathrm{ev}_{t_0}(f_0) + \alpha$.*

Proof. The proof is a series of straightforward computations using the explicit formulæ of Thm. **2.2** as applied to $Df = (a_0 f_0' + b_0 f_1, a_1 f_0' + a_0 f_1' + b_1 f_1)$, and distinguishing between the cases: $(a_0 \neq 0, \Delta(a;b) \neq 0)$, $(a_0 \neq 0, \Delta(a;b) = 0)$, and $(a_0 = 0, a_1 b_0 \neq 0)$. □

Now, to prove statement (2) of the theorem one computes $I_{(D;t_0,\alpha)}(Df) - f$ in the three cases $(a_0 \neq 0, \Delta(a;b) \neq 0)$, $(a_0 \neq 0, \Delta(a;b) = 0)$, and $(a_0 = 0, a_1 b_0 \neq 0)$, and one verifies that in all of them $\mathrm{ev}_{\tilde{t}_0}(I_{(D;t_0,\alpha)}(Df) - f)$ equals $I_{(D;t_0,\mathrm{ev}_{t_0}(\alpha - f))}(0)$.

To prove statement (3), notice that $I_{(D;t_0,\alpha)}(Df) - f \in \operatorname{Ker} D$. Therefore, by part (1), there exist $\tilde{t}_0$, and $\tilde{\alpha}$ such that $I_{(D;t_0,\alpha)}(Df) - f = I_{(D;\tilde{t}_0,\tilde{\alpha})}(0)$. The value of $\tilde{\alpha}$ is $\mathrm{ev}_{\tilde{t}_0}(I_{(D;t_0,\alpha)}(Df) - f)$. Let $\tilde{D} \in \mathfrak{gl}_{1|1}$, and let $\varphi \in G$ such that $D = \tilde{D} \cdot \varphi$. It follows from (2) that $I_{(\tilde{D};t_0,\alpha)}(\tilde{D}f) = f + I_{(\tilde{D};\tilde{t}_0,\tilde{\alpha})}(0)$. Finally, using Def. **2.8** we have:

$$\begin{aligned} I_{(D\cdot\varphi;t_0,\mathrm{ev}_{t_0}\alpha)}\big((D \cdot \varphi)f\big) &= \varphi^* I_{(D;\varphi_0(t_0),\mathrm{ev}_{\varphi_0(t_0)}(\alpha\circ\varphi^{-1}))}(D \circ \varphi^{-*} f) \\ &= \varphi^*(\varphi^{-*} f + I_{(D;\varphi_0(t_0),\mathrm{ev}_{\varphi_0(t_0)}(\alpha\circ\varphi^{-1} - f\circ\varphi^{-1}))}(0)) \\ &= f + I_{(D\cdot\varphi;t_0,\mathrm{ev}_{t_0}(\alpha - f))}(0) \end{aligned}$$

which proves part (3). □

3. Integrals on the Berezin sheaf and SUSY-curves

It was pointed out in **2.3** that the Berezin integral can be related to a particular I_D; namely, that where $D \in \mathfrak{gl}_{1|1}$ is defined by the constant coeficient values $a_0 = b_1 = 0$, and $a_1 = b_0 = 1$. Since the Berezin integral is still one of the most popular tools for performing 'fermionic integration', we shall pause here to compare it with such an I_D in more detail, both, in the smooth and the holomorphic categories.

Remark. In order to make the comparison available in the holomorphic case, note that the problem posed in **2.1** of finding a right inverse for the derivation D on $\mathbb{R}^{1|1}$, and the explicit formulæ in Thm. **2.2** make sense if $\mathbb{C}^{1|1}$ is replaced by $\mathbb{R}^{1|1}$, and one interpretes $\int_{t_0}$ as the indefinite integral of holomorphic functions on the complex plane with base point t_0, and initial condition $\alpha \in \mathbb{C}^2$ or $\alpha \in \mathbb{C}$ depending on the type of orbit. The integral $I_{(D,t_0,\alpha)}(f)$ is well-defined because $\mathbb{C}$ is simply connected, and $f = (f_0, f_1)$ is in this case, a pair of holomorphic functions. All our results in §2 are valid *mutatis mutandis* for simply-connected open holomorphic graded submanifolds of $\mathbb{C}^{1|1}$.

This section is organized as follows: We shall first recall in **3.1** below the coordinate-dependent, as well as the coordinate-free definition of the Berezin integral. We reproduce from [19] one of the best known examples that shows why, in general, one cannot hope for any notion of *indefinite Berezin integral*; that is, a Berezin integral with values in the structure sheaf of the graded manifold rather than in the base field. Section **3.2** recalls the modification to be made in the domain of the Berezin integral — as proposed by M. Rothstein in [19] — in order to avoid the ambiguity thrawn when sections with noncompact support are considered. As a consequence of the work developed in [19], we can prove in Prop. **3.3** that $(m, 1)$-dimensional graded manifolds do admit an indefinite Berezin integral. We then proceed to Thm. **3.4** where we fully compare that indefinite Berezin integral with I_D (for $D = \frac{\partial}{\partial\tau} + \tau\frac{\partial}{\partial t}$) on any $(1, 1)$-dimensional graded manifold with simply connected base. Finally, it is proved in Prop. **3.5** that when the $(1, 1)$-dimensional graded manifold is a SUSY-curve in the sense of [6] (see also [9]), there is a mapping from the Berezinian sheaf into the structure sheaf of the SUSY-curve which locally coincides with I_D.

Conventions and notation. We shall denote by $\mathbb{M}$ an arbitrary graded manifold defined by the ringed space $(M, \mathcal{A})$, where M is a smooth or holomorphic manifold in the usual sense, and $\mathcal{A}$ is its structure sheaf of graded commutative algebras. The nilpotent sheaf of ideals, $\mathcal{N}$, is defined by the exact sequence $0 \to \mathcal{N} \to \mathcal{A} \to \mathcal{O} \to 0$, where $\mathcal{O}$ denotes the sheaf of smooth or holomorphic functions on M, depending on whether $\mathbb{M}$ is a smooth or holomorphic graded manifold, respectively. As the case may be, Ω^k (resp., Ω^k_c) will denote the sheaf of smooth or holomorphic k-forms (resp., k-forms with compact support) on M. Finally, we shall adhere ourselves to the convention that when a set of local coordinates $x_1, \dots, x_m, \theta_1, \dots, \theta_n$ is given, the x's are even, and the θ's are odd. If other systems of coordinates need to be considered (as in Section **4** below), the convention is that latin character coordinates are even, and greek character coordinates are odd.

3.1 The Berezin integral. Recall first that the Berezinian sheaf of a given (m, n)-dimensional graded manifold $\mathbb{M}$ is a locally free sheaf of $\mathcal{A}$-modules of rank 1 whose sections are called *Berezin forms*. It is denoted by $\mathrm{Ber}_{\mathbb{M}}$, or simply by Ber if the graded manifold $\mathbb{M}$ is clear from the context. It can be described in terms of a given set of local coordinates $x_1, \dots, x_m, \theta_1, \dots, \theta_n$ defined on some open subset U of M as follows: Write $\mathrm{d}^*(x;\theta)$ for the generator of $\mathrm{Ber}(U)$. If $U' \subset M$ is some other open subset with local coordinates $x'_1, \dots, x'_m, \theta'_1, \dots, \theta'_n$, then the following relationship holds true in $\mathrm{Ber}(U' \cap U)$:

$$\mathrm{d}^*(x;\theta) = \mathrm{d}^*(x';\theta')\ \mathrm{Ber}\Big(\frac{\partial(x;\theta)}{\partial(x';\theta')}\Big)$$

where $\mathrm{Ber}\Big(\frac{\partial(x;\theta)}{\partial(x';\theta')}\Big)$ stands for the Berezinian of the jacobian matrix associated to the change of variables $x'_i \mapsto x_i$, $\theta'_\mu \mapsto \theta_\mu$ (see [11] for details). Now, the domain of the Berezin integral is the subsheaf Ber_c consisting of those sections from Ber having compact support, while its codomain is the base field (*ie,* $\mathbb{R}$ or $\mathbb{C}$). It is for this reason that the Berezin integral is a *definite integral.*

Let us now recall from [11] the description of the Berezin integral $\int_{\mathrm{Ber}} : \mathrm{Ber}_c \to \mathbb{R}$. One considers first the sheaf complex of integral forms $\delta : \Sigma^* \to \Sigma^{*+1}$ introduced in [2]-[3] (see [11: Ch.IV, §5.4] for details). It is proved in [11: Ch.IV, §6] that: (1) $\mathrm{Ber}_c = \Sigma^n = \mathrm{Im}(\delta) + \mathcal{N}^n \mathrm{Ber}_c$, and furthermore, $\int_{\mathrm{Ber}} \omega = 0$ if and only if $\omega \in \mathrm{Im}(\delta)$. (2) The submondule $\mathcal{N}^n \mathrm{Ber}_c$ is canonically isomorphic to Ω_c^m ($m = \dim M$), and furthermore $\mathrm{Ber}_c / \mathrm{Im}(\delta) \simeq \Omega_c^m / \mathrm{d}\, \Omega_c^{m-1}$. Therefore, the Berezin integral $\int_{\mathrm{Ber}}$ is defined as the composition of the canonical projection $\mathrm{Ber}_c \to \mathrm{Ber}_c / \mathrm{Im}(\delta)$ followed by the usual (definite) integration on top forms: $\int_M : \Omega_c^m / \mathrm{d}\, \Omega_c^{m-1} \to \mathbb{R}$. In terms of local coordinates this amounts to:

$$\begin{aligned} &\mathrm{Ber}_c(U) \ni \omega = \mathrm{d}^*(x;\theta)(f_0(x) + \sum_\mu f_\mu(x)\, \theta_\mu + \cdots + f_{1\cdots n}(x)\, \theta_1 \cdots \theta_n) \\ &\qquad\qquad \mapsto \int_{\mathrm{Ber}} \omega = \int_M f_{1\cdots n}(x)\, \mathrm{d}x_1 \cdots \mathrm{d}x_m \end{aligned} \tag{1}$$

It becomes clear from this description that the particularities of the Berezin integral arise from the construction of the appropriate volume sheaf Ber_c rather than from the integration procedure itself. At first sight, one might think that the reason for integrating only compactly supported Berezin forms is to ensure the convergence of $\int_M f_{1\cdots n}(x)\, \mathrm{d}x_1 \cdots \mathrm{d}x_m$. But if this was the case, it should be possible to define an indefinite Berezin integral on any $(1,n)$-dimensional graded manifold by the same procedure as before, provided that the base manifold is simply connected to allow a well-defined indefinite integral; namely, the canonical projection $\mathrm{Ber} \to \mathrm{Ber} / \mathrm{Im}(\delta)$ followed by the indefinite integration. Thus, for one even coordinate $x = x_1$, and a given choice of base point $t_0 \in M$, we would have,

$$\int_{t_0} \mathrm{d}^*(x;\theta)(f_0(x) + \sum_\mu f_\mu(x)\, \theta_\mu + \cdots + f_{1\cdots n}(x)\, \theta_1 \cdots \theta_n) = \int_{t_0} f_{1\cdots n}(x)\ \mathrm{d}x \tag{2}$$

This integral, however, is not well-defined. An easy example on the graded manifold $\mathbb{R}^{1|2}$ that points out the difficulty was given in [19]: Let us consider two different coordinate systems x, θ_1, θ_2, and $x', \theta_1', \theta_2'\}$ on $\mathbb{R}^{1|2}$, related between them by means of $x = x' + \theta_1'\theta_2'$, and $\theta_\mu = \theta_\mu'$ ($\mu = 1, 2$). Then, the prescription above would give (*cf,* [19]),

$$\int_0 \mathrm{d}^*(x;\theta)\, x = \frac{x^2}{2}, \qquad \text{whereas} \qquad \int_0 \mathrm{d}^*(x';\theta')\, (x' + \theta_1'\theta_2') = x' \tag{3}$$

3.2 The Rothstein correction. To correct this ambiguity M. Rothstein proposed in [19] an alternative volume sheaf for a given (m,n)-dimensional graded manifold $\mathbb{M}$; namely, the sheaf $\Omega^m \otimes \mathcal{D}$, where $\mathcal{D}$ is the sheaf of differential operators on $\mathbb{M}$. It is proved in [19] that this volume sheaf is a locally free sheaf of $\mathcal{A}$-modules generated on a coordinate neighborhood U with coordinates $x_1, \ldots, x_m, \theta_1, \ldots, \theta_n$ by $\mathrm{d}x \otimes \dfrac{\partial}{\partial x_I} \circ \dfrac{\partial}{\partial \theta}$ where $\mathrm{d}x$ stands for $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m$, $\dfrac{\partial}{\partial \theta}$ stands for $\dfrac{\partial}{\partial \theta_1} \circ \ldots \circ \dfrac{\partial}{\partial \theta_n}$, and for each multi-indexed sequence $I = \{i_1, \ldots, i_p\}$ $(p \geq 0)$, $\dfrac{\partial}{\partial x_I}$ stands for $\dfrac{\partial}{\partial x_{i_1}} \circ \ldots \circ \dfrac{\partial}{\partial x_{i_p}}$. The relationship between this volume sheaf, and the Berezinian sheaf, was given in [19] (see also [7]) through the short exact sequence of $\mathcal{D}$-modules,

$$0 \to (\Omega^m \otimes \mathcal{D})^+ \to \Omega^m \otimes \mathcal{D} \to \mathrm{Ber} \to 0 \tag{1}$$

where the $\mathcal{D}$-submodule $(\Omega^m \otimes \mathcal{D})^+$ has been characterized in [19] by the property that for any section ω from it, and for any section f from $\mathcal{A}$, the value $\omega(f)$ is actually an exact m-form on the base M (see [19] for details). The Rothstein integral of a section ω from $\Omega^m \otimes \mathcal{D}$ is then defined as $\int_{\mathrm{Roth}} \omega = \int_M \omega(1)$.

The important result for our purposes is that the Rothstein integral does correct the ambiguity pointed out in **3.1**(3) above. In fact, the value of the Rothstein integral of any volume form ω from $\Omega^m \otimes \mathcal{D}$ is independent of the coordinate system. This amounts to prove that $\int_{\mathrm{Roth}}$ commutes with pullback under graded manifold diffeomorphisms, regardless of whether the supports are compact or not (see [19] for the proofs). Once this point is settled, it follows that ambiguities like **3.1**(3) cannot arise within the Rothstein approach. Therefore, there is a well-defined indefinite Rothstein integral when the base manifold is one-dimensional and simply connected; namely, for a given choice of base point $t_0 \in M$,

$$(\Omega^1 \otimes \mathcal{D})(M) \ni \omega \mapsto \int_{t_0} \omega(1) \in \mathcal{C}^\infty(\mathbb{M}) \tag{2}$$

Having these results in mind we can now prove the following:

3.3 Proposition. *For any $(m,1)$-dimensional graded manifold,*

(1) *The short exact sequence of $\mathcal{D}$-modules* **3.2**(1) *splits.*

(2) *The restriction of the Rothstein integral to* $\mathrm{Ber}_c \subset \mathrm{Ber} \subset \Omega^m \otimes \mathcal{D}$ *is the Berezin integral. In particular, the Berezin integral can be extended from* Ber_c *to* Ber.

(3) *There is a subsheaf $\mathcal{D}' \subset \mathcal{D}$ generated over each coordinate neighborhood with odd coordinate θ, by the differential operator $\dfrac{\partial}{\partial\theta} \circ \theta$. Furthermore, $\Omega^m \otimes \mathcal{D}'$ is isomorphic to Ω^m, and the restriction of the Rothstein integral to $\Omega^m \otimes \mathcal{D}'$ is the usual integral of m-forms on the base manifold.*

Proof. (1) Let U be a coordinate neighborhood with even coordinates $x_1, \ldots, x_m$, and odd coordinate θ. Write $\mathrm{D}_I(x,\theta)$ for the local basis of $\Omega^m \otimes \mathcal{D}$ $\mathrm{d}x \otimes \dfrac{\partial}{\partial x_I} \circ \dfrac{\partial}{\partial \theta}$, $I = \{i_1, \ldots, i_p\}$. Now, $\mathrm{D}_0(x,\theta)$ is by definition $\mathrm{d}x \otimes \dfrac{\partial}{\partial\theta}$. Let U' be some other coordinate neighborhood with even coordinates $x'_1, \ldots, x'_m$, and odd coordinate θ'. Suppose $U' \cap U$ is nonempty, and that x' and θ' are expressed in terms of x, and θ. Then, $\mathrm{D}_I(x',\theta') = \sum_{|J| \geq 0} \mathrm{D}_J(x,\theta)\Phi^J_I$. Since the odd dimension is one, $x'_i = \varphi^i_0(x)$, and $\theta' = \varphi_1(x)\theta$. Thus, $\mathrm{D}_0(x',\theta') = \mathrm{Ber}\Big(\dfrac{\partial(x',\theta')}{\partial(x,\theta)}\Big)\mathrm{D}_0(x,\theta) =$

$\dfrac{1}{\varphi_1}\det\Bigl(\dfrac{\partial\varphi_0^i}{\partial x_j}\Bigr)\mathrm{D}_0(x,\theta)$. Therefore, the subsheaf locally generated by $\mathrm{D}_0(x',\theta')$ is precisely Ber, and the map $\mathrm{d}^*(x,\theta)\mapsto \mathrm{D}_0(x,\theta)$ gives the desired splitting.

(2) The Berezin integral of a section $\beta\in\mathrm{Ber}(U)$ with local expression $\mathrm{d}^*(x,\theta)(f_0(x) + \theta f_1(x))$ is the usual integral of the m-form $f_1(x)\,\mathrm{d}x$. Under the splitting given in (1), β corresponds to the section ω from $\Omega^m\otimes\mathcal{D}$ having local expression $\mathrm{D}_0(x,\theta)$ $(f_0(x)+\theta f_1(x))$. Since $\omega(1)$ is equal to $\mathrm{d}x\dfrac{\partial(f_0(x)+\theta f_1(x))}{\partial\theta} = \mathrm{d}x f_1(x)$, it follows that the Berezin integral of β and the Rothstein integral of its corresponding ω are the same.

(3) Under a change of coordinates like the one considered in (1), $\dfrac{\partial}{\partial\theta}\circ\theta$ is transformed into $\dfrac{\partial}{\partial\theta'}\circ\theta'$. Thus, $\mathcal{D}'$ is well defined. Since the pullback of $\mathrm{d}x\otimes\dfrac{\partial}{\partial\theta}\circ\theta$ under the given change of coordinates is $\mathrm{d}x'\otimes\dfrac{\partial}{\partial\theta'}\circ\theta'\det\Bigl(\dfrac{\partial\varphi_0^i}{\partial x_j}\Bigr)$, it follows that $\Omega^m\otimes\mathcal{D}'$ is isomorphic to Ω^m. The fact that the restriction of the Rothstein integral coincides with the usual integral is a direct consequence of the definition. $\square$

Remark. This proposition guarantees that indefinite Berezin integrals are well defined on any $(1,1)$-dimensional (smooth or holomorphic) graded manifold with simply connected base. Therefore, we can now proceed to compare the indefinite Berezin integral obtained via this proposition with I_D of §2, where $D=\theta\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial\theta}$ in the local coordinates x,θ. **Note:** This derivation will be kept fixed until the end of this section.

3.4 Theorem. *For any $(1,1)$-dimensional smooth or holomorphic graded manifold with simply connected base manifold M, and underlying base field $\mathbb{F}$ ($\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$),*

(1) *The map*

$$\mathcal{N}\,\mathrm{Ber}\longrightarrow\mathcal{O}/\mathbb{F}$$

which locally sends $\mathrm{d}^*(x,\theta)\theta f(x)$ into the class of $I_{(D;t_0,\alpha)}(\theta f(x))$ modulo the additive action of $\mathbb{F}$ in $\mathcal{O}$, is well defined, and if $\alpha=0$ it coincides with the class of the indefinite Berezin integral.

(2) *Define the definite integral $I_{(D;\alpha)}$ by the formula of Thm.* **2.2**(2)*, with the only difference that $\int_{t_0}$ gets replaced by $\int_M$. Then, the map*

$$\mathcal{N}\,\mathrm{Ber}_c\longrightarrow\mathbb{F}$$

which locally sends $\mathrm{d}^*(x,\theta)\theta f(x)$ into $I_{(D;\alpha)}(\theta f(x))$, is well defined. If $\alpha=0$, it coincides with the (definite) Berezin integral.

Proof. (1) is proved by showing that the difference

$$\varphi^*\Bigl(I_{(D;t_0,\alpha)}\theta f(x)) - I_{(D;t_0,\alpha)}\varphi^*(\theta f(x))\frac{\varphi_0'}{\varphi_1}\Bigr)$$

is constant, for any diffeomorphism φ of the graded manifold locally given by $x\mapsto x'=\varphi^*x=\varphi_0$, and $\theta\mapsto\theta'=\varphi^*\theta=\varphi_1\theta$. This, however, follows immediately from a straightforward computation using Thm. **2.2**(2). As a matter of fact, the same computation shows that the map in the statement is precisely the indefinite Berezin integral.

Similarly, that the map in statement (2) is well defined is proved by a strightforward computation: Let ω be a section from $\mathcal{N}\,\mathrm{Ber}_c$ locally written as $\mathrm{d}^*(x,\theta)\theta f(x)$, and let φ be any diffeomorphism of the graded manifold as in the proof of (1). Whence, on the one hand we have, $\varphi^*\big(I_{(D;\alpha)}(\theta f(x))\big) = \varphi^*(\alpha + \int_M f_1) = \alpha + \int_M \varphi_0'\varphi_0^* f_1$. On the other hand, $\varphi^*\big(\mathrm{d}^*(x,\theta)\theta f(x)\big) = \mathrm{d}^*(x,\theta)\dfrac{\varphi_0'}{\varphi_1}\varphi_1\theta\,\varphi^* f$. Hence, $I_{(D;\alpha)}\big(\dfrac{\varphi_0'}{\varphi_1}\varphi_1\theta\,\varphi^* f\big) = \alpha + \int_M \varphi_0'\,\varphi_0^* f_1$. $\square$

When the graded manifold is a SUSY-curve, Thm. **3.4** can be nicely improved. Recall from [6] that a SUSY-curve is a $(1,1)$-dimensional holomorphic graded manifold that admits an atlas of *canonical coordinates*. This is, by definition, an atlas such that if two systems of local coordinates z,θ and z',θ' are related by a biholomorphic map φ: $z \mapsto z' = \varphi^*(z) = \varphi_0$, and $\theta \mapsto \theta' = \varphi^*(\theta) = \varphi_1\theta$, then there exists a holomorphic superfunction ψ such that $D\cdot\varphi = \psi\,D'$. The local derivation $D = \theta\dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial\theta}$ is called the canonical local derivation associated to the canonical coordinates $\{z,\theta\}$ (see [9] for an equivalent definition in the more general context of SUSY-curves over graded manifolds. Note, however, that no condition on compactness is here assumed). Since the canonical local derivations are odd, ψ is necessarily even. In terms of the coefficients of the canonical local derivations D and D' associated to the canonical coordinates z,θ and z',θ', this condition means

$$D\cdot\varphi = \Big(0,\frac{\varphi_1}{\varphi_0'},\frac{1}{\varphi_1},0\Big) = \Big(0,\psi_0,\psi_0,0\Big) = \psi_0 D'$$

In other words, the transition from one system of coordinates into the other must satisfy the relationship $\varphi_0' = \varphi_1^2$.

3.5 Proposition. *For any SUSY-curve, there exists a canonical map*

$$\mathrm{Ber} \longrightarrow \mathcal{O}/\mathbb{C}$$

which locally coincides with the integral $I_{(D;t_0,0)}$ of Thm. **2.2**(2)*, D being the canonical local derivation.*

Proof. The map is locally defined as $\mathrm{d}^*(z,\zeta)f \to I_{(D;t_0,0)}(f)$, where the base point has been chosen arbitrarily. Denote the local image of $\mathrm{d}^*(z;\zeta)f$ by $I(\mathrm{d}^*(z;\zeta)f)$. A simple computation shows that

$$\varphi^*(I(\mathrm{d}^*(z;\zeta)f)) - I(\varphi^*(\mathrm{d}^*(z;\zeta)f)) = \int_{\varphi_0^{-1}(t_0)}^{t_0'} \varphi_0'\varphi_0^*(f_1)$$

where t_0' is the base point chosen to compute $I(\mathrm{d}^*(z',\zeta')\varphi^*(f))$, and the fact that $\varphi_0' = \varphi_1^2$ for a SUSY-curve has been used. $\square$

3.6 Corollary. *For any holomorphic curve with a choice of a spin structure* $\kappa^{1/2}$*, there exists a canonical integration map*

$$\kappa^{1/2} \otimes \Lambda\kappa^{1/2} \longrightarrow \Lambda\kappa^{1/2}/\mathbb{C}$$

which locally sends $\sqrt{dz} \otimes (f_0 + \sqrt{dz} f_1)$ *into* $\int f_1(z)\,dz + \sqrt{dz} f_0$*.*

Proof. This is a direct consequence of Prop. **3.5**, and of the first Example in Section 1 of [9]. $\square$

4. CALCULUS OF VARIATIONS FOR $I_{(D;t_0,\alpha)}$ WITH $D \in \mathfrak{gl}_{1|1}$

Our objective in this section is to write down the Euler-Lagrange equations derived from the calculus of variations as applied to the integral $I_{(D;t_0,\alpha)}$ for any constant coefficient, right-invertible derivation $D \in \mathfrak{gl}_{1|1}$ (see also [14]).

We shall need to consider several graded manifolds, all of which are obtained by restricting the structure sheaf of $\mathbb{R}^{1|1}$ to intervals $J \subset \mathbb{R}$. We shall therefore denote by $J^{1|1}$ the graded manifold $\big(J\,,\,\Gamma(\,\cdot\,,\Lambda T)|_J\big)$. The restriction to $J^{1|1}$ of the standard coordinates of $\mathbb{R}^{1|1}$ introduced in **1.1**, will be called *the standard coordinates on* $J^{1|1}$. We shall need to put different names on the standard coordinates arising this way in order to maintain a clear distinction between the various graded manifolds that appear. For the moment, we shall let t, τ be the standard coordinates on $[0,1]^{1|1}$, and let s, ξ be the standard coordinates on $(-\varepsilon,\varepsilon)^{1|1}$, where $\varepsilon > 0$. The standard coordinates in the whole $\mathbb{R}^{1|1}$ will be here denoted by x, θ.

We are only interested in the local appearance of the Euler-Lagrange equations, since the main issue here is the integral $I_{(D;t_0,\alpha)}$ itself. Therefore we shall work on the (m,n)-dimensional graded manifold $\mathbb{R}^{m|n}$ with coordinates x^i, θ^μ ($1 \le i \le m$, and $1 \le \mu \le n$).

We shall also need to consider the graded manifold associated to the tangent sheaf whose sections are derivations of sections of the structure sheaf of $\mathbb{R}^{m|n}$. This is the trivial graded vector bundle $\mathrm{T}\,\mathbb{R}^{m|n} = \mathbb{R}^{m|n} \times \mathbb{R}^{m+n|m+n} \to \mathbb{R}^{m|n}$. If x^i, θ^μ are the coordinates on the base, the coordinates on the fiber will be denoted by $\dot{x}^i, \pi\dot{\theta}^\mu, \dot{\theta}^\mu, \pi\dot{x}^i$, where $|\dot{x}^i| = 0 = |\pi\dot{\theta}^\mu|$, and $|\dot{\theta}^\mu| = 1 = |\pi\dot{x}^i|$ (see [13], [16], or [22] for details).

4.1 Variations of $(1,1)$-dimensional paths. Let $\sigma : [0,1]^{1|1} \to \mathbb{R}^{m|n}$ be a given curve in $\mathbb{R}^{m|n}$, and let $j\sigma : [0,1]^{1|1} \to \mathrm{T}\,\mathbb{R}^{m|n}$ be its natural lifting to the tangent bundle. Recall that $j\sigma$ is defined in terms of σ as follows:

$$(1) \qquad \begin{array}{lll} (j\sigma)^* x^i = \sigma^* x^i & (j\sigma)^* \dot{x}^i = \dfrac{\partial \sigma^* x^i}{\partial t} & (j\sigma)^* \pi\dot{x}^i = \dfrac{\partial \sigma^* x^i}{\partial \tau} \\ (j\sigma)^* \theta^\mu = \sigma^* \theta^\mu & (j\sigma)^* \dot{\theta}^\mu = \dfrac{\partial \sigma^* \theta^\mu}{\partial t} & (j\sigma)^* \pi\dot{\theta}^\mu = \dfrac{\partial \sigma^* \theta^\mu}{\partial \tau} \end{array}$$

A *variation* of $j\sigma$ is a morphism $\delta j\sigma : (-\varepsilon,\varepsilon)^{1|1} \times [0,1]^{1|1} \to \mathrm{T}\,\mathbb{R}^{m|n}$ such that $\delta j\sigma \circ \mathrm{e}_0 = j\sigma$, where $\mathrm{e}_0 : [0,1]^{1|1} \to (-\varepsilon,\varepsilon)^{1|1} \times [0,1]^{1|1}$ is defined so that the pullback under e_0 of the standard coordinates of $(-\varepsilon,\varepsilon)^{1|1}$ factor is identically zero, whereas the pullback of any section coming from the $[0,1]^{1|1}$ factor remains unchanged under e_0 (see [12] where this map was thoroughly used in the theory of differential equations on graded manifolds).

Let $L : \mathrm{T}\,\mathbb{R}^{m|n} \to \mathbb{R}^{1|1}$ be a given morphism (*Lagrangian*), and consider the composition $L\circ\delta j\sigma : (-\varepsilon,\varepsilon)^{1|1}\times[0,1]^{1|1} \to \mathbb{R}^{1|1}$. This morhpism defines a section of the structure sheaf of the graded manifold $(-\varepsilon,\varepsilon)^{1|1}\times[0,1]^{1|1}$ via the fundamental correspondence of Prop. **1.2**. According to Prop. **1.2**, this section is $(L\circ\delta j\sigma)^*(x+\theta)$. Now, let $\partial := \dfrac{\partial}{\partial s} + \dfrac{\partial}{\partial\xi} \in \operatorname{Der} C^\infty((-\varepsilon,\varepsilon)^{1|1})$, and consider,

$$\mathrm{e}_0^*\,\partial\,(L\circ\delta j\sigma)^*(x+\theta) \in C^\infty(\,[0,1]^{1|1}) \tag{2}$$

This section can now be integrated with respect to any right invertible D. The fundamental problem of the calculus of variations, as applied to the integral $I_{(D;t_0,\alpha)}$ is to actually determine σ after imposing the stationary condition,

$$I_{(D;t_0,\alpha)}\big(\mathrm{e}_0^*\,\partial\,(L\circ\delta j\sigma)^*(x+\theta)\big) = 0, \qquad \text{for any variation } \ \delta j\sigma\,. \tag{3}$$

The fact that this is precisely the statement of the calculus of variations becomes apparent upon interpreting the expression $\mathrm{e}_0^*\,\partial\,(L\circ\delta j\sigma)^*(x+\theta)$ in a geometrical fashion at the light of the theory of superdifferential equations and Lie derivatives (see [12] for details).

We now directly use the chain rule (see [8]) to compute $\mathrm{e}_0^*\,\partial\,(L\circ\delta j\sigma)^*(f)$, for an arbitrary $f\in C^\infty(\mathbb{R}^{1|1})$. We look first at,

$$\begin{aligned}\frac{\partial}{\partial s}\big\{(\delta j\sigma)^*(L^*f)\big\} &= \frac{\partial(\delta j\sigma)^*x^i}{\partial s}\,(\delta j\sigma)^*\frac{\partial L^*f}{\partial x^i} + \frac{\partial(\delta j\sigma)^*\theta^\mu}{\partial s}\,(\delta j\sigma)^*\frac{\partial L^*f}{\partial\theta^\mu}\\ &\quad+\frac{\partial(\delta j\sigma)^*\dot{x}^i}{\partial s}\,(\delta j\sigma)^*\frac{\partial L^*f}{\partial\dot{x}^i} + \frac{\partial(\delta j\sigma)^*\dot{\theta}^\mu}{\partial s}\,(\delta j\sigma)^*\frac{\partial L^*f}{\partial\dot{\theta}^\mu}\\ &\quad+\frac{\partial(\delta j\sigma)^*\pi\dot{x}^i}{\partial s}\,(\delta j\sigma)^*\frac{\partial L^*f}{\partial\pi\dot{x}^i} + \frac{\partial(\delta j\sigma)^*\pi\dot{\theta}^\mu}{\partial s}\,(\delta j\sigma)^*\frac{\partial L^*f}{\partial\pi\dot{\theta}^\mu}\end{aligned} \tag{4}$$

where we understand that sums are taken over repeated indices. We shall now introduce a convenient notation: For any coordinate Y of $\mathrm{T}\,\mathbb{R}^{m|n}$, we set,

$$Y_s = \mathrm{e}_0^*\,\frac{\partial(\delta j\sigma)^*Y}{\partial s} \qquad \text{and} \qquad Y_\xi = \mathrm{e}_0^*\,\frac{\partial(\delta j\sigma)^*Y}{\partial\xi} \tag{5}$$

Therefore, after applying e_0^* on both sides of (4), we obtain,

$$\begin{aligned}\mathrm{e}_0^*\frac{\partial}{\partial s}\big\{(\delta j\sigma)^*(L^*f)\big\} &= x_s^i\,(j\sigma)^*\frac{\partial L^*f}{\partial x^i} + \theta_s^\mu\,(j\sigma)^*\frac{\partial L^*f}{\partial\theta^\mu}\\ &\quad+\dot{x}_s^i\,(j\sigma)^*\frac{\partial L^*f}{\partial\dot{x}^i} + \dot{\theta}_s^\mu\,(j\sigma)^*\frac{\partial L^*f}{\partial\dot{\theta}^\mu}\\ &\quad+\pi\dot{x}_s^i\,(j\sigma)^*\frac{\partial L^*f}{\partial\pi\dot{x}^i} + \pi\dot{\theta}_s^\mu\,(j\sigma)^*\frac{\partial L^*f}{\partial\pi\dot{\theta}^\mu}\end{aligned} \tag{6}$$

and similarly,

$$\mathrm{e}_0^*\frac{\partial}{\partial\xi}\left\{(\delta j\sigma)^*(L^*f)\right\} = x_\xi^i\,(j\sigma)^*\frac{\partial L^*f}{\partial x^i} + \theta_\xi^\mu\,(j\sigma)^*\frac{\partial L^*f}{\partial\theta^\mu} \tag{7}$$
$$+\,\dot{x}_\xi^i\,(j\sigma)^*\frac{\partial L^*f}{\partial\dot{x}^i} + \dot\theta_\xi^\mu\,(j\sigma)^*\frac{\partial L^*f}{\partial\dot\theta^\mu}$$
$$+\,\pi\dot{x}_\xi^i\,(j\sigma)^*\frac{\partial L^*f}{\partial\pi\dot{x}^i} + \pi\dot\theta_\xi^\mu\,(j\sigma)^*\frac{\partial L^*f}{\partial\pi\dot\theta^\mu}$$

4.2 Equations of variation. A typical Taylor expansion for $(\delta j\sigma)^*Y$ is the following:

$$\begin{aligned}(\delta j\sigma)^*Y &= Y_0 + Y_1\xi + Y_2\tau + Y_{12}\xi\tau\\ &= Y_0(0,t) + \frac{\partial Y_0}{\partial s}(0,t)\,s + \mathrm{o}(s^2) + \left(\,Y_1(0,t) + \frac{\partial Y_1}{\partial s}(0,t)\,s + \mathrm{o}(s^2)\,\right)\xi\\ &\quad + \left(\,Y_2(0,t) + \frac{\partial Y_2}{\partial s}(0,t)\,s + \mathrm{o}(s^2)\,\right)\tau\\ &\quad + \left(\,Y_{12}(0,t) + \frac{\partial Y_{12}}{\partial s}(0,t)\,s + \mathrm{o}(s^2)\,\right)\xi\tau\end{aligned} \tag{1}$$

where $Y_0(0,t) + Y_2(0,t)\,\tau = (j\sigma)^*Y$. Therefore, typical expressions for Y_s, and Y_ξ, are as follows:

$$\begin{gathered} Y_s = \mathrm{e}_0^*\,\frac{\partial(\delta j\sigma)^*Y}{\partial s} = \frac{\partial Y_0}{\partial s}(0,t) + \frac{\partial Y_2}{\partial s}(0,t)\,\tau\\ \text{and}\\ Y_\xi = \mathrm{e}_0^*\,\frac{\partial(\delta j\sigma)^*Y}{\partial\xi} = Y_1(0,t) + Y_{12}(0,t)\,\tau\end{gathered} \tag{2}$$

Since Y stands for any of the coordinates in $\mathrm{T}\,\mathbb{R}^{m|n}$, the parity of both, Y_s, and Y_ξ is well defined, and therefore,

$$Y_s = \begin{cases}\dfrac{\partial Y_0}{\partial s}(0,t) & \text{If } |Y| = 0\\[2mm] \dfrac{\partial Y_2}{\partial s}(0,t)\,\tau & \text{If } |Y| = 1\end{cases} \qquad\text{and}\qquad Y_\xi = \begin{cases} Y_{12}(0,t)\,\tau & \text{If } |Y| = 0\\ Y_1(0,t) & \text{If } |Y| = 1\end{cases} \tag{3}$$

At any rate, for an arbitrary variation $\delta j\sigma$ of $j\sigma$, the functions $\dfrac{\partial Y_0}{\partial s}(0,t)$, $\dfrac{\partial Y_2}{\partial s}(0,t)$, $Y_1(0,t)$, and $Y_{12}(0,t)$ are arbitrary, and one may choose — as it is standard in the ordinary calculus of variations — the various functions Y_0, Y_1, Y_2, and Y_{12} so that the following relations hold true:

$$\begin{array}{llll} \dot{x}_s^i = \dfrac{\partial x_s^i}{\partial t} & \dot\theta_s^\mu = \dfrac{\partial\theta_s^\mu}{\partial t} & \pi\dot{x}_s^i = 0 & \pi\dot\theta_s^\mu = \dfrac{\partial\theta_s^\mu}{\partial\tau}\\[3mm] \dot{x}_\xi^i = \dfrac{\partial x_\xi^i}{\partial t} & \dot\theta_\xi^\mu = \dfrac{\partial\theta_\xi^\mu}{\partial t} & \pi\dot{x}_\xi^i = \dfrac{\partial x_\xi^i}{\partial\tau} & \pi\dot\theta_\xi^\mu = \tau\,\theta_\xi^\mu \end{array} \tag{4}$$

We mention in passing that the equations $\pi\dot{x}_s^i = 0$ are forced by the fact $j\sigma^*\pi\dot{x}^i = \dfrac{\partial\sigma^* x^i}{\partial\tau} = 0$, as there is only one odd variable in $[0,1]^{1|1}$. We can now substitute these equations into **4.1**(6)-(7) above so as to conclude that,

$$
\begin{aligned}
\mathrm{e}_0^*\frac{\partial}{\partial s}\left\{(\delta j\sigma)^*(L^*f)\right\} &= x_s^i\left((j\sigma)^*\frac{\partial L^*f}{\partial x^i}-\frac{\partial}{\partial t}(j\sigma)^*\frac{\partial L^*f}{\partial \dot{x}^i}\right)\\
&\quad+\theta_s^\mu\left((j\sigma)^*\frac{\partial L^*f}{\partial \theta^\mu}-\frac{\partial}{\partial t}(j\sigma)^*\frac{\partial L^*f}{\partial \dot{\theta}^\mu}+\frac{\partial}{\partial \tau}(j\sigma)^*\frac{\partial L^*f}{\partial \pi\dot{\theta}^\mu}\right)\\
&\quad+\frac{\partial}{\partial t}\left(x_s^i\,(j\sigma)^*\frac{\partial L^*f}{\partial \dot{x}^i}+\theta_s^\mu\,(j\sigma)^*\frac{\partial L^*f}{\partial \dot{\theta}^\mu}\right)\\
&\quad+\frac{\partial}{\partial \tau}\left(\theta_s^\mu\,(j\sigma)^*\frac{\partial L^*f}{\partial \pi\dot{\theta}^\mu}\right)
\end{aligned}
\tag{5}
$$

and

$$
\begin{aligned}
\mathrm{e}_0^*\frac{\partial}{\partial \xi}\left\{(\delta j\sigma)^*(L^*f)\right\} &= x_\xi^i\left((j\sigma)^*\frac{\partial L^*f}{\partial x^i}-\frac{\partial}{\partial t}(j\sigma)^*\frac{\partial L^*f}{\partial \dot{x}^i}+\frac{\partial}{\partial \tau}(j\sigma)^*\frac{\partial L^*f}{\partial \pi\dot{x}^i}\right)\\
&\quad+\theta_\xi^\mu\left((j\sigma)^*\frac{\partial L^*f}{\partial \theta^\mu}-\frac{\partial}{\partial t}(j\sigma)^*\frac{\partial L^*f}{\partial \dot{\theta}^\mu}+\tau\,(j\sigma)^*\frac{\partial L^*f}{\partial \pi\theta^\mu}\right)\\
&\quad+\frac{\partial}{\partial t}\left(x_\xi^i\,(j\sigma)^*\frac{\partial L^*f}{\partial \dot{x}^i}+\theta_\xi^\mu\,(j\sigma)^*\frac{\partial L^*f}{\partial \dot{\theta}^\mu}\right)\\
&\quad+\frac{\partial}{\partial \tau}\left(x_\xi^i\,(j\sigma)^*\frac{\partial L^*f}{\partial \pi\dot{x}^i}\right)
\end{aligned}
\tag{6}
$$

We now follow the standard procedure of the calculus of variations: Since the variation is arbitrary, we may choose one for which $x_s^i=\psi\delta_{ij}$, while $x_\xi^i=\theta_s^\mu=\theta_\xi^\mu=0$ with ψ being an arbitrary smooth function, and δ_{ij} the Kronecker delta. Similarly, one may choose a variation for which $x_\xi^i=\psi\delta_{ij}\tau$, while $x_s^i=\theta_s^\mu=\theta_\xi^\mu=0$, etc. We will therefore have four types of possibilities for the integrand $\mathrm{e}_0^*\,\partial\left\{(\delta j\sigma)^*(L^*(x+\theta))\right\}$. In each of these four cases we may write the integrand in the form

$$
\left(\psi F_0+\frac{\partial}{\partial t}(\psi G_0)\,,\,\psi F_1+\frac{\partial}{\partial t}(\psi G_1)\right)\in C^\infty(\,[0,1]^{1|1})
\tag{7}
$$

and it is a straightforward matter to verify that the functions F_0, F_1, G_0, and G_1 are given as in the following table (where we have not written the pullback $(j\sigma)^*$ anymore, but it must be understood):

	F_0	G_0	F_1	G_1
$x_s^i\neq 0$	$\frac{\partial L^*x}{\partial x^i}-\frac{\partial}{\partial t}\frac{\partial L^*x}{\partial \dot{x}^i}$	$\frac{\partial L^*x}{\partial \dot{x}^i}$	$\frac{\partial}{\partial \tau}\left(\frac{\partial L^*\theta}{\partial x^i}-\frac{\partial}{\partial t}\frac{\partial L^*\theta}{\partial \dot{x}^i}\right)$	$\frac{\partial}{\partial \tau}\frac{\partial L^*\theta}{\partial \dot{x}^i}$
$x_\xi^i\neq 0$	$\frac{\partial L^*\theta}{\partial \pi\dot{x}^i}$	0	$\frac{\partial}{\partial \tau}\frac{\partial L^*x}{\partial \pi\dot{x}^i}$	$\frac{\partial L^*x}{\partial \dot{x}^i}$
$\theta_s^\mu\neq 0$	$\frac{\partial L^*x}{\partial \pi\dot{\theta}^\mu}$	0	$\frac{\partial}{\partial \tau}\frac{\partial L^*\theta}{\partial \pi\dot{\theta}^\mu}$	$\frac{\partial L^*\theta}{\partial \dot{\theta}^\mu}$
$\theta_\xi^\mu\neq 0$	$\frac{\partial L^*\theta}{\partial \theta^\mu}-\frac{\partial}{\partial t}\frac{\partial L^*\theta}{\partial \dot{\theta}^\mu}$	$\frac{\partial L^*\theta}{\partial \dot{\theta}^\mu}$	$\frac{\partial}{\partial \tau}\left(\frac{\partial L^*x}{\partial \theta^\mu}-\frac{\partial}{\partial t}\frac{\partial L^*x}{\partial \dot{\theta}^\mu}\right)$	$\frac{\partial}{\partial \tau}\frac{\partial L^*x}{\partial \dot{\theta}^\mu}$

It is now a straightforward matter to check that for a right-invertible derivation $D \in \mathfrak{gl}_{1|1}$ of type (1) (*i.e.*, with $a_0 \neq 0$) the condition **4.1**(3) leads to the Euler-Lagrange equations,

$$(8) \qquad F_0 = 0 \qquad \text{and} \qquad a_0 F_1 = \frac{\Delta(a;b)}{a_0^2}\Delta(a;G)$$

On the other hand, for a right-invertible derivation $D \in \mathfrak{gl}_{1|1}$ of type (2) (*i.e.*, with $a_0 = 0$, and $a_1 b_0 \neq 0$), the condition **4.1**(3) leads to,

$$(9) \qquad F_0 = 0, \qquad F_1 = 0, \qquad G_0 = 0$$

We shall leave to the reader the easy task of writing in each case all the equations in full detail.

APPENDIX

A.1 $\mathbb{Z}_2$-graded linear algebra background. Let k be a field. A $\mathbb{Z}_2$-graded k-vector space V is a a vector space over k, together with a direct sum decomposition $V = V_0 \oplus V_1$, and a parity function $|\cdot|: V_0 - \{0\} \cup V_1 - \{0\} \to \mathbb{Z}_2 = \{0,1\}$, so that $|v| = \mu$ whenever $v \in V_\mu - \{0\}$. The $\mathbb{Z}_2$-graded k-vector space $V = V_0 \oplus V_1$ is finite-dimensional of dimension (m,n) if $\dim V_0 = m$ and $\dim V_1 = n$.

A.2 $\mathbb{Z}_2$-graded associative algebras. A $\mathbb{Z}_2$-graded associative algebra over k is an ordinary associative k-algebra having the structure of a $\mathbb{Z}_2$-graded k-vector space, $A = A_0 \oplus A_1$, and such that its multiplication map $m_A : A \times A \to A$ satisfies the homogeneity conditions, $m_A(A_\mu, A_\nu) \subset A_{\mu+\nu}$ for all μ, ν in $\mathbb{Z}_2$. In particular, the unit element 1_A lies in the subspace A_0. A morphism of $\mathbb{Z}_2$-graded k-algebras is a k-linear map $T : A \to \tilde{A}$ of the underlying k-vector spaces, such that: (1) $T(A_\mu) \subset \tilde{A}_\mu$ for $\mu = 0, 1$, (2) $T(m_A(x,y)) = m_{\tilde{A}}(T(x), T(y))$, and (3) $T(1_A) = 1_{\tilde{A}}$. The following result classifies the $\mathbb{Z}_2$-graded algebras of dimension $(1,1)$ over k.

A.3 Proposition. *Let $A = A_0 \oplus A_1$ be a $\mathbb{Z}_2$-graded associative algebra over k of dimension $(1,1)$. Then, there exists a real number q, such that A is isomorphic to the subalgebra $A[q]$ of 2×2-matrices with entries in k given by,*

$$A[q] = \left\{ \begin{pmatrix} a & qb \\ b & a \end{pmatrix} \; : \; a, b \in \mathbb{R} \right\}$$

where, $A_0 \simeq k\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $A_1 \simeq k\left(\begin{smallmatrix} 0 & q \\ 1 & 0 \end{smallmatrix}\right)$. Moreover, if $k = \mathbb{R}$ there are up to isomprophism, only three $\mathbb{Z}_2$-graded associative algebras of dimension $(1,1)$, and the representatives are the algebras $A[q]$ parametrized by the values $q = -1$, 0, and $+1$. If $k = \mathbb{C}$, there are up to isomorphism, only two, and correspond to the values $q = 0$, and $q = +1$.

Proof. Since $\dim A = (1,1)$, and $1 \in A_0$, we may write $A = k\,1 \oplus k\,\zeta$, with $\zeta \in A_1 - \{0\}$. The homogeneity conditions on the multiplication map imply, $m_A(1,\zeta) =$

$\zeta = m_A(\zeta,1)$, and $m_A(\zeta,\zeta) = q1_A$ for some constant $q \in k$. It is easy to check that the correspondence,

$$A \ni a1 + b\,\zeta \longleftrightarrow \begin{pmatrix} a & qb \\ b & a \end{pmatrix} \in A[q] \qquad a,b \in \mathbb{R}$$

is in fact an algebra isomorphism. Now, let $T : A \to \tilde{A}$ be a $\mathbb{Z}_2$-graded algebra isomorphism between the two $(1,1)$-dimensional algebras A, and $\tilde{A}$. Then, $T(1_A) = 1_{\tilde{A}}$, and $T(\zeta) = c\tilde{\zeta}$ where $\tilde{\zeta} \in \tilde{A}_1 - \{0\}$, and c is some nonzero element in k. It follows that $\tilde{q}1_{\tilde{A}} = m_{\tilde{A}}(\tilde{\zeta},\tilde{\zeta}) = c^{-2}m_{\tilde{A}}(T(\zeta),T(\zeta)) = c^{-2}T(m_A(\zeta,\zeta)) = c^{-2}q1_{\tilde{A}}$. The special assertions in the statement for $k = \mathbb{R}$, and $k = \mathbb{C}$ follow by looking at the possibilities for the equation $\tilde{q} = c^{-2}q$. $\square$

A.4 $\mathbb{Z}_2$-graded Lie algebras. A $\mathbb{Z}_2$-graded Lie algebra over k is a $\mathbb{Z}_2$-graded k-vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with a k-bilinear map $[\,\cdot\,,\,\cdot\,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying,

$$[x,y] = -(-1)^{|x|\,|y|}[y,x]$$

and

$$(-1)^{|x|\,|z|}\big[\,x,[y,z]\big] + (-1)^{|z|\,|y|}\big[\,z,[x,y]\big] + (-1)^{|y|\,|x|}\big[\,y,[z,x]\big] = 0$$

for any homogeneous elements x, y, and z. A morphism of $\mathbb{Z}_2$-graded Lie algebras ia a linear map $T\colon \mathfrak{g} \to \tilde{\mathfrak{g}}$, such that, (1) $T(\mathfrak{g}_\mu) \subset \tilde{\mathfrak{g}}_\mu$ for $\mu = 0,1$, and (2) $T[x,y]_{\mathfrak{g}} = [T(x),T(y)]_{\tilde{\mathfrak{g}}}$. Just as in the associative case, the $\mathbb{Z}_2$-graded Lie algebras of dimension $(1,1)$ can be immediately divided into exactly three different isomorphism classes labeled by $q = -1$, $q = 0$, and $q = 1$ (*cf,* [12]); namely,

$$[x_0,x_1] = \frac{q(q+1)}{2}x_1 \qquad \text{and} \qquad [x_1,x_1] = \frac{q(q-1)}{2}x_0$$

where $x_\mu \in \mathfrak{g}_\mu - \{0\}$ $(\mu = 0,1)$.

A.5 Examples. If $V = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded vector space over k, the space $\operatorname{End}_k V$, of k-linear maps $T : V \to V$ exhibits a natural $\mathbb{Z}_2$-grading:

$$(\operatorname{End}_k V)_0 = \{T \in \operatorname{End}_k V \mid T(V_0) \subset V_0 \text{ and } T(V_1) \subset V_1\,\}$$
$$(\operatorname{End}_k V)_1 = \{T \in \operatorname{End}_k V \mid T(V_0) \subset V_1 \text{ and } T(V_1) \subset V_0\,\}$$

Clearly, $\operatorname{End}_k V = (\operatorname{End}_k V)_0 \oplus (\operatorname{End}_k V)_1$. Let $|\;\cdot\;|$ be its parity function. Then, $\operatorname{End}_k V$ also has the structure of a $\mathbb{Z}_2$-graded Lie algebra by letting, $[\,\cdot\,,\,\cdot\,]\colon \operatorname{End}_k V \times \operatorname{End}_k V \to \operatorname{End}_k V$ be defined on homogeneous elements as,

$$[S,T] = S \circ T - (-1)^{|S|\,|T|}T \circ S$$

and extending it k-bilinearly to $\operatorname{End}_k V$. When endowed with this $\mathbb{Z}_2$-graded Lie algebra structure, $\operatorname{End}_k V$ is usually denoted by $\mathfrak{gl}_k(V_0 \mid V_1)$. By choosing a basis of V, ordered so that the first $\dim V_0$ elements of it form a basis of V_0, and the last $\dim V_1$ form a basis of V_1, $\mathfrak{gl}_k(V_0 \mid V_1)$ can be realized in terms of $(\dim V_0 + \dim V_1) \times$

$(\dim V_0 + \dim V_1)$ matrices with entries in k. When so realized, the notation for the resulting $\mathbb{Z}_2$-graded Lie algebra of matrices is $\mathfrak{gl}_{\dim V_0|\dim V_1}(k)$.

The simplest nontrivial example is obtained from $V = \mathbb{R}^2 \simeq \mathbb{R} \oplus \mathbb{R}$, with the choice that the first direct summand on the right is V_0, and the second one is V_1. Thus, $\dim V_0 = 1 = \dim V_1$, and the homogeneous subspaces,

$$\mathfrak{gl}_{1|1}(\mathbb{R})_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Big| \; a, d \in \mathbb{R} \right\} \qquad \text{and} \qquad \mathfrak{gl}_{1|1}(\mathbb{R})_1 = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \Big| \; b, c \in \mathbb{R} \right\}$$

are 2-dimensional each. A convenient basis for $\mathfrak{gl}_{1|1}(\mathbb{R}) = \mathfrak{gl}_{1|1}(\mathbb{R})_0 \oplus \mathfrak{gl}_{1|1}(\mathbb{R})_1$ is given by

$$\mathbf{j}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{h}_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathbf{e}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{f}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and the $\mathbb{Z}_2$-graded commutation relations are explicitly given by

$$[\,\mathbf{h}_0\,,\,\mathbf{e}_1\,] = \mathbf{e}_1 \qquad [\,\mathbf{h}_0\,,\,\mathbf{f}_1\,] = -\mathbf{f}_1 \qquad [\,\mathbf{e}_1\,,\,\mathbf{f}_1\,] = \mathbf{j}_0$$

with all other equal to zero.

A.6 Proposition. *The $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{gl}_{1|1}(\mathbb{R})$ is isomorphic to the $\mathbb{Z}_2$-graded Lie subalgebra consisting of constant coefficient $\mathbb{Z}_2$-graded derivations of the structure ring of the graded manifold $\mathbb{R}^{1|1}$.*

It is in fact clear that the isomorphism is provided by the map,

$$\mathbf{j}_0 \mapsto \frac{\partial}{\partial t} \qquad \mathbf{h}_0 \mapsto \tau\frac{\partial}{\partial \tau} \qquad \mathbf{e}_1 \mapsto \tau\frac{\partial}{\partial t} \qquad \mathbf{f}_1 \mapsto \frac{\partial}{\partial \tau}$$

A.7 The automorphism group of $\mathfrak{gl}_{1|1}(\mathbb{R})$. Let $\{\mathbf{j}_0, \mathbf{h}_0, \mathbf{e}_1, \mathbf{f}_1\}$ be the homogeneous basis of $\mathfrak{gl}_{1|1}(\mathbb{R})$ considered before. Let $\Psi\colon \mathfrak{gl}_{1|1}(\mathbb{R}) \to \mathfrak{gl}_{1|1}(\mathbb{R})$ be an automorphism of the graded Lie algebra. Then,

$$\begin{aligned} \Psi\mathbf{j}_0 &= A\mathbf{j}_0 + C\mathbf{h}_0 & \qquad \Psi\mathbf{e}_1 &= \alpha\mathbf{e}_1 + \gamma\mathbf{f}_1 \\ \Psi\mathbf{h}_0 &= B\mathbf{j}_0 + D\mathbf{h}_0 & \qquad \Psi\mathbf{f}_1 &= \beta\mathbf{e}_1 + \delta\mathbf{f}_1 \end{aligned}$$

From the equation $[\Psi\mathbf{e}_1, \Psi\mathbf{f}_1] = \Psi\mathbf{j}_0$ one concludes that,

$$\alpha\delta + \beta\gamma = A \qquad \text{and} \qquad C = 0$$

Similarly, one obtains,

$$[\Psi\mathbf{e}_1, \Psi\mathbf{e}_1] = 0 \iff \alpha\gamma = 0 \qquad \text{and} \qquad [\Psi\mathbf{f}_1, \Psi\mathbf{f}_1] = 0 \iff \beta\delta = 0$$

And,

$$[\Psi\mathbf{h}_0, \Psi\mathbf{e}_1] = \Psi\mathbf{e}_1 \qquad \iff \qquad \begin{cases} D\alpha = \alpha \\ D\gamma = -\gamma \end{cases}$$

$$[\Psi\mathbf{h}_0, \Psi\mathbf{f}_1] = -\Psi\mathbf{f}_1 \qquad \iff \qquad \begin{cases} D\beta = -\beta \\ D\delta = \delta \end{cases}$$

It can be checked that these are the only nontrivial relations that can be imposed on Ψ to be an automorphism of the given graded Lie algebra. We may summarize our findings in the following:

A.8 Proposition. *Let* $\{\mathbf{j}_0, \mathbf{h}_0, \mathbf{e}_1, \mathbf{f}_1\}$ *be the homogeneous basis of* $\mathfrak{gl}_{1|1}(\mathbb{R})$ *considered before. The automorphism group of this graded Lie algebra consists of all the linear transformations* $\Psi\colon \mathfrak{gl}_{1|1}(\mathbb{R}) \to \mathfrak{gl}_{1|1}(\mathbb{R})$ *of the following form:*

$$\begin{aligned} \Psi\mathbf{j}_0 &= A\mathbf{j}_0 & \qquad \Psi\mathbf{e}_1 &= \frac{1+\varepsilon}{2}\,\alpha\,\mathbf{e}_1 + \frac{1-\varepsilon}{2}\,\frac{A}{\beta}\,\mathbf{f}_1 \\ \Psi\mathbf{h}_0 &= B\mathbf{j}_0 + \varepsilon\mathbf{h}_0 & \qquad \Psi\mathbf{f}_1 &= \frac{1-\varepsilon}{2}\,\beta\,\mathbf{e}_1 + \frac{1+\varepsilon}{2}\,\frac{A}{\alpha}\,\mathbf{f}_1 \end{aligned}$$

where $\varepsilon = \pm 1$*, and the real parameters* A*,* B*,* α*, and* β *are subject to the constraints,* $(\varepsilon = 1 \Rightarrow \alpha A \neq 0)$*, and* $(\varepsilon = -1 \Rightarrow \beta A \neq 0)$*.*

Let us now consider the correspondence,

$$\operatorname{Der} C^\infty(\mathbb{R}^{1|1}) \quad \ni \quad D \qquad \leftrightarrow \qquad \begin{pmatrix} a_0 + \dfrac{b_1}{2} & a_1 \\ b_0 & a_0 - \dfrac{b_1}{2} \end{pmatrix} \quad \in \quad \mathfrak{gl}_{1|1}(\mathbb{R})$$

An automorphism Ψ transforms this element into,

$$D^\Psi \quad \leftrightarrow \quad \begin{cases} \begin{pmatrix} a_0A + b_1B + \dfrac{b_1}{2} & a_1\alpha \\ \dfrac{b_0A}{\alpha} & a_0A + b_1B - \dfrac{b_1}{2} \end{pmatrix} & \text{If} \quad \varepsilon = 1 \\ \begin{pmatrix} a_0A + b_1B - \dfrac{b_1}{2} & b_0\beta \\ \dfrac{a_1A}{\beta} & a_0A + b_1B + \dfrac{b_1}{2} \end{pmatrix} & \text{If} \quad \varepsilon = -1 \end{cases}$$

Remark. From this description one may also obtain the same orbit structure found in Cor. **1.9** above for $\mathfrak{gl}_{1|1}(\mathbb{R})$ under the H-action. We leave the details to the reader.

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