Exact Controllability and Perturbation Analysis for Elastic Beams

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Abstract

The Rayleigh beam is a perturbation of the Bernoulli-Euler beam. We establish convergence of the solution of the Exact Controllability Problem for the Rayleigh beam to the correspondig solution of the Bernoulli-Euler beam.The problem of convergence is related to a Singular Perturbation Problem. The main tool in solving this problem is a weak version of a lower bound for hyperbolic polynomials.

1 Introduction

Let us recall the Timoshenko equation

$$\rho w_{tt} - I_\rho w_{ttxx} + EIw_{xxxx} + \frac{\rho}{K}\left(I_\rho w_{tttt} - EIw_{ttxx}\right) = 0$$

it models the vibrations of an elastic beam, see Russell [12]. The physical constans are $\rho \equiv$ density, $EI \equiv$ flexural rigidity, $I_\rho \equiv$ rotary inertia, and $K \equiv$ shear modulus. When shear effect is neglected we are led to the Rayleigh equation

$$\rho w_{tt} - I_\rho w_{ttxx} + EIw_{xxxx} = 0$$

If rotary inertia is neglected in the Rayleigh equation we obtain the Bernoulli-Euler (B-E) equation

$$\rho w_{tt} + EIw_{xxxx} = 0\,.$$

After a change of variables we may write the Rayleigh anb B-E equations in the form

$$w_{tt} - Iw_{ttxx} + w_{xxxx} = 0$$

and

$$w_{tt} + w_{xxxx} = 0\,.$$

We consider the Exact Controllability Problem (ECP) and the method of solution for these equations as proposed by Littman [7, 8].

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In Moreles [10] we deal with the ECP for the Timoshenko and Rayleigh equations. Control time is established independent of the shear modulus, and more importantly, it is shown that the solution of the ECP for the Timoshenko equation converges *strongly* to that of Rayleigh when $K \to +\infty$.

We may pose the correponding problems for plate equations. For instance, there are the Mindlin-Timoshenko and Kirchhoff plate equations. These are generalizations of the Timoshenko an Rayleigh beam equations. It can be seen that the Mindlin-Timoshenko plate is obtained by uncoupling the Reissner-Mindlin Plate, see Lagnese [4], Lagnese and Lions [5] and Lagnese, Leugering and Schmidt [6].

Results on Perturbation Analysis in Control Theory are not new. Our work was motivated by results in Lagnese & Lions [5]. There, it is shown that solutions of the Timoshenko beam converge in a weak* topology to those of the Rayleigh beam, similarly for plates. We also mention the work in Komornik [3] where the Reissner-Mindlin Plate is studied establishing control in time independent of the shear modulus K. A similar result follows for the corresponding beam system. In Moreles [11] we go a step further and besides control in time independent of K, we establish also strong convergence of the solution of the ECP for the Mindlin-Timoshenko equation to that of Kirchhoff in several space dimensions, in particular for plates. This generalizes our earlier result.

In this work we establish the analogue for the Rayleigh and B-E equations. That is, we establish control in time independent of the rotary inertia and convergence of the solution of the ECP for the Rayleigh equation to the solution of the ECP for the Bernoulli-Euler equation. The outline is as follows.

The content of Section 2 is the statement of the main result, the proof is carried out in the next two sections. In Section 3, using the method of Littman we solve the ECP for Rayleigh and Bernoulli-Euler equations.

In order to describe the content of the last section, a brief description of the method will prove useful.

Let L any of the operator involved in the equations above. In Littman's method, the solution of the control problem is given in the form

$$W = w\psi(t) - v \tag{1}$$

Here t_0 and t_1 with $t_1 > t_0 > 0$, depending on L, are chosen appropiately to define $\psi(t)$ a cut-off function

$$\psi(t) = \begin{cases} 1 & t \le t_0 \\ \\ 0 & t \ge t_1 \end{cases}$$

In the decomposition (1) w satisfies $L[w] = 0$ with Cauchy data in the x-axis. On the other hand v is of the form $v = v_+ + v_-$ and satisfies $L[v] = f$, $f = L[w\psi]$. Observe that f is supported in the strip $t_0 \le t \le t_1$. The functions v_+ and v_- are constructed as solutions of the inhomogeneous Cauchy problems $L[v_+] = f$, $x \ge 0$, $L[v_-] = f$, $x \le 0$, with zero Cauchy data in the t-axis. Moreover, the support of v_+ and v_- is compact in t away from $t = 0$. Since the method is constructive, it suffices to read off boundary conditions (controls) that guarantee uniqueness.

For the Rayleigh equation we have a solution for the ECP of the form

$$W^I = u^I \psi(t) - v^I$$

whereas for the Bernoulli-Euler equation we have

$$W^0 = u^0 \psi(t) - v^0$$

In Section 4 we deal with the perturbation problems arising from these decompositions. That is, we prove convergence of v^I to v^0, and of u^I to u^0.

¿From Littman & Markus [9] we see that the function v^0 is solution of an ill-posed problem. The function is found by taking ψ in a Gevrey class. Hence, convergence of v^I to v^0 may be considered a singular perturbation problem. We remark that for the corresponding perturbation problem for Timoshenko and Rayleigh equation in Moreles [10], convergence is established by finding energy type inequalities for a first order hyperbolic system. Although the Rayleigh operator is hyperbolic with respect to the x-direction, such approach does not work here. We propose an alternative proof. An important step is deriving a weak version of the lower bound for hyperbolic polinomials in Hörmander [1].

The proof of convergence of u^I to u^0 is straightforward, it follows from basic properties of Fourier Transform and Ordinary Differential Equations.

2 Main Result

Let Ω be a bounded interval in $\mathbb{R}$. Recall that $\mathcal{W}_2^l(\Omega)$ is the Sobolev space of all functions $f \in L_2(\Omega)$ for which the distributional derivatives $\partial_x^k f$ are again elements of $L_2(\Omega)$ for $k \leq l$. The norm is denoted by $\|f\|_l$. When $\Omega = \mathbb{R}$ we regard $\mathcal{W}_2^l$ as $\mathcal{H}_l$ and

$$\|f\|_l = \left(\int \Lambda_{2l} \left| \widehat{f} \right|^2 \right)^{\frac{1}{2}}$$

where $\widehat{f}$ is the Fourier Transform of f and

$$\Lambda_s(\xi) = \left(1 + |\xi|^2 \right)^{\frac{s}{2}} , \quad s \in \mathbb{R}$$

Let $\Omega = (-1, 1)$, and

$$u^0 \in \mathcal{W}_2^8(\Omega),\ u^1 \in \mathcal{W}_2^6(\Omega)$$

Let $\delta > 0$ be given. Extend the Cauchy Data to have compact support in a δ-neighborhhod of Ω which we call again Ω .

Theorem 1 *Let $T > 0$ be given, and let Q be a bounded interval in* $\mathbb{R}$ containing Ω. *Then, there exists* $I_0 > 0$ sufficiently small such that

(1) there exist solutions $W^I(x,t)$ and $W^0(x,t)$ of the Cauchy problems

$$W^I_{tt} - IW^I_{ttxx} + W^I_{xxxx} = 0 \ , \qquad \text{in } \Omega \times \{t \geq 0\} \, ;$$

$$W^I(x,0) = u_0(x), \quad W^I_t(x,0) = u_1(x) \qquad \text{in } \Omega$$

and

$$W^0_{tt} + W^0_{xxxx} = 0 \ , \qquad \text{in } \Omega \times \{t \geq 0\} \, ;$$

$$W^0(x,0) = u_0(x), \quad W^0_t(x,0) = u_1(x) \qquad \text{in } \Omega$$

both vanishing in $\overline{\Omega} \times \{t \geq T\}$.
(2) W^I converges to W^0 when $I \to 0$ in the L^∞-norm of bounded subsets of $Q \times [0,T]$.
(3) there exists T_0, $0 < T_0 < T$, such that *W^I_t converges to W^0_t when $I \to 0$ in the L^∞-norm of bounded subsets of $Q \times [0,T_0]$.*

Remark. (1) A drawback of this result is the regularity required for the Cauchy data. This regularity is imposed to be able to prove strong convergence. It will become apparent that weaker convergence can be established with less regularity of the Cauchy data. See remarks in Section 4.
(2) To solve the ECP it suffices to read-off boundary conditions (controls) that make a well posed Initial-Boundary Value Problem. We may mimic the proof of Theorem 1 and make the appropiate changes to obtain convergence of the controls as well.

3 Exact Controllability

Let $T > 0$ be given. Here we establish controllability of both elastic systems in time T. More precisely, we construct two functions, $W^I(x,t)$, $W^0(x,t)$, both vanishing for $t \geq T$, which are solutions of the Cauchy Problems

$$W^I_{tt} - IW^I_{ttxx} + W^I_{xxxx} = 0$$

$$W^I(x,0) = u_0(x), \quad W^I_t(x,0) = u_1(x)$$

and

$$W^0_{tt} + W^0_{xxxx} = 0$$

$$W^0(x,0) = u_0(x), \quad W^0_t(x,0) = u_1(x)$$

The construction is not new, we just collect results from several places.

Bernoulli-Euler System

To carry out the first step in Littman's method we solve the Cauchy problem

$$w^0_{tt} + w^0_{xxxx} = 0$$

$$w^0(x,0) = u_0(x), \quad w^0_t(x,0) = u_1(x)$$

The solution is analytic in t, for $t > 0$, see Littman [8].

For the second step we mimic the proof of Theorem 4 in Littman & Markus [9]. We use the Gevrey classes, γ^2, and $\underline{\gamma}^2$ introduced therein.

Let ε, with $0 < \varepsilon < T$. Let $\psi(t)$ be a cut-off function such that

$$\psi(t) = \begin{cases} 1 & t \leq t_0 \\ \\ 0 & t \geq t_1 \end{cases}$$

where $t_0 = \dfrac{\varepsilon}{2}$ and $t_1 = \dfrac{3}{4}\varepsilon$. It is necessary to choose $\psi \in \gamma^2$. Define

$$f^0(x,t) = L^0[w^0\psi]$$

f^0 is supported in the set

$$\left\{(x,t) \in \mathbb{R}^2 : t_0 \leq t \leq t_1\right\}$$

and $f^0 \in \underline{\gamma}^2$.

This leads to a solution v^0_+ of the Cauchy problem

$$L^0 v^0_+ = f^0$$

$$\partial^j_x v^0_+(0,t) = 0, \quad j = 0,1,2,3$$

with v^0_+ supported in the set

$$\left\{(x,t) \in \mathbb{R}^2 : t_0 \leq t \leq t_1\right\}$$

A similar construction works for v^0_-, $x \leq 0$.

Finally, the solution of the control problem is given by

$$W^0 = w^0\psi - (v^0_+ + v^0_-) \tag{2}$$

To complete the solution of the ECP, we just read off boundary conditions that ensure uniqueness. Notice that $W^0(x,t)$ coincides with $w^0(x,t)$ for $|x| \leq 1$ and $t < t_0$.

Rayleigh System

Now consider the Cauchy Problem

$$w^I_{tt} - Iw^I_{ttxx} + w^I_{xxxx} = 0$$

$$w^I(x,0) = u_0(x), \quad w^I_t(x,0) = u_1(x)$$

It turns out that the solution $w^I(x,t)$, $(x,t) \in \mathbb{R}^2$, is smooth in the set

$$\left\{ t > -\sqrt{I}\,(x+1+\delta)\,,\ t>0 \right\} \cap \left\{ t > \sqrt{I}\,(x-1-\delta)\,,\ t>0 \right\}$$

which is the subset of the upper half-plane bounded by the interval $(-1-\delta, 1+\delta)$ in the $x-$axis, and the characteristic lines $t = -\sqrt{I}\,(x+1+\delta)\,,\ t = \sqrt{I}\,(x-1-\delta)\,.$

Indeed, let $G(x,t)$ be the fundamental solution of the Rayleigh operator

$$L^I = \partial_t^2 - I\partial_t^2\partial_x^2 + \partial_x^4$$

with principal part

$$L_4^I = -I\partial_t^2\partial_x^2 + \partial_x^4 = -I\partial_x^2\left(\partial_t^2 - \frac{1}{I}\partial_x^2\right)$$

Notice the wave operator in L_4^I. It follows that the singular support of G is contained in the cone

$$\Gamma^0 = \left\{ (x,t) \in \mathbb{R}^2 : t^2 = Ix^2, \quad t \geq 0 \right\}\ .$$

See Corollary 3.1 in Moreles [11]. Thus, in particular G is smooth in the set

$$\left\{ (x,t) \in \mathbb{R}^2 : t^2 > Ix^2, \quad t > 0 \right\}\ .$$

Since the Cauchy data u_0, u_1 is supported in the interval $-1-\delta \leq x \leq 1+\delta$ the claim follows.

Let us study the corresponding problem in the x-direction. Following Hörmander [2], let us associate to the Rayleigh operator the polynomial

$$P(\xi,\tau) = \xi^4 - I\tau^2\xi^2 - \tau^2$$

with principal part

$$P_4(\xi,\tau) = \xi^4 - I\tau^2\xi^2$$

Let $N = (1,0)$. It follows that the roots σ of $P_4\left((\xi,\tau)+\sigma N\right) = P_4(\xi + i\sigma,\tau) = 0$ are real for ξ,τ real. Hence from Theorem 12.4.3 in Hörmander [2] P_4 is hyperbolic with respect to N. On the other hand, it is readily seen that

$$\left| \frac{P(\xi+i\sigma,\tau)}{P_4(\xi+i\sigma,\tau)} \right| \leq 1 + \frac{2}{I}$$

¿From Theorem 12.4.6 in Hörmander [2], we conclude that P is hyperbolic with respect to N. Moreover, its fundamental solution is supported in the cone

$$\left\{ (x,t) \in \mathbb{R}^2 : t^2 \leq Ix^2, \quad t > 0 \right\}.$$

Define

$$f^I(x,t) = L^I[w^I\psi],$$

f^I is supported in the set

$$\left\{(x,t) \in \mathbb{R}^2 : t_0 \leq t \leq t_1\right\}$$

Hence, the solution of the Cauchy problem

$$L^I v_+^I = f^I$$

$$\partial_x^j v_+^I(0,t) = 0, \quad j = 0,1,2,3$$

is supported in the set

$$\left\{(x,t) \in \mathbb{R}^2 : t \leq \sqrt{I}\,x + t_1,\ x \geq 0\right\} \cap \left\{(x,t) \in \mathbb{R}^2 : t \geq -\sqrt{I}\,x + t_0,\ x \geq 0\right\}.$$

We require $v_+^I \equiv 0$ for $|x| \leq 1$ and in a neighborhood of $t = 0$, and $t = T$. Because of finite speed of propagation, this holds if I is small enough. More precisely, we require that $\sqrt{I} + t_1 < \varepsilon$ and $-\sqrt{I} + t_0 > 0$, since $t_0 = \dfrac{\varepsilon}{2}$ and $t_1 = \dfrac{3\varepsilon}{4}$, it suffices to take I such that

$$I < \frac{\varepsilon^2}{16}\,.$$

A similar argument works for $x \leq 0$ leading to a function v_-^I. The function

$$W^I = w^I\psi - (v_+^I + v_+^I) \tag{3}$$

solves the control problem with control time T as before. As before, observe that $W^I(x,t)$ coincides with $w^I(x,t)$ for $|x| \leq 1$ and $0 \leq t \leq \dfrac{\varepsilon^2}{4}$.

Defining $I_0 = \dfrac{\varepsilon^2}{16}$ we fullfil the initial requirement in Theorem 1. Later, we shall see that $T_0 = \dfrac{\varepsilon^2}{4}$ is what we need to prove part (3) of Theorem 1.

4 Perturbation Analysis

The decompositions (2) and (3) lead us to two perturbation problems. The study of these problems is the content of the results below. Consequently, we have a proof of Theorem 1.

We require convergence of v_+^I to v_+^0 and v_-^I to v_-^0 . Since both cases are similar it suffices to consider convergence of v_+^I to v_+^0.

In this section, c denotes a constant independent of the rotary inertia I.

Theorem 2. *Let $v = v_+^I - v_+^0$. Then, v satisfies the inhomogeneous Cauchy problem*

$$\begin{aligned} &v_{xxxx} - Iv_{ttxx} + v_{tt} = f \\ &\partial_x^j v(0,t) = 0, \quad j = 0,1,2,3 \end{aligned} \tag{4}$$

where $f = f^I - f^0 + Iv^0_{+ttxx}$. *Moreover, with* $f_\alpha(x,t) = e^{-\alpha x} f(x,t)$, $\alpha = \dfrac{\sqrt{3}}{2\sqrt{2}}$, *$v$ satisfies*

$$|v(x,t)| \leq ce^{cx}\left(\int_0^x\int |f(y,t)|^2\,dtdy + \left(\int\int |(1+\partial_t)\,f_\alpha(x,t)|^2\,dxdt\right)^{1/2}\right) \tag{5}$$

Theorem 3. *The solution w^I of the Cauchy problem*

$$\begin{aligned} &w^I_{tt} - Iw^I_{ttxx} + w^I_{xxxx} = 0 \\ &w^I(x,0) = u_0(x), \quad w^I_t(x,0) = u_1(x) \end{aligned} \tag{6}$$

satisfies the estimates

$$\left\|\partial_x^m\partial_t^n w^I(\cdot,t)\right\| \leq \|u_0\|_{m+2n} + \|u_1\|_{m+2(n-1)} \tag{7}$$

$$\left|\partial_x^m\partial_t^n w^I(x,t)\right| \leq \|u_0\|_{m+2n+1} + \|u_1\|_{m+2n-1}$$

Theorem 4. *The solution w^0 of the Cauchy problem*

$$w^0_{tt} + w^0_{xxxx} = 0$$

$$w^0(x,0) = u_0(x), \quad w^0_t(x,0) = u_1(x)$$

satisfies

$$\left\|\partial_x^m\partial_t^n w^0(\cdot,t)\right\| \leq \|u_0\|_{m+2n} + \|u_1\|_{m+2(n-1)} \tag{8}$$

$$\left|\partial_x^m\partial_t^n w^0(x,t)\right| \leq \|u_0\|_{m+2n+1} + \|u_1\|_{m+2n-1} \tag{9}$$

As a consequence of these results, we may derive the estimates

$$\left\|\partial_t^n\left(w^I(\cdot,t) - w^0(\cdot,t)\right)\right\| \leq I\,(1+t)\left(\|u_0\|_{5+n} + \|u_1\|_{3+n}\right) \tag{10}$$

$$\left|\partial_t^n\left(w^I(x,t) - w^0(x,t)\right)\right| \leq I\,(1+t)\left(\|u_0\|_{6+n} + \|u_1\|_{4+n}\right) \tag{11}$$

for $n = 0, 1, 2$.

Before proving these theorems, let us establish the main result. From the exspressions

$$W^I = w^I\psi - (v^I_+ + v^I_-)$$

$$W^0 = w^0\psi - (v^0_+ + v^0_-)$$

we have that

$$\begin{aligned}\left|W^I(x,t) - W^0(x,t)\right| &\leq |\psi(t)| \left|w^I(x,t) - w^0(x,t)\right| + \\ &\left|v_+^I(x,t) - v_+^0(x,t)\right| + \left|v_-^I(\cdot,t) - v_-^0(\cdot,t)\right|\end{aligned}$$

but from (11)

$$\left|w^I(x,t) - w^0(x,t)\right| \leq I\,(1+t)\left(\|u_0\|_6 + \|u_1\|_4\right)$$

and from (5)

$$\begin{aligned}\left|v_+^I(x,t) - v_+^0(x,t)\right| &\leq ce^{cx}\left(\int_0^x \int |f(y,t)|^2\, dtdy + \right. \\ &\left. \left(\int\int |(1+\partial_t)\, f_\alpha(x,t)|^2\, dxdt\right)^{1/2}\right)\end{aligned}$$

Similarly for $\left|v_-^I(x,t) - v_-^0(x,t)\right|$.

Recall that

$$f = 2\psi_t\left(w_t^I - w_t^0\right) + \psi_{tt}\left(w^I - w^0\right) - I\left(2w_{xxt}^I\psi_t + w_{xx}^I\psi_{tt}\right) + Iv_{+ttxx}^0$$

but from the estimates (7), (8), (10) and the fact that $\psi(t)$ is compactly supported we have

$$\int_0^x \int |f(y,t)|^2\, dtdy \leq cI\left(\|u_0\|_6 + \|u_1\|_4\right)$$

We may apply the same argumemnt to $\partial_t f_\alpha$ to obtain

$$\left(\int\int |(1+\partial_t)\, f_\alpha(x,t)|^2\, dxdt\right)^{1/2} \leq cI\left(\|u_0\|_8 + \|u_1\|_6\right)$$

Summarizing

$$\left|W^I(x,t) - W^0(x,t)\right| \leq cI(1+t)e^{cx}\left(\|u_0\|_8 + \|u_1\|_6\right).$$

Recall that $W^I(x,t)$ coincides with $w^I(x,t)$ for $|x| \leq 1$ and $0 \leq t \leq T_0$. Thus, the last assertion in Theorem 1 follows from (11). Hence, The proof is complete.

Remark. Due to Littman'method we require boundary controls in the whole boundary. It will be of interest to establish a similar result for clamped beams. By the work of Littman & Markus [9] it suffices to solve the ECP for the clamped Rayleigh beam and prove the analogue to theorems 3 and 4. Theorem 2 will apply.

Singular Perturbation

Here we prove Therorem 2. In this paragraph, any Fourier Transform is denoted with capital letters, that is, if a function $h(x,t)$ is given, we denote

$$H(x,\tau) = \mathcal{F}_t[h(x,\cdot)](\tau)$$

$$H(\xi,\tau) = \mathcal{F}[h](\xi,\tau)$$

Recall the polynomial

$$P(\xi,\tau) = \xi^4 - I\tau^2\xi^2 - \tau^2$$

Denote $D_x = -i\partial_x$, and $D_t = -i\partial_t$. Since $P(D_x, D_t)$ is hyperbolic with respect to $N = (1,0)$ there exists $\sigma_0 > 0$ such that

$$P(\xi - i\sigma, \tau) \neq 0, \quad (\xi,\tau) \in \mathbb{R}^2, \quad \sigma > \sigma_0\,.$$

It will become apparent that necessarily $\sigma_0 \geq \dfrac{1}{\sqrt{I}}$.

¿From Hormander [1] we have for some positive numbers c, μ that

$$|P(\xi - i\sigma,\tau)| \geq c\left(1 + \left(|\xi|^2 + |\tau|^2 + |\sigma|^2\right)^{1/2}\right)^{-\mu}, \quad (\xi,\tau) \in \mathbb{R}^2 \tag{12}$$

Our proof is based on a similar estimate, but independent of I.

Consider the Cauchy problem (4). Since f, and v in (4) are compactly supported in t, we may perform Fourier Transform with respect to t to obtain

$$V_{xxxx} + I\tau^2 V_{xx} - \tau^2 V = F$$

$$\partial_x^j V(0,\tau) = 0, \quad j = 0,1,2,3$$

We seek $V = e^{\sigma x}G$, $\sigma > 0$. Then G satisfies

$$P(D_x - i\sigma, \tau)G = e^{-\sigma x}F$$

Let $F_\sigma = e^{-\sigma x}F$. Fourier Transform with respect to x is permitted obtaining

$$P(\xi - i\sigma,\tau)G = F_\sigma \tag{13}$$

As remarked before, if $\sigma_0 = \dfrac{1}{\sqrt{I}}$, it is possible to solve for G if $\sigma > \dfrac{1}{\sqrt{I}}$. However, we need to solve for G even if this condition does not hold. To do so, we now derive a weaker version of (12).

Let $z = \xi - i\sigma$. The roots of the polynomial $P(z,\tau) = 0$ are

$$\begin{aligned}
z_1 &= -\frac{1}{\sqrt{2}}\sqrt{I\tau^2 + \sqrt{I^2\tau^4 + 4\tau^2}} \\
z_2 &= z_1 \\
z_3 &= -i\frac{1}{\sqrt{2}}\sqrt{\sqrt{I^2\tau^4 + 4\tau^2} - I\tau^2} \\
z_4 &= -z_3
\end{aligned}$$

Then
$$P(\xi - i\sigma, \tau) = (\xi - i\sigma - z_1)(\xi - i\sigma - z_2) \cdot$$
$$\cdot (\xi - i\sigma - z_3)(\xi - i\sigma - z_4)$$

Since
$$|(\xi - i\sigma - z_1)(\xi - i\sigma - z_2)| \geq \sigma \left(\sigma + |\xi| + \frac{1}{\sqrt{2}} \sqrt{I\tau^2 + \sqrt{I^2\tau^4 + 4\tau^2}} \right)$$

and
$$|(\xi - i\sigma - z_3)(\xi - i\sigma - z_4)| \geq \left(\sigma + |\xi| + \frac{1}{\sqrt{2}} \sqrt{\sqrt{I^2\tau^4 + 4\tau^2} - I\tau^2} \right) \cdot$$
$$\cdot \left(|\xi| + \left| \sigma - \frac{1}{\sqrt{2}} \sqrt{\sqrt{I^2\tau^4 + 4\tau^2} - I\tau^2} \right| \right)$$

we have
$$|P(\xi - i\sigma, \tau)| \geq \sigma \left(\sigma + |\xi| + \frac{1}{\sqrt{2}} \sqrt{I\tau^2 + \sqrt{I^2\tau^4 + 4\tau^2}} \right) \cdot$$
$$\cdot \left(\sigma + |\xi| + \frac{1}{\sqrt{2}} \sqrt{\sqrt{I^2\tau^4 + 4\tau^2} - I\tau^2} \right) \cdot$$
$$\cdot \left(|\xi| + \left| \sigma - \frac{1}{\sqrt{2}} \sqrt{\sqrt{I^2\tau^4 + 4\tau^2} - I\tau^2} \right| \right)$$

It is readily seen that the function $\frac{1}{\sqrt{2}} \sqrt{\sqrt{I^2\tau^4 + 4\tau^2} - I\tau^2}$ is increasing in $|\tau|$ and converges to $\frac{1}{\sqrt{I}}$ when $|\tau| \to +\infty$.

Fix I_a with $0 < I_a << 1$. Choose $\sigma = \frac{1}{4\sqrt{I_a}}$. It follows at once that
$$\frac{1}{\sqrt{2}} \sqrt{\sqrt{I^2\tau^4 + 4\tau^2} - I\tau^2} \geq 2\sigma$$
for $|\tau| \geq \frac{1}{2\sqrt{3}I_a}$ and $0 < I < I_a$.

Hence, we have the lower bound
$$|P(\xi - i\sigma, \tau)| \geq c\sigma \left(\sigma + |\xi| + \sqrt{I}\,|\tau| + |\tau|^{1/2} \right) (\sigma + |\xi|)^2 \tag{14}$$
for $|\tau| \geq \frac{1}{2\sqrt{3}I_a}$ and $0 < I < I_a$.

We proceed to estimate $V(x, \tau)$. Consider first the case $|\tau| \geq 1$. This corresponds to $I_a = \frac{1}{2\sqrt{3}}$, and $\sigma = \frac{\sqrt{3}}{2\sqrt{2}}$. This is our α in Theorem 2.

Denote $F_\alpha = e^{-\alpha x} F$. With these restrictions we have the next bound independent of I

$$|P(\xi - i\alpha, \tau)| \geq c \left(1 + |\tau|^{1/2}\right) (1 + |\xi|)^3 , \quad \xi \in \mathbb{R}, \;\; |\tau| \geq 1 \tag{15}$$

Now we may solve for G in (13)

$$G = \frac{F_\alpha}{P(\xi - i\alpha, \tau)}$$

and by inverse Fourier Transform

$$G(x, \tau) = \frac{1}{2\pi} \int \frac{e^{ix\xi} F_\alpha(\xi, \tau)}{P(\xi - i\alpha, \tau)} d\xi$$

which satisfies

$$|G(x, \tau)| \leq c \int \frac{|F_\alpha(\xi, \tau)|}{\left(1 + |\tau|^{1/2}\right) (1 + |\xi|)^3} d\xi$$

thus

$$|V(x, \tau)| \leq c e^{\alpha x} \int \frac{|F_\alpha(\xi, \tau)|}{\left(1 + |\tau|^{1/2}\right) (1 + |\xi|)^3} d\xi \tag{16}$$

Let us consider the case $|\tau| \leq 1$. After the change of variables

$$V_1 = V, \quad V_2 = V_x, \quad V_3 = V_{xx}, \quad V_4 = V_{xxx}$$

in (4) we obtain the system

$$\begin{array}{rcl} V_{1x} & = & V_2 \\ V_{2x} & = & V_3 \\ V_{3x} & = & V_4 \\ V_{4x} & = & \tau^2 V_1 - I\tau^2 V_3 + F \end{array}$$

with zero initial data.

Let $\overrightarrow{V} = (V_1, V_2, V_3, V_4)$, then

$$\frac{d}{dx} \left(\left| \overrightarrow{V} \right|^2 \right) = 2V_1V_2 + 2V_2V_3 + 2V_3V_4 + 2\tau^2 V_1 V_4 - 2I\tau^2 V_3 V_4 + F V_4$$

Using the fact that $I < 1$ and $|\tau| \leq 1$ we obtain

$$\frac{d}{dx} \left(\left| \overrightarrow{V} \right|^2 \right) \leq c \left| \overrightarrow{V} \right|^2 + F^2$$

and

$$\left|\overrightarrow{V}(x,\tau)\right|^2 \le e^{cx}\int_0^x |F(y,\tau)|^2\,dy \tag{17}$$

Now we apply both estimates to

$$v(x,t)=\frac{1}{2\pi}\int e^{i\tau t}V(x,\tau)d\tau\,.$$

Let us write

$$v(x,t)=\frac{1}{2\pi}\int_{|\tau|\le 1} e^{i\tau t}V(x,\tau)d\tau+\int_{|\tau|\ge 1} e^{i\tau t}e^{\alpha x}G(x,\tau)d\tau$$

$$v(x,t)\equiv\frac{1}{2\pi}\int_{|\tau|\le 1} J_1 d\tau+\int_{|\tau|\ge 1} J_2 d\tau\,.$$

¿From (17)

$$|J_1|\le e^{cx}\int_0^x |F(y,\tau)|^2\,dy$$

On the other hand, (16), implies

$$|J_2|\le ce^{\alpha x}\int\frac{|F_\alpha(\xi,\tau)|}{\left(1+|\tau|^{1/2}\right)(1+|\xi|)^3}d\xi$$

that is

$$\begin{aligned}|v(x,t)| \quad &\le \quad \tfrac{1}{2\pi}\textstyle\int_{|\tau|\le 1} e^{cx}\int_0^x |F(y,\tau)|^2\,dy d\tau+\\ &\textstyle\int_{|\tau|\ge 1} ce^{\alpha x}\int\dfrac{|F_\alpha(\xi,\tau)|}{\left(1+|\tau|^{1/2}\right)(1+|\xi|)^3}d\xi d\tau\end{aligned}$$

or

$$\begin{aligned}|v(x,t)| \quad &\le \quad ce^{cx}\Bigg(\textstyle\int_{|\tau|\le 1}\int_0^x |F(y,\tau)|^2\,dy d\tau\,+\\ &\textstyle\int_{|\tau|\ge 1}\int\dfrac{|F_\alpha(\xi,\tau)|}{\left(1+|\tau|^{1/2}\right)(1+|\xi|)^3}d\xi d\tau\Bigg)\end{aligned}$$

Applying Plancherel Theorem in the first term of the right hand side and Hölder Inequality in the second we have

$$|v(x,t)|\le ce^{cx}\left(\int_0^x\int |f(y,t)|^2\,dtdy+\left(\int\int\left(1+|\tau|^2\right)|F_\alpha(\xi,\tau)|^2\,d\xi d\tau\right)^{1/2}\right)$$

Again by Plancherel Theorem, it follows that

$$|v(x,t)| \leq ce^{cx} \int_0^x \int |f(y,t)|^2 \, dtdy + \left(\int \int |(1 + \partial_t) \, f_\alpha(x,t)|^2 \, dxdt \right)^{1/2} \qquad (18)$$

and the proof of Theorem 2 is complete.

Remark. Notice that instead of (15) we could derive from (14) the estimate

$$|P(\xi - i\alpha, \tau)| \geq c\sqrt{I} \, (1 + |\tau|) \, (1 + |\xi|)^3 \, , \quad \xi \in \mathbb{R}, \;\; |\tau| \geq 1$$

This lead us to require less regularity from the Cauchy data u_0, u_1, but implying weaker convergence in Theorem 1. The analogue for (18) will be I dependent.

Regular Perturbation

Here we carry out the proof of theorems 3 and 4. The proof is straightforward, we just perform Fourier Transform with respect to x and solve the resulting second order equations explicitly.

After Fourier Transform with respect to x in (6) we have

$$\left(1+I\xi^2\right)\widehat{w^I}_{tt}+\xi^4\widehat{w^I}=0$$

$$\widehat{w^I}(\xi,0)=\widehat{u}_0(\xi),\quad \widehat{w^I_t}(\xi,0)=\widehat{u}_1(\xi)$$

with solution

$$\widehat{w^I}(\xi,t)=\widehat{u}_0(\xi)\cos\lambda(\xi)t+\widehat{u}_1(\xi)\frac{\sin\lambda(\xi)t}{\lambda(\xi)}$$

where

$$\lambda(\xi)=\frac{\xi^2}{\sqrt{1+I\xi^2}} \tag{19}$$

easily

$$\left|\partial_t^n\widehat{w^I}(\xi,t)\right|\le\left|\lambda(\xi)^n\widehat{u}_0(\xi)\right|+\left|\lambda(\xi)^{n-1}\widehat{u}_1(\xi)\right| \tag{20}$$

We prefer a bound independent of I, to do so we require more regularity for the Cauchy Data. We obtain

$$\left|\partial_t^n\widehat{w^I}(\xi,t)\right|\le\left|\xi^{2n}\widehat{u}_0(\xi)\right|+\left|\xi^{2(n-1)}\widehat{u}_1(\xi)\right| \tag{21}$$

Similarly we apply Fourier transform with respect to x in the B-E problem to obtain

$$\widehat{w^0_{tt}}+\xi^4\widehat{w^0}=0$$

$$\widehat{w^0}(\xi,0)=\widehat{u}_0(\xi),\quad \widehat{w^0_t}(\xi,0)=\widehat{u}_1(\xi)$$

whose solution is

$$\widehat{w^0}(\xi,t)=\widehat{u}_0(\xi)\cos\xi^2t+\widehat{u}_1(\xi)\frac{\sin\xi^2t}{\xi^2}$$

with estimates

$$\left|\partial_t^n\widehat{w^0}(\xi,t)\right|\le\left|\xi^{2n}\widehat{u}_0(\xi)\right|+\left|\xi^{2(n-1)}\widehat{u}_1(\xi)\right|$$

For the difference we have the estimates

$$\left|w^I(\xi,t)-w^0(\xi,t)\right|\le It\left(\left|(1+|\xi|)^5\,\widehat{u}_0(\xi)\right|+\left|(1+|\xi|)^3\,\widehat{u}_1(\xi)\right|\right)$$

$$\left|w^I_t(\xi,t)-w^0_t(\xi,t)\right|\le It\left(\left|(1+|\xi|)^6\,\widehat{u}_0(\xi)\right|+\left|(1+|\xi|)^4\,\widehat{u}_1(\xi)\right|\right)$$

$$\left|w_{tt}^I(\xi,t) - w_{tt}^0(\xi,t)\right| \le I\,(1+t)\left(\left|(1+|\xi|)^7\,\widehat{u}_0(\xi)\right| + \left|(1+|\xi|)^5\,\widehat{u}_1(\xi)\right|\right)$$

Remark. Estimate (21) is independent of I. ¿From (19) and (20), we have again an alternative estimate dependent on I, namely

$$\left|\partial_t^n \widehat{w^I}(\xi,t)\right| \le \left|\frac{1}{\left(\sqrt{I}\right)^n}(1+|\xi|)^n \widehat{u}_0(\xi)\right| + \left|\frac{1}{\left(\sqrt{I}\right)^{n-1}}(1+|\xi|)^{n-1}\widehat{u}_1(\xi)\right|$$

with the same conclusion as the remark above.

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