UNIVERSAL HOMOGENEOUS DERIVATIONS OF GRADED $\mathcal{E}$-COMMUTATIVE ALGEBRAS

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Abstract

Let K be a commutative ring, let Δ be an abelian group, and let $\varepsilon : \Delta \times \Delta \to K$ be a commutation factor over Δ. A Δ-graded K-algebra is said to be ε-commutative if its ε-bracket is identically zero. (K, ε)-derivations from a given ε-commutative Δ-graded K-algebra A into bimodules are studied. It is proved that for each $\lambda \in \Delta$ there exists a universal initial (K, ε)-derivation of degree λ of A. For each $\lambda \in \Delta$ a natural module of $(K, \varepsilon, \lambda)$-differentials of A along with a differential map is constructed. It is proved that each derivation of A canonically equipps this module with a structure of differential module. Applications and examples are given. It is shown that the first order exterior differentials which are known from the theory of smooth graded manifolds are universal initial homogeneous derivations of the sort considered hereby.

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Introduction

Let K be a commutative ring, and let Δ be an abelian group written additively. Let $A = \bigoplus_{\lambda\in\Delta} A_\lambda$ be a Δ-graded K-algebra (*cf,* [3, II §11.2]), and let $|\cdot| : \bigcup_{\lambda\in\Delta}(A_\lambda \setminus \{0\}) \to \Delta$ the map which sends each $a \in A_\lambda \setminus \{0\}$ into its degree $|a| = \lambda \in \Delta$. Elements in the domain of $|\cdot|$ are called homogeneous and $|a|$ is called the degree of the element a. Let $\varepsilon : \Delta \times \Delta \to K$ be a commutation factor over Δ (we refer the reader to Section 1 below for its defining properties). The K-algebra A is called ε-commutative if $ab = \varepsilon(|a|,|b|)\, ba$ for any homogeneous elements $a, b \in A$. The context of this article is that of graded bimodules over a given ε-commutative Δ-graded K-algebra. More concisely, we focus our attention into the (K,ε)-derivations of degree $\lambda \in \Delta$ from A into a given A-bimodule M; these are K-linear maps $\mathrm{d} : A \longrightarrow M$ satisfying $\mathrm{d}(xy) = (\mathrm{d}x)y + \varepsilon(\lambda, |x|)x\mathrm{d}y$. The main purpose of this article is to prove that, for each $\lambda \in \Delta$, there exists a (necessarily unique) universal initial (K,ε)-derivation of degree λ (*cf,* Theorem 2.2 below).

We have essentially three reasons for addressing this question: First, whereas the general concept of (K,ε)-derivation of Δ-graded algebras can be introduced with a fairly high level of abstraction (see for instance [3, III §10]), the only known results concerning the existence of universal derivations deal with the very particular case of the trivial grading (*ie,* $A_\lambda = 0, \forall\lambda \neq 0$) and ε being the constant function whose value is equal to 1 (*cf,* [3, III §10.10 Prop. 17]). Second, the theory of smooth graded manifolds (*cf,* [6], and [7]) provides a natural setting where examples of graded homogeneous derivations of different degrees appear in a natural fashion; as far as the authors know, no universal object for them has been studied before. Third, some of the basic tools in non-commutative geometry such as, cyclic cohomology [4], and differential calculus on quantum groups [12], seem to deal almost exclusively with derivations satisfying the usual Leibniz rule, thus excluding some important homogeneous (K,ε)-derivations of degree other than 0. In particular, even though graded manifolds are examples of non-commutative geometric objects, the $\mathbb{Z}_2$-graded differential calculus they give rise to cannot be understood in a categorically adequate fashion, without the fundamental algebraic understanding of the derivations of degree 1 (see for example [10]; see also [7]). It is only reasonable then to wonder whether universal models exist for homogeneous (K,ε)-derivations of *any* degree.

This paper is organized as follows: Section 1 recalls the basic definitions following essentially [3]. Section 2 is devoted to the proof of some of the main results. The existence of the searched universal initial homogenous (K,ε)-derivation is proved in Theorem 2.2. Then, for a given $\lambda \in \Delta$, the notion of (K,ε,λ)-differential is introduced in Definition 2.2 and the universal property for them is proved in Proposition 2.4. This result generalizes what is so far known about K-derivations (compare with [3, III §10.10 Lemma 1, and Prop. 17], and [3, III §10.11 Prop. 18]). Section 3 proves that each derivation of A naturally induces a derivation of its (K,ε,λ)-differentials (Proposition 3.1). Finally, Section 4 shows that the known first order exterior differentials coming from the theory of smooth graded manifolds are actual examples of universal initial homogeneous derivations.

1 Preliminaries

Let Δ be an abelian group written additively with identity element denoted by 0. The group Δ will be kept fixed throughout all the general arguments given in this work. We shall then refer to the various Δ-graded objects that will appear in the sequel, simply as graded. We shall also make the convention that each time we refer to the degree of an element, it will always be assumed that such an element is homogeneous; in other words, it will always be assumed that it lies in the domain of the corresponding degree map associated to the object in question. For simplicity, all degree maps will be denoted by $|\cdot|$, so that $|x|$ stands for the degree of the element x, regardlees of whether it belongs to a graded ring, a graded module, or any other graded object.

We shall now recall from [3] some basic results about graded rings: Given a graded ring $R = \bigoplus_{\lambda\in\Delta} R_\lambda$, the homogeneous component R_0 is a subring of R, and $1 \in R_0$ (Note: All rings in this paper have a multiplicative unit element $1 \neq 0$). Let R and S be two graded rings and let $f : R \longrightarrow S$ be a homogeneous morphism of graded abelian groups. If furthermore f is a ring morphism, its degree is necessarily 0, since $f(1) = 1$, and $|f(1)| = |f| + |1| = |1|$. Thus, all graded ring morphisms will be hereby assumed to be of degree 0.

Let K be a commutative ring. A graded K-algebra is a graded ring A having the structure of an algebra over the commutative ring K, with $K \subset A_0$ and $K \subset \mathrm{Z}(A)$ — the center of A. A commutation factor over Δ with values in K is a map $\varepsilon : \Delta\times\Delta \longrightarrow K$ such that $\varepsilon(\mu+\nu,\lambda) = \varepsilon(\mu,\lambda)\varepsilon(\nu,\lambda)$, $\varepsilon(\lambda,\mu+\nu) = \varepsilon(\lambda,\mu)\varepsilon(\lambda,\nu)$, $\varepsilon(\mu,\nu)\varepsilon(\nu,\mu) = 1$ for all $\lambda,\mu,\nu \in \Delta$ (*cf,* [3, III §4.7]). Now suppose that a commutation factor ε has been fixed. The ε-bracket of two homogeneous elements $x, y \in A$ is defined by,

$$[x, y]_\varepsilon = xy - \varepsilon(|x|, |y|)yx$$

This map can be extended K-bilinearly so as to obtain a map $[\,\cdot\,,\,\cdot\,]_\varepsilon \colon A\times A \to A$. Two elements $x, y \in A$ are said to ε-commute if $[x, y]_\varepsilon = 0$. The graded K-algebra is ε-commutative if any two elements from A ε-commute.

Conventions: Since the commutation factor ε is always used with respect to some given grading, we shall agree that when we refer ourselves to 'an ε-commutative K-algebra A', the latter will be assumed to be graded, even though no explicit reference be made to the adjective 'graded'.

The main objects of study in this paper, are graded bimodules over ε-commutative algebras whose left and right module structures are compatible in the sense specified in Definition 1.1 below. Let A be a graded K-algebra (*i.e.,* no commutation factor ε has been defined *a priori* on it, yet). The definition of graded left (resp., right) A-module adopted here is the standard one (*cf,* [3, II §1]): It is a graded abelian group $M = \bigoplus_{\lambda\in\Delta} M_\lambda$ together with a map $A\times M \longrightarrow M$, $(a,x)\mapsto ax$ (resp., $M\times A\longrightarrow M$, $(x,a)\mapsto xa$), satisfying $A_\mu M_\lambda \subset M_{\mu+\lambda}$ (resp., $M_\mu A_\lambda \subset M_{\mu+\lambda}$), and the usual properties of distributivity, and associativity of the module operations. Now recall that a right morphism of left modules $M \longrightarrow N$ is a morphism of abelian groups such that $(ax)f = a(x)f$ (such morphisms are usually written on the right; that is, $f : x \longmapsto (x)f$). A right morphism of left modules is homogeneous if so is the underlying morphism of abelian groups. The group of graded right morphisms is the direct sum of the groups of homogeneous right morphisms of degree λ, where λ runs over Δ. The composition of two homogeneous right morphisms f and g is defined to be the right morphism $(x)(f\cdot g) = ((x)f)g$. Notice that $f\cdot g$ is homogeneous and that $|f\cdot g| = |f| + |g|$. This definition is extended additively to nonhomogeneous right morphisms. In this way, one constructs the category of left A-modules with right morphisms. Similarly, a left morphism of right modules $M \longrightarrow N$ is a morphism of abelian groups such that $f(xa) = f(x)a$ (such morphisms are usually written on the left; that is $f : x \longmapsto f(x)$). A left morphism of right modules is homogeneous if so is the underlying morphism of abelian groups. The group of graded left morphisms is the direct sum of the groups of homogeneous left morphisms of degree λ, where λ runs over Δ. The composition of two homogeneous left morphisms f and g is defined to be the left morphism $(f\cdot g)(x) = f(g(x))$. As before, $f\cdot g$ is homogeneous and $|f\cdot g| = |f| + |g|$, and the definition is extended additively to nonhomogeneous left morphisms. In this way, one obtains the category of right A-modules with left morphisms.

If the base K-algebra A is commutative, the category of left modules with right morphisms is equivalent to the category of right modules with left morphisms; whence, the distinction is unnecessary. We are interested, however, in the case when the base K-algebra A is ε-commutative.

Definition 1.1 *Let A be an ε-commutative K-algebra. Let M be a graded abelian group with given left and right graded A-module structures. These structures are*

1. *compatible, if $(ax)b = a(xb)$ for all $a, b \in A$, and all $x \in M$*

2. *ε-compatible, if they are compatible and moreover $xa = \varepsilon(|x|, |a|)ax$ for all homogeneous elements $a \in A$ and $x \in M$.*

If the given left and right structures are compatible, M is called an A-bimodule (according to [3, II §1.14 def. 13]). If both structures are ε-compatible, M is called an A-ε-bimodule.

Notice that for the definition of A-ε-bimodule to make sense, the assumption that the base algebra A is ε-commutative is essential. If A is ε-commutative, then A is an A-ε-bimodule.

Lemma 1.1 *Let M and N be A-ε-bimodules.*

1. *If f is a homogeneous right morphism such that $(xa)f = (x)fa\varepsilon(|a|,|f|)$ for all $x \in M$ and all homogeneous $a \in A$, then the formula $f^l(x) = (x)f\varepsilon(|f|,|x|)$ defines a homogeneous left morphism f^l of degree $|f^l| = |f|$*

2. *If f is a homogeneous left morphism such that $f(ax) = \varepsilon(|f|,|a|)af(x)$ for all $x \in M$ and all homogeneous $a \in A$, then the formula $(x)f^r = \varepsilon(|x|,|f|)f(x)$ defines a homogeneous right morphism f^r of degree $|f^r| = |f|$*

3. *$(f^l)^r = f$ and $(f^r)^l = f$.*

4. *If f and g are right morphisms, $(f \cdot g)^l = \varepsilon(|f|,|g|)g^l \cdot f^l$*

5. *If f and g are left morphisms, $(f \cdot g)^r = \varepsilon(|f|,|g|)g^r \cdot f^r$*

Proof. For the first part, $f^l(xa) = (xa)f\varepsilon(|f|,|x|+|a|) = (x)fa\varepsilon(|a|,|f|)\,\varepsilon(|f|,|x|+|a|) = (x)fa\varepsilon(|f|,|x|) = (x)f\varepsilon(|f|,|x|)a = f^l(x)a$. The other parts are proved by similar simple verifications. $\square$

Corollary 1.1 *The category of A-ε-bimodules with graded right morphisms, each of whose homogeneous components satisfy $(xa)f = (x)fa\varepsilon(|a|,|f|)$, is graded equivalent to the category of A-ε-bimodules with graded left morphisms, each of whose homogeneous components satisfy $f(ax) = \varepsilon(|f|,|a|)af(x)$.*

Proof. The assignment from the first category into the second given by sending each object into itself, and each morphism f into f^l is easily seen to be a graded contravariant functor. The same is true for the assignment from the second category into the first that sends each object into itself, and each morphism f into f^r. These two functors are mutually inverse by Lemma 1.1. $\square$

Henceforth, by the category of A-ε-bimodules we will indistinctly understand any of the two categories in the statement of Corollary 1.1.

Lemma 1.2 *Let A be an ε-commutative K-algebra. The category of left (resp., right) graded A-modules with right (resp., left) morphisms is equivalent to the category of graded A-ε-bimodules.*

Proof. Note first that the natural forgetful functor from the category of A-ε-bimodules into the category of graded left A-modules has an inverse (the case of right A-modules is similar). If M is a graded left A-module, define on it the unique ε-compatible right product: $xa = \varepsilon(|x|,|a|)\,ax$ (see Definition 1.1). Since A is ε-commutative, this defines a right A-module structure on M which is ε-compatible with the given left structure. So, an A-ε-bimodule M' is obtained. Now, a right morphism $f : M \longrightarrow N$ of left modules may be viewed as a map f' from M' into N'. It obviously satisfies $(xa)f' = \varepsilon(|x|,|a|)\, a(x)f' = \varepsilon(|x|,|a|)\, \varepsilon(|a|,|x|+|f|)\, (x)f'\, a = \varepsilon(|a|,|f|)\, (x)f'\, a$. So, f' becomes a morphism of A-ε-bimodules. It is now clear that this construction defines an inverse for the natural forgetful functor as claimed. □

Lemma 1.3 *The category whose objects are A-ε-bimodules, and whose morphisms are those which are homogeneous of degree 0, is a subcategory of the category of graded A-bimodules.*

Proof. One only has to verify that any morphism f of A-ε-bimodules is a morphism of A-bimodules. In fact, let f be a right morphism. Then f is morphism of left A-modules, since by definition $(ax)f = a(x)f$; moreover, $(xa)f = \varepsilon(|a|,0)(x)\, fa = (x)fa$. Thus, f is also a morphism of right A-modules. □

We close this section with a few more definitions that will be used in the sequel. Let M be a graded module. For each $\lambda \in \Delta$ the λ-shift of M is a graded module, denoted by $M(\lambda)$, where $M(\lambda)_\mu = M_{\lambda+\mu}$. Recall that if M, and N are A-ε-bimodules its graded tensor product $M\hat{\otimes}_A N$ is the usual tensor product equipped with the grading for which the elements of degree λ are sums of products $x \otimes y$, with $x \in M_\mu$, $y \in M_\nu$, and $\mu + \nu = \lambda$. It is a simple fact that the trivial left and right A-module structures of the graded tensor product of two A-ε-bimodules are ε-compatible. So the graded tensor product of A-ε-bimodules is an A-ε-bimodule.

2 Derivations

Let Δ, K, and ε be as in Section 1, and let A be a graded K-algebra. The existence of left and right morphisms gives rise to left and right deriviations according to the following:

Definition 2.1 *Let A be an ε-commutative K-algebra, and let M be an A-ε-bimodule. A map $\mathrm{d} : A \longrightarrow M$ is a*

1. *left (K,ε)-derivation of degree λ of A into M, if it is a left morphism of degree λ between K-modules, and*

$$\mathrm{d}(xy) = (\mathrm{d}x)y + \varepsilon(\lambda,|x|)x\mathrm{d}y \tag{1}$$

   for any $x, y \in A$; x homogenous.

2. *right (K,ε)-derivation of degree λ of A into M, if it is a right morphism of degree λ between K-modules, and*

$$(xy)\mathrm{d} = x(y)\mathrm{d} + \varepsilon(|y|,\lambda)(x\mathrm{d})y \tag{2}$$

for any $x, y \in A$; y homogenous.

If A is ε-commutative, these two concepts are essentially the same, as the following lemma proves.

Lemma 2.1 *Let A be an ε-commutative K-algebra. Consider the category $\mathcal{D}(K,\varepsilon;A)_\lambda$, (resp., $\mathcal{D}(K,\varepsilon;A)'_\lambda$) whose objects are left (resp., right) (K,ε)-derivations of degree λ from A into some A-ε-bimodule, and whose morphisms are given as follows: Given any two objects, $\mathrm{d}_1 : A \longrightarrow M$, and $\mathrm{d}_2 : A \longrightarrow N$, a morphism from the first into the second is a left (resp., right) morphism of A-ε-bimodules $f : M \longrightarrow N$, which is of degree zero and such that $f \circ \mathrm{d}_1 = \mathrm{d}_2$. Then the categories $\mathcal{D}(K,\varepsilon;A)_\lambda$ and $\mathcal{D}(K,\varepsilon;A)'_\lambda$ are (gradedly) equivalent under the functor given by the relation,*

$$(x)\mathrm{d}^r = \varepsilon(|x|,\lambda)\mathrm{d}^l(x)$$

which defines a right derivation d^r out from a left derivation d^l and viceversa.

Proof. It is similar to that of Corollary 1.1. □

Therefore, when the base algebra is ε-commutative there is no loss of generality in considering only left (K,ε)-derivations. We shall, therefore, restrict ourselves to the latter and will refer to them simply as (K,ε)-derivations.

Let us now recall the following well known result which states the existence of a universal initial object in the catgeory $\mathcal{D}(K;A)$ whose objects are K-derivations from A into A-bimodules (*ie*, K-linear maps d from A into some A-bimodule such that $\mathrm{d}(xy) = \mathrm{d}(x)y + x\mathrm{d}(y)$) and whose morphisms are defined in the usual way as in Lemma 2.1.

Theorem 2.1 *Let A be an associative K-algebra. Let I be the kernel of the multiplication map regarded as a sub-A-bimodule of $A \otimes_K A$.*

1. *The map $\delta : x \longmapsto x \otimes 1 - 1 \otimes x$ is a K-derivation from A into I. Moreover, I is generated by the image of δ as a left A-module.*
2. *δ is the universal initial object of $\mathcal{D}(K;A)$; that is, for any A-bimodule M and any K-derivation $\mathrm{d} : A \longrightarrow M$, there exists a unique A-bimodule morphism $f : I \longrightarrow M$, such that $\mathrm{d} = f \circ \delta$.*

Proof. The first part is [3, III §10.10 Lemma 1]. The second part is [3, III §10.10 Prop. 17]. □

We are really interested in the existence of universal initial objects for the categories $\mathcal{D}(K,\varepsilon;A)_\lambda$, under the assumption that A is some ε-commutative graded K-algebra. The easiest case corresponds to $\lambda = 0$:

Lemma 2.2 *$\mathcal{D}(K,\varepsilon;A)_0$ has a universal initial object.*

Proof. Let the notation be as in the proof of Lemma 1.2 and Theorem 2.1. First note that: *(1)* The ideal I is graded; *(2)* (K,ε)-derivations of degree $\lambda = 0$ are the same thing as K-derivations; *(3)* If M' stands for the A-ε-bimodule constructed from the left A-module M (see the proof of Lemma 1.2), then $A = A'$, because A is ε-commutative. Now, consider the K-derivation of $A = A'$ into I' defined by $\delta' : x \longmapsto x \otimes 1 - 1 \otimes x$. It is obviously homogeneous of degree 0. We shall now see that it is the universal derivation. In fact, if $\mathrm{d}' : A \longrightarrow M'$ is any K-derivation of degree 0 from A into the A-ε-bimodule M', then it defines a unique derivation d from A into the A-bimodule M. By Theorem 2.1, there exists a unique morphism $f : I \longrightarrow M$ of A-bimodules, such that $f \circ \delta = \mathrm{d}$. Therefore, by the proof of Lemma 1.2, $f' \circ \delta' = \mathrm{d}'$. The uniqueness of such an f' follows from the facts that $x \otimes 1 - 1 \otimes x$ generates I' as a left A-module (by Theorem 2.1 again), and that the image of such an element is necessarily $\mathrm{d}'(x)$. Notice that f' is a morphism of A-bimodules which is of degree zero, and then it is a morphism of A-ε-bimodules of the same degree. □

We now want to settle the question of whether or not there is a universal initial object in $\mathcal{D}(K,\varepsilon;A)_\lambda$.

Theorem 2.2 *The category $\mathcal{D}(K,\varepsilon;A)_\lambda$ has a universal initial object.*

Proof. We consider the $-\lambda$-shift $A(-\lambda)$ of A (*cf*, Section 1). Now, in the A-module $A(-\lambda)\hat{\otimes}_K A$ consider the left A-submodule generated by $1 \otimes a - \varepsilon(\lambda, |a|)\, a \otimes 1$, where a runs over all homogeneous elements of A. This is easily seen to be a graded left A-submodule. Let $\tilde{A}$ be the quotient of $A(-\lambda)\hat{\otimes}_K A$ modulo this submodule, with its induced left A-module structure (therefore, with its compatible right A-module, and A-ε-bimodule structures by Lemma 1.1) and with its induced grading (that is, elements of degree α in the quotient are precisely the equivalence classes of elements of degree α in $A(-\lambda)\hat{\otimes}_K A$). The sub-$A$-bimodule I of $A \otimes_K A$ in Theorem 2.1 is an A-bimodule but it is not, in general, an A-ε-bimodule: Its right structure does not coinicide in general with the unique right structure which is ε-compatible with the given left A-module structure (recall Lemma 1.1). Thus, to avoid confusion, and following the proof of Lemma 1.2 we shall write again I' for the A-ε-bimodule constructed from the left A-module I. We now define I_λ as the graded tensor product $\tilde{A}\hat{\otimes}_A I'$. It is easy to verify that its natural left and right A-module structures are ε-compatible, and they thus define an A-ε-bimodule. Let δ_λ be the map of A into I_λ given by $\delta_\lambda : x \longmapsto 1 \otimes 1 \otimes \delta(x)$, where $\delta(x) = x \otimes 1 - 1 \otimes x$ is as in Theorem 2.1. It is clearly K-bilinear and homogeneous of degree λ. Moreover, it satisfies: $\delta_\lambda(xy)$ $= 1 \otimes 1 \otimes (\delta(x)y + x\delta(y)) = \delta_\lambda(x)y + 1 \otimes x \otimes \delta(y) = \delta_\lambda(x)y + \varepsilon(\lambda, |x|)\, x \otimes 1 \otimes \delta(y)$ $= \delta_\lambda(x)y + \varepsilon(\lambda, |x|)\, x\delta_\lambda(y)$. The fact that the degree of $1 \otimes 1 \in A(-\lambda)\hat{\otimes}_K A$ is λ has been used here. This follows from the grading of the graded tensor product, and from the fact that $1 \in A(-\lambda)$ has degree λ, since $1 \in A$ has degree 0. Therefore, δ_λ is a (K,ε)-derivation of degree λ. We shall now show that δ_λ is universal; that is, for any given (K,ε)-derivation of degree λ, say $\mathrm{d} : A \to M$, into any given A-ε-bimodule M, we must prove the existence of a morphism of A-ε-bimodules having degree 0, $f_\lambda : I_\lambda \longrightarrow M$, such that $\mathrm{d} = f_\lambda \circ \delta_\lambda$. Note, however, that the right map $f_\lambda : a \otimes b \otimes x \otimes y \longmapsto -\varepsilon(\lambda, |b| + |x|)abx\mathrm{d}y$ satisfies:

1. $((1 \otimes a - \varepsilon(\lambda, |a|)a \otimes 1)x \otimes y)f_\lambda = 0$; and for $w \in \tilde{A}$, $\tilde{x} \in I'$, $a \in A$: $(w \otimes a\tilde{x})f_\lambda = (wa \otimes \tilde{x})f_\lambda$. Thus f_λ is well defined.
2. f_λ is $\mathbb{Z}$-linear and of degree 0
3. $(a\tilde{x})f_\lambda = a(\tilde{x})f_\lambda$ for all $a \in A$ and all $\tilde{x} \in I_\lambda$
4. $(\tilde{x}a)f_\lambda = (\tilde{x})f_\lambda a$ for all homogeneous $a \in A$ and all $\tilde{x} \in I_\lambda$
5. $f_\lambda \circ \delta_\lambda(x) = \mathrm{d}(x)$

Parts (2), (3) and (4) prove that f_λ is an A-ε-bimodule morphism. For the uniqueness of f_λ amongst all the A-ε-bimodule morphisms satisfying (5), just note that I_λ is generated as an A-ε-bimodule by all the elements of the form $1 \otimes 1 \otimes \delta(x)$ with $x \in A$. □

Remark 2.1 *The most general setting for considering graded objects in algebra only requires that Δ be a commutative monoid. If we had started from this assumption we would have been led to a statement of the form:* For each $\lambda \in \Delta$ having an inverse $-\lambda \in \Delta$, the category $\mathcal{D}(K,\varepsilon;A)_\lambda$ has a universal initial object. *However, in all the examples and applications we have in mind Δ is actually an abelian group.*

An immediate consequence of Theorem 2.2 is this

Corollary 2.1 *Let $\delta_\lambda : A \longrightarrow I_\lambda$ be the universal initial object of $\mathcal{D}(K,\varepsilon;A)_\lambda$ and M be an A-ε-bimodule. Let $\mathrm{Hom}(I_\lambda, M)_0$ be the K-module of morphisms of A-ε-bimodules wich are homogeneous of degree 0 and $\mathrm{D}(K,\varepsilon;A,M)_\lambda$ be the K-module of (K,ε)-derivations from A into M which are homogeneous of degree λ. Then, the map $\mathrm{Hom}(I_\lambda, M)_0 \ni f \mapsto f \circ \delta_\lambda \in \mathrm{D}(K,\varepsilon;A,M)_\lambda$ is a canonical K-module isomorphism.*

The following two Propositions 2.1 and 2.2 show how in the particular cases $\lambda = 0$ or ε constant and equal to $1 \in K$, the result of Theorem 2.2 can be related to well known results about the universal derivation over a given commutative algebra.

Proposition 2.1 *Let $\mathcal{D}(K,\varepsilon;A)_0 \longrightarrow \mathcal{D}(K;A)$ be the following forgetful functor: On objects, it sends each (K,ε)-derivation from A into an A-ε-bimodule M into the same map viewed as a K-derivation from A into the A-bimodule M on which the grading has been forgotten (this is well defined since the (K,ε)-derivations are of degree 0); on maps, the functor is the identity. This functor is faithful, and sends the universal initial object into the universal initial object.*

Proof. All morphisms in the category $\mathcal{D}(K,\varepsilon;A)_0$ are morphisms of A-ε-bimodules which are of degree 0; therefore, all of them are also morphisms of A-bimodules. (Note that a morphism f of A-ε-bimodules satisfies $(ax)f = a(x)f$, so it is a morphism of left A-modules; moreoever, $(xa)f = (x)fa\varepsilon(|a|,|f|) = (x)fa\varepsilon(|a|,0) = (x)fa$, so it is also a morphism of right A-modules). Since $\lambda = 0$, the quotient $\tilde{A}$ in the proof of Theorem 2.2 is $A \otimes_A A \simeq A$; thus, I_0 is canonically isomorphic to I. So, the functor sends I_0 into I, and δ_0 into δ, where δ is as in Theorem 2.1. That this functor is faithful is clear. $\square$

If ε is constant and equal to 1 then, $(K,1)$-derivations are K-derivations, and by forgetting the grading, a functor from $\mathcal{D}(K,1;A)_\lambda$ into $\mathcal{D}(K;A)$ is defined. This functor is used in the next proposition to prove that Theorem 2.1 is a particular case of Theorem 2.2.

Proposition 2.2 *Let $\mathcal{D}(K,1;A)_\lambda \longrightarrow \mathcal{D}(K;A)$ be the following forgetful functor: On objects, it sends each derivation from A into a graded A-1-bimodule M into the same map viewed as a K-derivation from A into the A-bimodule M on which the grading has been forgotten (this is well defined because $\varepsilon \equiv 1$); on maps, the functor is the identity. This functor is faithful, and sends the universal initial object into the universal initial object.*

Proof. Since $\varepsilon \equiv 1$, all morphisms of A-ε-bimodules are morphisms of A-bimodules. (Note that a morphism f of A-ε-bimodules satisfies $(ax)f = a(x)f$, so it is a morphism of left A-modules; moreover, $(xa)f = (x)fa\varepsilon(|a|,|f|) = (x)fa$, so it is a morphism of right A-modules). Also, $\varepsilon \equiv 1$ implies that the quotient $\tilde{A}$ in the proof of Theorem 2.2 is $A(-\lambda) \otimes_A A \simeq A(-\lambda)$; thus, I_λ is canonically isomorphic to $A(-\lambda) \otimes_A I \simeq I(-\lambda)$. Furthermore, the universal derivation δ_λ sends $x \in A$ into $x \otimes 1 - 1 \otimes x \in I(-\lambda)$. Its image by the functor of the statement just forgets the grading. Thus, the image of δ_λ is δ which is the universal derivation of Theorem 2.1. $\square$

Let A be a (not necessarily graded) commutative K-algebra. It is a well known fact that the kernel I of the multiplication map is an ideal of $A \otimes_K A$ which satisfies $\mathrm{Hom}_A(I,M) \simeq \mathrm{Hom}_A(I/I^2,M)$. Furthermore, Theorem 2.1 readily implies that there is a canonical K-module isomorphism from $\mathrm{Hom}_A(I,M)$ into the K-derivations of A into M, $\mathrm{D}(K;A,M)$, which gives rise to a canonical isomorphism of A-modules,

$\mathrm{Hom}_A(I/I^2, M) \simeq \mathrm{D}(K; A, M)$. We shall adopt the standard notation for the A-module I/I^2, and shall denote it by $\Omega(K; A)$. It is called the module of Kähler differentials of A over K. The canonical differential map $\mathrm{d}_{A/K}$ defined on it is just δ modulo I^2 (δ has been defined in Theorem 2.1); that is, $\mathrm{d}_{A/K} = \pi \circ \delta$, where $\pi : I \longrightarrow I/I^2$ is the canonical projection. $\mathrm{d}_{A/K}$ is called the Kähler differential map of A. The key point that makes this construction work is precisely that I is an ideal of $A \otimes_K A$ (*cf*, [3, III §10.11 Prop. 18]). This, however, is not true in general when A is replaced by some non-commutative graded K-algebra. However, if the graded K-algebra is ε-commutative, then I — the kernel of the multiplication map — can be shown to be a two-sided graded ideal of $A\hat{\otimes}_K A$. Once this simple fact can be stated, the A-module of (K, ε)-differentials on A can be constructed as in the commutative case, and similar isomorphisms can be established.

Proposition 2.3 *1. The formula* $(a \otimes b)(x \otimes y) = \varepsilon(|b|, |x|)\, ax \otimes by$ *defines a structure of ε-commutative K-algebra on $A\hat{\otimes}_K A$*

2. The kernel I of the multiplication map $\mathrm{m} : A\hat{\otimes}_K A \ni x \otimes y \mapsto xy \in A$ *is a two sided graded ideal of $A\hat{\otimes}_K A$ with respect to the algebra structure defined in part 1, and it is generated by $x \otimes 1 - 1 \otimes x$, with $x \in A$.*

Proof.

1. That the given formula defines an algebra structure on $A\hat{\otimes}_K A$ is proved in [3, III §4.7 Prop. 10]. It is immediate to verify that this product is ε-commutative.

2. Note that I is clearly graded. Let us now see that I is a left ideal. Take $a \otimes b \in A\hat{\otimes}_K A$, and $\sum_i x_i \otimes y_i \in I$: $\mathrm{m}((a \otimes b)(\sum_i x_i \otimes y_i)) = \mathrm{m}(\sum_i \varepsilon(|b|, |x_i|) a x_i \otimes b y_i)$ $= \sum_i \varepsilon(|b|, |x_i|) a x_i b y_i = \sum_i \varepsilon(|b|, |x_i|)\varepsilon(|x_i|, |b|) ab\, x_i y_i = ab\, \mathrm{m}(\sum_i x_i y_i) = 0$. Thus, I is a left ideal. The verification that I is a right ideal is similar. If $x = \sum_i x_i \otimes y_i \in I$, then $x = \sum_i (x_i \otimes 1)(1 \otimes y_i - y_i \otimes 1)$. Then, the elements $y \otimes 1 - 1 \otimes y$, with $y \in A$ generate I.

□

Note that [3, III §10.11 Prop. 18] can be now generalized as follows:

Proposition 2.4 *Let A be an ε-commutative K-algebra, let I be as in Proposition 2.3 and, let I_λ be as in the proof of Theorem 2.2. The map* $\mathrm{d}_{A/K,\varepsilon,\lambda} : A \longrightarrow I_\lambda / I I_\lambda$ *which sends x into the class of $\delta_\lambda(x)$ is a homogenous (K, ε)-derivation of degree λ which is universal; that is, for each A-ε-bimodule M, and each (K, ε)-derivation* $\mathrm{d} : A \longrightarrow M$ *of degree λ, there exists a unique A-ε-bimodule morphism* $f : I_\lambda / I I_\lambda \longrightarrow M$ *of degree* 0 *such that* $f \circ \mathrm{d}_{A/K,\varepsilon,\lambda} = \mathrm{d}$.

Proof. By Theorem 2.2, δ_λ is a homogenous (K,ε)-derivation of degree λ. Since $\mathrm{d}_{A/K,\lambda}$ is δ_λ modulo II_λ, it is also a derivation of the same type. The universal property can be proved by adapting the arguments given in the proof of [3, III §10.11 Prop. 18]:

$$\begin{aligned}\mathrm{D}(K,\varepsilon;A,M)_\lambda &\simeq \mathrm{Hom}_{A-\varepsilon-\mathrm{bimod}}(I_\lambda,M)_0 \simeq \mathrm{Hom}_{A\hat{\otimes}_K A-\varepsilon-\mathrm{bimod}}(I_\lambda,M)_0\\ &\simeq \mathrm{Hom}_{A-\varepsilon-\mathrm{bimod}}((A\hat{\otimes}_K A/I)\hat{\otimes}_A I_\lambda,M)_0 \simeq \mathrm{Hom}_{A-\varepsilon-\mathrm{bimod}}(I_\lambda/II_\lambda,M)_0\end{aligned}$$

The first isomorphism is Theorem 2.2. For the second one, note that any A-ε-bimodule can be viewed as an $A\hat{\otimes}_k A$-ε-bimodule by the multiplication map $A\hat{\otimes}_k A \longrightarrow A$. Thus, a natural map from the left hand side into the right hand side is defined. Since the structure of M as $A\hat{\otimes}_K A$-ε-bimodule is the only one induced by its structure as A-ε-bimodule, a map from the right hand side into the left hand side is also defined, and it is easily seen to be the inverse of the one defined first. The third isomorphism sends a morphism f of $A\hat{\otimes}_K A$-ε-bimodules into the map of A-ε-bimodules $a_1\otimes a_2\otimes x \longmapsto a_1a_2f(x)$; and conversely, it sends a morphism g of A-ε-bimodules into the morphism of $A\hat{\otimes}_K A$-ε-bimodules $x \longmapsto g(1\otimes 1\otimes x)$ (since all morphism of bimodules are of degree 0 these maps are well defined). The last isomorphism follows from the fact $A\hat{\otimes}_K A/I \otimes_A M \simeq M/IM$ which is the graded ε-commutative version of the well known result in linear commutative algebra. □

Definition 2.2 *The A-ε-bimodule I_λ/II_λ will be called the module of (K,ε,λ)-differentials of A and will be denoted by $\Omega(K,\varepsilon,\lambda;A)$. The map $\mathrm{d}_{(A/K,\varepsilon,\lambda)}$ in Proposition 2.4 will be called the (K,ε,λ)-differential map of A.*

By endowing each K-algebra A (resp., each module M), with the trivial grading (that is, $A=A_0$, resp. $M=M_0$), the category of K-algebras (resp., modules), can be viewed as a subcategory of the category of graded K-algebras (resp., of graded modules). The next corollary proves in particular that the module of differentials and the differential map of a trivially graded commutative algebra are the module of its Kähler differentials and its Kähler differential map respectively, in the sense of Definition 2.2.

Corollary 2.2 *Let A be an ε-commutative K-algebra.*

1. *If $\lambda=0$ then, $\Omega(K,\varepsilon,0;A)=I/I^2$.*

2. *If $\lambda=0$ and $\varepsilon\equiv 1$ then, A is a commutative K-algebra and $(\Omega(K,1,0;A), \mathrm{d}_{(A/K,1,0)}) = (\Omega(K;A),\mathrm{d}_{A/K})$, where $\Omega(K;A)$ is the module of Kähler differentials of A over K and $\mathrm{d}_{A/K}$ is the Kähler differential map.*

Proof. For the first part, notice that if $\lambda = 0$ then, $I = I_0$ and therefore $I_0/II_0 = I/I^2$. If moreover $\varepsilon \equiv 1$ then, we are actually dealing with derivations of bimodules over the commutative algebra A. Thus, it makes sense to consider the usual module of I/I^2 along with the Kähler differential map $A \ni \mapsto x \otimes 1 - 1 \otimes x$. Then part 1 ensures that the two modules on part 2 are the same. Moreover, the two Kähler differential maps in part 2 are also the same by the construction of $\mathrm{d}_{A/K,1,0}$ given in Proposition 2.4. □

In the non graded case, any $\mathbb{Z}$-linear combination of K-derivations is again a K-derivation. Similarly, one may take $\mathbb{Z}$-linear combinations of homogeneous (K,ε)-derivations. The problem is that if (K,ε)-derivations of different degrees appear in such a linear combination , the result may no longer be a (K,ε)-derivation. One way out of dealing with this new situation is to enlarge the category of derivations to a category of braided derivations. Alternatively, (but not equivalently) one may consider the product category. We shall explore the former possibility in a subsequent article.

Let $\mathcal{D}(K,\varepsilon;A)$ stand for the product category $\Pi_{\lambda\in\Delta}\mathcal{D}(K,\varepsilon;A)_\lambda$. Recall that the product category contains each $\mathcal{D}(K,\varepsilon;A)_\lambda$ as subcategory. Moreover, $\mathcal{D}(K,\varepsilon;A)$ is an additive category, since each factor subcategory is. Therefore, the following result is immediate:

Theorem 2.3 *$\mathcal{D}(K,\varepsilon;A)$ has a universal initial object.*

Proof. The product of the universal initial objects of each factor subcategory is a universal initial object. □

Theorem 2.4 below also concerns with the problem of considering (K,ε)-derivations of different degrees as objects of a single category. It is stated in the particular case where Δ is $\mathbb{Z}_2 = \{0,1\}$ and $\varepsilon(x,y) = (-1)^{xy}$ because this is the precise situation in which we shall apply it in Section 4. However, the proof still remains valid if Δ is any abelian group whose elements are all of order 2 and if ε is any symmetric commutation factor.

Theorem 2.4 *Let Δ be $\mathbb{Z}_2 = \{0,1\}$ and $\varepsilon(x,y) = (-1)^{xy}$. The category whose objects are homogeneous (K,ε)-derivations from A into some A-ε-bimodule and whose morphisms from $\mathrm{d}_1 : A \longrightarrow M$ into $\mathrm{d}_2 : A \longrightarrow N$ are homogeneous morphisms $f : M \longrightarrow N$ of A-ε-bimodules such that $f \circ \mathrm{d}_1 = \mathrm{d}_2$, has a universal initial object.*

Proof. Fix any λ. Since $\mathrm{D}(K,\varepsilon;A)_\lambda$ is a subcategory of the category in the statement of the theorem, the universal initial object $\delta_\lambda : A \longrightarrow I_\lambda$ of the former is an object of the latter, which is actually universal. In fact, given a (K,ε)-derivation $\mathrm{d} : A \to M$ of degree μ consider the morphism $f : I_\lambda \to M$ given by $f : a \otimes b \otimes \sum_i x_i \otimes y_i \longmapsto -\sum_i \varepsilon(\lambda, |b| + |x_i|) a b x_i \mathrm{d} y_i$. This is evidently homogeneous of degree $\mu+\lambda$. Moreover, f satisfies items (1), (2), (3), and (5) in the proof of Theorem 2.2, and instead of item (4) f satisfies $(\tilde{x}a)f=(\tilde{x})fa\varepsilon(|a|,\mu)\varepsilon(\lambda,|a|)$, and (since ε is symmetric) $(\tilde{x}a)f=(\tilde{x})fa\,\varepsilon(|a|,|f|)$. (Note that if $|f|=0$, one obtains exactly the cited item (4)). Thus, f is a well defined morphism of A-ε-bimodules of degree $\mu+\lambda$. □

Perhaps it is worth comparing Theorems 2.2 and 2.4. The first difference, is that the category in the statement of Theorem 2.2 is additive, whereas that in Theorem 2.4 is not. Besides, although the definition of the morphism f_λ in the proof of Theorem 2.2 and that of the morphism f in the proof of Theorem 2.4 seem quite similar, notice that the first one is a morphism of degree 0 whereas the second one is of degree $\mu+\lambda$. It is then said that f is the $(\mu+\lambda)$-shift of f_λ. In the context of graded manifolds (see Section 4) the functor which shifts degrees by $1\in\mathbb{Z}_2$ is denoted by Π (*cf,* [7, Chp. 3 §1.5]).

The module of Kähler differentials of a commutative K-algebra satisfies some well known functorial properties (*cf,* [3, III §10.12]) which are immediately deduced from the universal property. The interested reader will find no mishap in generalizing these functorial properties to (K,ε,λ)-differentials.

3 Differential structures

In this section we generalize to the setting of graded derivations the following well known result: If A is a commutative differential K-algebra then, the module of its Kähler differentials over K has a canonical structure of differential module over A (*cf,* [5]). Let us explain this result in some detail for the reader's convenience. Although the definition can be made with some more generality, a differential K-algebra is, for our purposes, a commutative K-algebra A equipped with a derivation ∂ over K. In this situation it is proved (*cf,* [5]) that there exists a unique derivation (also called) ∂ of the A-module $\Omega(K;A)$ of Kähler differentials of A over K which commutes with the Kähler differential map. The module $\Omega(K;A)$ along with this derivation is said to be a differential module. Propositions 3.1 and 3.2 below generalize this result to the module of (K,ε,λ)-differentials. To prove them, we need some definitions and lemmas.

Definition 3.1 *Let δ be a left, resp. right, (K,ε)-derivation of the ε-commutative K-algebra A and, M be an A-ε-bimodule. A K-linear map $\partial : M \longrightarrow M$ is a left (resp., right), derivation w.r.t. δ if*

1. *∂ is homogeneous and $|\partial|=|\delta|$*
2. *$\partial(ax)=\delta(a)x+\varepsilon(|\partial|,|a|)a\partial(x)$ (resp., $(ax)\partial=a(x)\partial+\varepsilon(|x|,|\partial|)(a)\delta x$).*

Since M is an A-ε-bimodule, condition 2 is equivalent to $\partial(xa) = \partial(x)a + \varepsilon(|\partial|,|x|)x\delta(a)$ (resp., $(xa)\partial = x(a)\delta + \varepsilon(|a|,|\partial|)(x)\partial a$). On the other hand, since M is an A-ε-bimodule it is also a K-ε-bimodule. So by Corollary 1.1 there is no need to specify if ∂ is a left or right K-linear map.

Lemma 3.1 *The category of left derivations of A-ε-bimodules w.r.t. δ is equivalent to the category of right derivations of A-ε-bimodules w.r.t. δ^r, where δ and δ^r are related as in Lemma 2.1.*

Proof. The formula which functorially relates a derivation on one side with its corresponding derivation on the opposite side is

$$(x)\partial = \varepsilon(|x|,|\partial|)\partial(x).$$

The needed verifications are similar to those of the proof of Lemma 2.1. $\square$

Then without any lose of generality we shall restrict ourselves to left derivations.

Lemma 3.2 *Let δ be a homogeneous (K,ε)-derivation of the ε-commutative K-algebra A. For any two derivations ∂_1 and ∂_2 w.r.t. δ of the A-ε-bimodules M_1 and M_2 respectively, the formula*

$$\partial_1 \otimes_A \partial_2(x_1 \otimes x_2) = \partial_1(x_1) \otimes x_2 + \varepsilon(|\delta|,|x_1|)x_1 \otimes \partial_2(x_2)$$

defines a derivation $\partial_1 \otimes_A \partial_2 : M_1 \longrightarrow M_2$ w.r.t. δ.

Proof. First, it must be shown that $\partial_1 \otimes_A \partial_2$ is a well defined K-linear map. The only thing which is not obvious at first sight is that $\partial_1 \otimes_A \partial_2(x_1 a \otimes x_2) = \partial_1 \otimes_A \partial_2(x_1 \otimes a x_2)$. It is a simple computation to verify that this equality is actually true and therefore, that the well defined K-linear map satisfies condition 2 of Definition 3.1. $\square$

If A is an ε-commutative K-algebra, $A\hat{\otimes}_K A$ is also an ε-commutative K-algebra with product $(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2 \varepsilon(|a_2|,|b_1|)$. If δ_1 and δ_2 are (K,ε)-derivations of A both of the same degree, it is easy to verify that the formula

$$\delta_1 \otimes \delta_2(a_1 \otimes a_2) = \delta_1(a_1) \otimes a_2 + \varepsilon(|\delta_2|,|a_1|)a_1 \otimes \delta_2(a_2)$$

defines a left (K,ε)-derivation $\delta_1 \otimes \delta_2$ of $A\hat{\otimes}_K A$ of degree $|\delta_1| = |\delta_2|$.

Lemma 3.3 *Let δ_1 and δ_2 be two homogeneous (K,ε)-derivations of the ε-commutative K-algebra A, both of the same degree. For any pair ∂_1 and ∂_2 of derivations of the A-ε-bimodules M_1 and M_2 respectively (with ∂_1 w.r.t. δ_1 and ∂_2 w.r.t. δ_2), the map*

$$x_1 \otimes x_2 \longmapsto \partial_1(x_1) \otimes x_2 + \varepsilon(|\partial_2|,|x_1|)x_1 \otimes \partial_2(x_2)$$

is a well defined derivation $\partial_1 \otimes_K \partial_2$ from the $A\hat{\otimes}_K A$-ε-bimodule $M_1\hat{\otimes}_K M_2$ into itself w.r.t. $\delta_1 \otimes \delta_2$.

Proof. Similarly to the proof of Lemma 3.2, the first thing that must be shown is that $\partial_1 \otimes_K \partial_2$ is a well defined map. That $x_1 a \otimes x_2$ and $x_1 \otimes a x_2$ have both the same image is due to the fact that now a belongs to K and therefore $|a| = 0$. $M_1 \hat{\otimes}_K M_2$ is an $A\hat{\otimes}_K A$-ε-bimodule whose left and right multiplications are respectively given by,

$$(a_1 \otimes a_2, x_1 \otimes x_2) \longmapsto a_1 x_1 \otimes a_2 x_2 \varepsilon(|a_2|, |x_1|)$$

and

$$(x_1 \otimes x_2, a_1 \otimes a_2) \longmapsto x_1 a_1 \otimes x_2 a_2 \varepsilon(|x_2|, |a_1|)$$

Finally, the verification of condition 2 in Definition 3.1 to prove that $\partial_1 \otimes_K \partial_2$ is a derivation w.r.t. $\delta_1 \otimes \delta_2$ is a formal computation. □

Proposition 3.1 *Any homogenoeus (K, ε)-derivation of the ε-commutative K-algebra A naturally induces a derivation of the A-module of $(K, \varepsilon, \lambda)$-differentials of A which ε-commutes with the differential map.*

Proof. It has been seen in the proof of Proposition 2.4 that the module of $(K, \varepsilon, \lambda)$-differentials of A is canonically isomorphic to

$$\big(A\hat{\otimes}_K A/I\big) \,\hat{\otimes}_A\, \big(A(-\lambda)\hat{\otimes}_K A/C\big) \,\hat{\otimes}_A I \tag{3}$$

where C is the left A-module generated by $1 \otimes a - \varepsilon(\lambda, |a|) a \otimes 1$, with a running over all the homogeneous elements of A. If ∂ is a (K, ε)-derivation of A, then $\partial \otimes_K \partial$ is a derivation of $A\hat{\otimes}_K A$ (by Lemma 3.3) which sends I into I. Then, it factors through a well defined derivation of the quotient $(A\hat{\otimes}_K A)/I$, also called $\partial \otimes_K \partial$. On the other hand, the map $A(-\lambda) \ni x \mapsto \partial(x) \in A(-\lambda)$ is a (K, ε)-derivation w.r.t. ∂. Then, also by Lemma 3.3, $\partial \otimes_K \partial$ is a well defined (K, ε)-derivation w.r.t ∂ of $A(-\lambda)\hat{\otimes}_K A$. It moreover maps C into C. Then a derivation w.r.t. ∂ of the quotient $(A(-\lambda)_K \otimes A)/C$ is obtained. Finally, as it has been said, the derivation $\partial \otimes_K \partial$ of $A\hat{\otimes}_K A$ maps I into itself, and therefore it induces a derivation of I w.r.t. ∂. Then by Lemma 3.2, we can tensorialize over A all these derivations w.r.t. ∂ and obtain a derivation of the module (3) w.r.t. ∂. Since this derivation w.r.t. ∂ is the natural extension of ∂, we shall denote it also as ∂. The only thing that remains to be proved is that the Kähler differential map ε-commutes with ∂. For short, let us here denote the differential map $\mathrm{d}_{A/K,\varepsilon,\lambda}$ simply as d. Then, $\partial \circ \mathrm{d}(x) = \partial(1 \otimes 1 \otimes 1 \otimes 1 \otimes (x \otimes 1 - 1 \otimes x)) = 1 \otimes 1 \otimes 1 \otimes 1 \otimes (\partial(x) \otimes 1 \; - 1 \otimes \partial(x))\varepsilon(|\partial|, \lambda)$ and $\mathrm{d} \circ \partial(x) = 1 \otimes 1 \otimes 1 \otimes 1 \otimes (\partial(x) \otimes 1 - 1 \otimes \partial(x))$. □

Proposition 3.2 $\Omega(K, \varepsilon, \lambda; A)$ *has a natural structure of $\mathcal{D}$-module, where $\mathcal{D}$ is the ring of differential operators with coefficients in A.*

Proof. This is a straightforward consequence of Proposition. 3.1 □

4 Examples in graded geometry

In this section we apply the algebraic constructions of Sections 2 and 3 to $\mathbb{Z}_2$-graded geometry. In this context, Δ is $\mathbb{Z}_2 = \{0,1\}$ and $\varepsilon(x,y) = (-1)^{xy}$. Roughly speaking, every (classical) geometric object can be viewed as a (trivially) $\mathbb{Z}_2$-graded geometric object of degree 0. On the other hand, in $\mathbb{Z}_2$-graded geometry there are objects which have no classical counterpart (for instance, odd vector fields, right connections,...). These objects are typically of degree 1. One of the basic motivations of $\mathbb{Z}_2$-graded geometry is the purpose of putting on equal footing geometric objects of degree 0 (called even) and those of degree 1 (called odd). Then, examples of derivations, differentials,... of both degrees, 0 and 1, naturally appear in $\mathbb{Z}_2$-graded geometry. It is to be said that what follows can be generalized to the finite dimensional $\mathbb{Z}_2^N$-graded geometry proposed in [9]. In this case Δ is $\mathbb{Z}_2^N$ and $\varepsilon((x_1,\ldots,x_N),(y_1,\ldots,y_N)) = (-1)^{x_1y_1+\ldots+x_Ny_N}$. Since all the elements of $\mathbb{Z}_2^N$ are of degree 2 and ε is symmetric, all our previous results, including Theorem 2.4, can be applied in this situation in a similar way as they will be now applied to $\mathbb{Z}_2$-graded geometry. We may safely leave to the interested reader the generalization of the following results stated in $\mathbb{Z}_2$-graded geometry to $\mathbb{Z}_2^N$-graded geometry for $N > 1$. From now, we fix Δ to be $\mathbb{Z}_2$ and the commutation factor $\varepsilon(x,y) = (-1)^{xy}$ and, as in the previous sections, graded will mean Δ-graded.

The two main ingredients of a smooth graded manifold (*cf,* [1], [6] and references therein) are: A smooth manifold (called the body) and a sheaf $\mathcal{A}$ of ε-commutative $\mathbb{R}$-algebras on it (whose sections are by definiton the functions on the graded manifold). Any smooth graded manifold admits (by definition) an atlas such that any of its system of local coordinates $x_1,\ldots,x_m;\xi_1,\ldots,\xi_n$ consists of local coordinates $x_1,\ldots,x_m$ on an open set U of the body and a basis $\xi_1,\ldots,\xi_m$ of an $\mathbb{R}$-vector space. With respect to such a system of local coordinates, $\mathcal{A}(U) \simeq \mathcal{C}^\infty(U)\hat{\otimes}_{\mathbb{R}}\Lambda(\xi_1,\ldots,\xi_m)$, where $\Lambda(\xi_1,\ldots,\xi_m)$ is the exterior algebra of the $\mathbb{R}$-vector space with basis $\xi_1,\ldots,\xi_m$ along with the $\mathbb{Z}_2$-grading induced by its natural $\mathbb{Z}$-grading modulo 2 and, $\mathcal{C}^\infty(U)$ is trivially graded. Since $\mathcal{C}^\infty(U)$ is commutative and $\Lambda(\xi_1,\ldots,\xi_m)$ is ε-commutative, their graded tensor product is ε-commutative (or supercommutative, as it is said in the geometric literature). Although the appearence of even and odd vector fields goes back to the foundational papers on graded manifolds (*cf,* [1]), the first reference known to the authors where an even as well as an odd analog of the classical exterior differential map appear is [7]. In this reference, even and odd differential maps are introduced algebraically without any explicit relationship with the existence of even and odd vector fields. It is in [10] where even and odd differential maps are constructed according to the existence of even and odd vector fields. As far as the authors know, all the well defined examples of even and odd generalizations of the exterior differential are:

1. The degree 0 derivation $\hat{\mathrm{d}} : A \longrightarrow \hat{\Omega}(A)$ introduced in [2].

2. The degree 0 derivation $\mathrm{d} : A \longrightarrow \Omega(A)$ introduced in [6].

3. The degree $\lambda = 0, 1$ derivations $\mathrm{d}_\lambda : A \longrightarrow \Omega_\lambda(A)$ introduced in [7, chp 4 §4.4] as d_{ev}, and $\mathrm{d}_{\mathrm{odd}}$, respectively.

4. The degree $\lambda = 0, 1$ derivations $\mathrm{D}_\lambda : A \longrightarrow \Omega^{(1-\lambda,\lambda)}(A)$ introduced in [10].

Here A is the ε-commutative $\mathbb{R}$-algebra of global sections of $\mathcal{A}$ and, as a matter of notation $\Omega(A)$, $\Omega_\lambda(A)$, $\Omega^{(1-\lambda,\lambda)}(A)$ denote the global sections of each corresponding sheaf of 1-forms (**Warning:** forms of higher order are not considered here, and then there is no need of any super or subindex 1 with the Ω's; in contrast, a super or subindex 1 is used in the refered articles where the distinction between 1-forms and higher order forms is needed). Several relationships between the four items above are known: By [10, Prop. 7.1], $\mathrm{D}_\lambda : A \longrightarrow \Omega^{(1-\lambda,\lambda)}(A)$ and $\mathrm{d}_\lambda : A \longrightarrow \Omega_\lambda(A)$ are isomorphic, for $\lambda = 0, 1$. It is also known that $\mathrm{d}_0 : A \longrightarrow \Omega_0(A)$ and $\mathrm{d} : A \longrightarrow \Omega(A)$ are isomorphic, which follows from the local expressions for d_0 and d (*cf,* [6], [7], and [10]). Finally, the constructions given in [2] can be shown to be equivalent to those in [6] and [7], because among other things $\hat{\mathrm{d}}(A) \subset \Omega(A) \subset \hat{\Omega}(A)$ (*cf,* [10] for details); whence, $\hat{d} : A \longrightarrow \Omega(A)$ and $\mathrm{d} : A \longrightarrow \Omega(A)$ are isomorphic, too. These isomorphisms can then be summarized as $\hat{\mathrm{d}} \simeq \mathrm{d} \simeq \mathrm{d}_0 \simeq \mathrm{D}_0$ and $\mathrm{d}_1 \simeq \mathrm{D}_1$. Thus in spite of their different constructions, these examples provide a unique (up to isomorphism) example of derivation of degree 0 and one of degree 1. In other words, the various constructions just mentioned provide isomorphic first order differential calculi. The difference between them lies in the fact that the higher order differential calculus associated to each one of them are no longer isomorphic. That isomorphic first order calculus may have nonisomorphic higher order extensions is a typical situation in noncommutative geometry; see for instance [12], and [8].

Proposition 4.1 *Let A be the ε-commutative $\mathbb{R}$-algebra of differentiable functions on a graded manifold. Then, $\mathrm{d}_\lambda : A \longrightarrow \Omega_\lambda(A)$ is isomorphic to $\mathrm{d}_{A/\mathbb{R},\varepsilon,\lambda} : A \longrightarrow \Omega(\mathbb{R}, \varepsilon, \lambda; A)$ for $\lambda = 0, 1$.*

Proof. Let us first consider the case $\lambda = 0$. Since $\mathrm{d}_0 : A \longrightarrow \Omega_0(A)$ is an $(\mathbb{R}, \varepsilon)$-derivation of degree 0 (*cf,* [7, Chp. 3 §4.4]), Theorem 2.4 says that there exists a unique morphism of A-ε-bimodules f such that $f \circ \mathrm{d}_{A/\mathbb{R},\varepsilon,0} = \mathrm{d}$. To construct f^{-1}, recall that $\Omega_0(A)$ is the A-ε-bimodule of global sections of the sheaf of 1-forms $\Omega_0(\mathcal{A})$ whose sections on an open set U of the body is denoted by $\Omega^1_{\mathrm{ev}}(U, \mathcal{A})$ in [7, Chp. 4§1.10]. It is well known that $\Omega^1_{\mathrm{ev}}(U, \mathcal{A})$ is a free $\mathcal{A}(U)$-ε-bimodule with basis $\mathrm{d}_0x_1, \dots , \mathrm{d}_0x_m, \mathrm{d}_0\xi_1, \dots , \mathrm{d}_0\xi_n$ (*cf,* [7, Chp. 4 §1.10]) and that for $\varphi \in \mathcal{A}(U)$, $\mathrm{d}_0\varphi = \sum_{i=1}^{m} \mathrm{d}_0x_i \frac{\varphi}{x_i} + \sum_{j=1}^{n} \mathrm{d}_0\xi_j \frac{\varphi}{x_j}$. Then f^{-1} is the morphism which locally sends d_0x_i, $\mathrm{d}_0\xi_j \longmapsto \mathrm{d}_{A/\mathrm{R},\varepsilon,0}x_i$, $\mathrm{d}_{A/\mathrm{R},\varepsilon,0}\xi_j$, respectively. The case of $\lambda = 1$ is similar.

□

Therefore, the well defined generalizations of the exterior differential defined on functions of graded manifolds which are known to the authors are isomorphic to the universal differential maps introduced here. It is to be said that some other generalizations can be found in the literature which unfortunately are not well defined (*cf*, [11]). This is the reason why we have not considered them here.

As an easy consequence of Propositions 3.2 and 4.1 we obtain

Corollary 4.1 *With the previous notations,* $\Omega(A)$*,* $\Omega_\lambda(A)$ *and* $\Omega^{(1-\lambda,\lambda)}(A)$ *for* $\lambda = 0, 1$ *have, all of them, a natural structure of* $\mathcal{D}$*-module, where* $\mathcal{D}$ *is the ring of differential operators with coefficients in* A*.*

As it has been said, any non graded object can be viewed as a graded object by means of the trivial grading. Thus by introducing the trivial $\mathbb{Z}_2$-grading on the ring of functions on a smooth manifold, every smooth manifold can be regarded as a graded manifold (with $m = 0$ in the local description given in the third paragraph of this Section). With respect to this trivial grading, the only possible derivation of degree 1 is the zero derivation. Therefore, in the case of smooth manifolds, all the results specialize to well known facts about the universality of the classical exterior differential.

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