DIHEDRAL GROUPS ACTING ON JACOBIANS

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ABSTRACT. We study curves with action of a dihedral group, giving an equivariant decomposition of their Jacobians into geometric factors, on each of which acts precisely one of the rational irreducible representations of the dihedral group.

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1. INTRODUCTION

In this paper we give a geometric equivariant decomposition for the Jacobian of curves where a dihedral group acts.

The case when the dihedral group D_p with p an odd prime acts on a curve W such that the quotient curve W/D_p is the Riemann sphere $\mathbb{P}^1$ and such that the number of fixed points on W of any element r of order p in D_p is zero or two was studied by Ries in [Ri].

There he concluded that the Prym variety $P(W/W_r)$ is isomorphic to twice the Jacobian JW_k, where W_r is the quotient of W by the cyclic

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group generated by r and W_k is the quotient of W by any involution k in D_p.

We prove that the Jacobian JW of a curve W with D_n-action is isogenous to a product of abelian sub-varieties where the group acts on each factor by a suitable multiple of precisely one of the rational irreducible representations of G.

For the particular case of $n = p$ and $n = 2p$, where p is a prime, we also compute the kernel of the isogeny, obtaining Ries result as a special case.

It is interesting to mention here that some of the factors we obtain may be described as Jacobians or Prym varieties of intermediate covers, while others are new kinds of sub-varieties; namely, orthogonal complements of Pryms inside Pryms.

In Section 2 we set up the notation and collect some general results which will be useful later on.

In Section 3 we describe the irreducible representations of the dihedral group over the rational numbers.

In Section 4 we describe the action for the case n odd and in Section 5 we give the decomposition for JW in this case.

In Section 5.1 we study the special case n an odd prime in detail, computing the kernel of the equivariant isogeny and re-obtaining Ries' result.

In Section 6 and in Section 7 we give the decomposition for the cases when n is a power of two and when is even but not a power of two, respectively.

In Section 7.1 we describe the case $n = 2p$ with p an odd prime in detail, including the kernel for a special isogeny.

2. Curves with D_n-action

Throughout this paper, W will denote be a smooth connected complete curve over the complex numbers $\mathbb{C}$ which admits the dihedral group D_n as group of automorphisms.

For any subgroup H of D_n we will denote by W_H the quotient curve W/H and by $f_H : W \to W_H$ the natural quotient map. If g is any element of D_n, we will abbreviate $W_{\langle g\rangle}$ (the quotient by the cyclic group generated by g) by W_g and $f_{\langle g\rangle}$ by f_g.

If H is a subgroup of a subgroup N of D_n, then $f_{H,N} : W_H \to W_N$ will denote the natural projection between the corresponding quotient curves.

JW_H will denote the Jacobian of W_H and $P(W_H/W_N)$ the Prym variety of $f_{H,N}$; namely, the orthogonal component of $f^*_{H,N}(JW_N)$ in JW_H with respect to the natural Hermitian form on JW_H.

Also, JW^H will denote the subvariety of JW fixed by H. For any natural number s, $JW[s]$ will denote the subgroup of JW formed by the points of order s.

If $f : W_H \to W_N$ is an unramified cyclic cover of curves of degree s, we will denote by η_f a point in $JW_N[s]$ which determines the cover f; equivalently, a generator for the kernel of $f^* : JW_N \to JW_H$ (see [L–B]).

We have the following diagram of curves and covers.

(2.1)
$$\begin{tikzcd}
 & W \arrow[dl, "f_{r^js}"'] \arrow[dr, "f_r"] \arrow[dd, "\gamma"] & \\
W_{r^js} \arrow[dr, "f_{r^js,D_n}"'] & & W_r \arrow[dl, "f_{r,D_n}"] \\
 & W_{D_n} &
\end{tikzcd}$$

The cases n odd and n even will be treated separately in the next sections.

We now collect some general results that will be used later.

The following result appeared in [SA] for $G = A_5$ and was generalized as follows in [C–R].

Proposition 2.1. *Let G be a finite group and let $\{\mathcal{W}_j\}_{j=1}^r$ denote its irreducible representations over the rational numbers.*

Assume there exist $H \leq N \leq G$ subgroups of G and $1 \leq \ell \leq r$ such that

1. $\rho_H^G - \rho_N^G = \sum_{j=1}^{\ell} \mathcal{W}_j$, *and*
2. $\langle \rho_H^G, \mathcal{W}_j \rangle_G = 1$ *for all* $1 \leq j \leq \ell$,

where ρ_H^G denotes the permutation representation of G induced by the trivial representation of H and $\langle , \rangle_G$ denotes the inner product between representations of G.

Let $\mathcal{W}$ denote any rational representation of G containing as different isotypical components $\mathcal{W}_1^{n_1} \oplus \oplus \dots \mathcal{W}_\ell^{n_\ell}$ and denote by $\mathcal{W}^H$ the subspace of $\mathcal{W}$ fixed by H.

Then

$$\mathcal{W}^H = \mathcal{W}^N \oplus (\mathcal{W}_1^{n_1})^H \oplus \dots \oplus (\mathcal{W}_\ell^{n_\ell})^H$$

Furthermore, if for every $1 \leq j \leq \ell$ we consider the (complex) dimension v_j of a complex irreducible representation $\mathcal{V}_j$ associated to $\mathcal{W}_j$, then, under condition 2) above, there are v_j elements $g_1^j, \dots, g_{v_j}^j$ of G such that

$$\mathcal{W}_j = \bigoplus_{k=1}^{v_j} \mathcal{W}_j^{g_k^j\, H\, (g_k^j)^{-1}}$$

Corollary 2.2. *Let G, H and N be as in Proposition 2.1 and let W be a curve with G-action.*

Denote by $f_H : W \to W_H = W/H$ the natural projection.

Then

$$(df_H{}^*)(P(W_H/W_N)) = (\mathcal{W}_1^{n_1})^H \oplus \ldots \oplus (\mathcal{W}_\ell^{n_\ell})^H$$

where the sum is orthogonal with respect to any Hermitian product invariant under G.

Furthermore, if $\ell = 1$, then $\mathcal{W}_1^{n_1}$ acts on

$$f_H^*(P(W_H/W_N))^{s_1}$$

Proof. Consider the rational representation of G on $T_0(JW)$ and apply the Proposition.

The last assertion is proved by observing that, for each $1 \leq k \leq v_1$, g_k^1 gives an isomorphism

$$g_k^1 : P(W_H/W_N) \to P(W_{g_k^1\, H\, (g_k^1)^{-1}}/W_{g_k^1\, N\, (g_k^1)^{-1}})$$

□

3. Irreducible rational representations of the dihedral group

We denote by $D_n = \langle r, s : r^n = s^2 = (rs)^2 = 1\rangle$ the dihedral group of order $2n$.

It is known that this group has two complex irreducible representations of degree one, χ_0 and χ_1, if n is odd, and four, χ_0, χ_1, χ_2 and χ_3, if n is even.

They are given as follows.

Representation	r	s	Case
χ_0	1	1	any
χ_1	1	-1	any
χ_2	-1	1	n even
χ_3	-1	-1	n even

The other complex irreducible representations of D_n all have degree two, and there are $\dfrac{n-1}{2}$ of them for n odd, respectively $\dfrac{n-2}{2}$ for n even, given as follows.

$$\mathcal{V}_j(r) = \begin{pmatrix} \omega^j & 0 \\ 0 & \omega^{-j} \end{pmatrix} \quad \text{and} \quad \mathcal{V}_j(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\omega = \omega_n = \exp \frac{2\pi i}{n}$ and $1 \leq j < \frac{n}{2}$.

We are interested in the irreducible rational representations of D_n, which are described in the next result; we were unable to find them explicitly in the literature.

Remark 3.1. Let D_n denote the dihedral group of order $2n$, and let

$$\Omega(n) = \{d : d \text{ is a divisor of } n\}$$

For each d in $\Omega(n)$ and $d < n/2$, consider the complex irreducible representation of degree two $\mathcal{V}_d$ given above.

Its field of definition coincides with the field generated by its characters

$$K_d = \mathbb{Q}(w_n^d + w_n^{-d}) = \mathbb{Q}(w_{\frac{n}{d}} + w_{\frac{n}{d}}^{-1})$$

and furthermore

$$[K_d : \mathbb{Q}] = \frac{\varphi(\frac{n}{d})}{2}$$

where φ denotes the Euler function.

Hence the irreducible rational representation $\mathcal{W}_d$ of D_n generated by $\mathcal{V}_d$ is given by

$$K_d \otimes_{\mathbb{Q}} \mathcal{W}_d = \bigoplus_{\sigma \in \mathrm{Gal}(K_d/\mathbb{Q})} (K_d \otimes_\sigma \mathcal{V}_d)$$

and has degree

$$\deg(\mathcal{W}_d) = \dim_{K_d} \mathcal{V}_d \cdot [K_d : \mathbb{Q}] = 2 \cdot \frac{\varphi(\frac{n}{d})}{2} = \varphi\left(\frac{n}{d}\right)$$

Theorem 3.2. *Let D_n denote the dihedral group of order $2n$ and let $\Omega(n) = \{d : d$ is a divisor of $n\}$.*

Then the rational irreducible representations of D_n are given as follows.

If n is odd, they are χ_0 and χ_1, of degree one, and $\{\mathcal{W}_d\}$ with d in $\Omega(n)$ and $d \neq n$, of respective degrees $\deg(\mathcal{W}_d) = \varphi\left(\frac{n}{d}\right)$.

If n is even, they are χ_0, χ_1, χ_2 and χ_3, of degree one, and $\{\mathcal{W}_d\}$ with d in $\Omega(n)$ and $d \neq n, n/2$, of respective degrees $\deg(\mathcal{W}_d) = \varphi\left(\frac{n}{d}\right)$.

Proof. We already know that the given representations are irreducible over the rationals and that they have the given degrees.

All that is left to show is that there are no other irreducible representations.

For n odd, just observe that

$$\sum_{\substack{d \in \Omega(n) \\ d \neq n}} \deg(\mathcal{W}_d) = \sum_{\substack{d \in \Omega(n) \\ d \neq n}} \varphi\left(\frac{n}{d}\right) = n - 1 = \frac{n-1}{2} \cdot 2 = \sum_{j=1}^{\frac{n-1}{2}} \deg(\mathcal{V}_j)$$

A similar argument for n even finishes the proof. □

4. The case n odd

When n is odd, all curve $W_{r^j s}$ are isomorphic to each other since the involutions $r^j s$ are conjugate in D_n. Hence the (common) number of fixed points of each of these involutions on W must be even, since it corresponds to the number of ramification points of the double cover $W_r \to W_{D_n}$; we will denote it by 2β.

Furthermore, the fixed points of r are permuted among themselves by s, say, and therefore the number of fixed points of r must also be even; we will denote it by 2α.

For the case of $n = p$ an odd prime, there are no other fixed points.

A calculation proves the following result.

Proposition 4.1. *Let n be an odd number and let W be a curve with $D_n \subseteq \mathrm{Aut}(W)$ and associated Diagram (2.1).*

We denote by 2β the number of fixed points of any involution of the form $r^j s$ in D_n and by 2α the number of fixed points of any element of order n in D_n.

Then the following table describes the genera of the corresponding curves and the total ramification order for the corresponding covers, where g denotes the genus of W_{D_n}.

genus	*order of ramification* $\lvert B\rvert$
$g_W = 2ng + n\beta + (n-1)\alpha + 1 - 2n$	
$g_{W_r} = 2g - 1 + \beta$	$\lvert B(W \to W_r)\rvert = 2(n-1)\alpha$ $\lvert B(W_r \to W_{D_n})\rvert = 2\beta$
$g_{W_{r^j s}} = ng + \dfrac{n-1}{2}(\alpha + \beta - 2)$	$\lvert B(W \to W_{r^j s})\rvert = 2\beta$ $\lvert B(W_{r^j s} \to W_{D_n})\rvert = (n-1)(\alpha + \beta)$

For the particular case when $n = p$ is an odd *prime* number, the curves in Diagram (2.1) give all the possible quotient curves. In this case, we can also prove the following results.

Proposition 4.2. *Let p denote an odd prime and let W be a curve such that $D_p \subseteq \mathrm{Aut}(W)$, with associated Diagram (2.1).*

Let k denote any involution in D_p. Then the following results hold.

i) $f^*_{k,D_n} : JW_{D_n} \to JW_k$ *is injective.*

ii) $\ker f^*_{r,D_n} = \ker \gamma^*$.

iii) $\ker f^*_k = f^*_{k,D_n}(\ker f^*_{r,D_n})$.

iv) $f^*_{r,D_n}(JW_{D_n}) \cap \ker f^*_r = \{0\}$.

Proof. i) holds because the cover $f_{k,D_n} : JW_k \to JW_{D_n}$ is non-cyclic of prime order (see [L–B, p. 337]).

As for ii), it is clear that $\ker f^*_{r,D_n} \subseteq \ker \gamma^*$. For every $P \in \ker \gamma^*$ we have that $f^*_{k,D_n}(P) \in \ker f^*_k \subseteq JW_k[2]$. By i) f^*_{k,D_n} is injective, and therefore $2P = 0$.

But also $f^*_{r,D_n}(P) \in \ker f^*_r \subseteq JW_r[p]$; since p is odd we must have $f^*_{r,D_n}(P) = 0$ and therefore $P \in \ker f^*_{r,D_n}$.

To prove iii) just note that $\ker f^*_{r,D_n} = \ker \gamma^* = (f^*_{k,D_n})^{-1}(\ker f^*_k)$, where the first equality holds from ii) and the second one from the commutativity of Diagram (2.1).

The proof of iv) is direct from ii): if $Q \in f^*_{r,D_n}(JW_{D_n}) \cap \ker f^*_r$ then $Q = f^*_{r,D_n}(P)$ for some $P \in JW_{D_n}$ and $0 = f^*_r(Q) = \gamma^*(P)$. Hence P must be in $\ker f^*_{r,D_n}$ and therefore $Q = 0$. □

We also have the following result.

Corollary 4.3. *Let W be as in Proposition 4.1 and let $k = r^j s$ denote an involution in D_n.*

Then f^*_k is injective if and only if f^*_{r,D_n} is injective.

Proof. The result follows immediately from Proposition 4.2 when n is an odd prime.

For the general case of n any odd natural number, note that since both f_k and f_{r,D_n} are of degree two, then f^*_k (respectively f^*_{r,D_n}) is injective if and only if f_k (respectively f_{r,D_n}) is ramified.

But the involution $k : W \to W$ descends to the involution $\sigma : W_r \to W_r$ that gives the morphism $f_{r,D_n} : W_r \to W_{D_n}$.

Therefore, if $w \in W$ is a ramification point of f_k, or, equivalently, if the stabilizer of w in D_n is the group generated by k, then $f_r(w)$ is a fixed point of σ, or, equivalently, it is a ramification point of f_{r,D_n}.

Conversely, a ramification point of f_{r,D_n} is a fixed point of σ of the form $f_r(w)$, and hence w is a fixed point of some conjugate $r^l s$ of k. Then the image of w under an appropriate power of r produces a fixed point of k. □

The next result will be very useful later on.

Corollary 4.4. *Let W be as in Proposition 4.1 and let $k = r^j s$ denote an involution in D_n.*

Then $f^*_k|_{P(W_k/W_{D_n})}$ is injective.

Proof. If f^*_k is not injective on JW_k, then

$$\ker f^*_{r,D_n} = \{0, \eta_{f_{r,D_n}}\} \subseteq JW_{D_n}[2]$$

with $\eta_{f_{r,D_n}} \neq 0$.

Furthermore, $f^*_{k,D_n}(\eta_{f_{r,D_n}}) = \eta_{f_k} \in f^*_{k,D_n}(JW_{D_n}[2])$ is the non zero element of $\ker f^*_k$, because $\ker f^*_{k,D_n} \subseteq JW_{D_p}[n]$ consists of points of odd order.

But $f^*_{k,D_n}(JW_{D_n}) \cap P(W_k/W_{D_n}) \subseteq JW_k[n]$ also consists of points of odd order, and therefore $\eta_{f_k} \notin P(W_k/W_{D_n})$. □

5. The decomposition for n odd

Theorem 5.1. *Let $n = p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ be the prime decomposition of the odd natural number n and let W be a curve with D_n action.*

For each $d \in \Omega(n)$ *different from* n*, write* $d = \frac{n}{p_1^{i_1} \cdot \ldots \cdot p_r^{i_r}}$ *and let* D_d *be a dihedral subgroup of* D_n *of order* $2d$*, which are all conjugate among themselves for fixed* d*.*

Also let

$$B_d = \bigcap_{\substack{j \in \{1, \ldots, r\} \\ i_j \geq 1}} P(W_{D_d}/W_{D_{p_j \cdot d}}) \subseteq JW_{D_d} \ .$$

Then the isogeny decomposition of JW *with respect to* D_n *is the following.*

$$JW \sim JW_{D_n} \times P(W_r/W_{D_n}) \times \prod_{\substack{d \in \Omega(n) \\ d \neq n}} 2B_d$$

where the D_n-*action on each factor on the right hand side is given by (appropriate multiples of) the trivial action* χ_0 *on* JW_{D_n}*, the alternating action* χ_1 *on* $P(W_r/W_{D_n})$*, and* $\mathcal{W}_d$ *on* $2B_d$ *for each* $d \in \Omega(n)$ *with* $d \neq n$*.*

Furthermore, for each j *such that* $i_j \geq 1$

$$2P(D_d/D_{p_j \cdot d}) \text{ is isogenous to } P(W_{r^{\frac{n}{d}}}/W_{r^{\frac{n}{p_j \cdot d}}}) \ .$$

Proof. For each subgroup H of D_n we denote by ρ_H the representation of D_n induced by the trivial representation of H.

Then calculations show that

$$\rho_{D_d} = \chi_0 + \sum_{\substack{k \in \Omega(n), k \neq n \\ k \cdot p_1^{i_1} \cdot \ldots \cdot p_r^{i_r} \equiv 0 (n)}} \mathcal{W}_k$$

and that if $i_1 \geq 1$ then

$$\rho_{D_{p_1 \cdot d}} = \chi_0 + \sum_{\substack{\ell \in \Omega(n), \ell \neq n \\ \ell \cdot p_1^{i_1 - 1} \cdot \ldots \cdot p_r^{i_r} \equiv 0 (n)}} \mathcal{W}_\ell$$

Hence if $i_1 \geq 1$ then

$$\rho_{D_d} - \rho_{D_{p_1 \cdot d}} = \sum_{\substack{j \in \{2, \ldots, r\} \\ s_j \in \{0, \ldots, i_j\}}} \mathcal{W}_{d \cdot p_2^{s_2} \cdot \ldots \cdot p_r^{s_r}} \tag{5.2}$$

and therefore (by 2.2)

$$P(W_{D_d}/W_{D_{p_1 \cdot d}}) \sim \prod_{\substack{j \in \{2, \ldots, r\} \\ s_j \in \{0, \ldots, i_j\}}} B_{d \cdot p_2^{s_2} \cdot \ldots \cdot p_r^{s_r}}$$

with similar results for the other $p_j's$ for which $i_j \geq 1$.

Furthermore, it follows from Corollary 2.2 that then

(5.3) (an appropriate multiple of) $\mathcal{W}_d$ acts on

$$2B_d = 2 \bigcap_{\substack{j\in\{1,\dots,r\}\\ i_j\geq 1}} P(W_{D_d}/W_{D_{p_j\cdot d}})$$

We also compute that

$$\rho_{\langle r^{\frac{n}{d}}\rangle} = \chi_0 + \chi_1 + 2 \sum_{\substack{k/n, k\neq n\\ k\cdot p_1^{i_1}\cdot\ldots\cdot p_r^{i_r}\equiv 0(n)}} \mathcal{W}_k = \rho_{\langle r\rangle} + 2 \sum_{\substack{k/n, k\neq n\\ k\cdot p_1^{i_1}\cdot\ldots\cdot p_r^{i_r}\equiv 0(n)}} \mathcal{W}_k$$

from where we obtain

$$\rho_{\langle r\rangle} - \rho_{D_n} = \chi_1 \tag{5.4}$$

and

$$\rho_{\langle r^{\frac{n}{d}}\rangle} - \rho_{\langle r^{\frac{n}{p_1\cdot d}}\rangle} = 2 \sum_{\substack{s_j\in\{0,\dots,i_j\}\\ j\in\{2,\dots,r\}}} \mathcal{W}_{d\cdot p_2^{s_2}\cdot\ldots\cdot p_r^{s_r}} \tag{5.5}$$

Now we apply Corollary 2.2 to conclude from (5.4) that

(5.6) (an appropriate multiple of) χ_1 acts on $P(W_r/W_{D_n})$

and also, by comparing (5.2) and (5.5), that if $i_1 \geq 1$ then

$$2P(W_{D_d}/W_{D_{p_1\cdot d}} \text{ is isogenous to } P(W_{r^{\frac{n}{d}}}/W_{r^{\frac{n}{p_1\cdot d}}})$$

The results (5.3) and (5.6) give the isogeny decomposition, thus completing the proof. □

5.1. **The case $n = p$, an odd prime.** In this case, we can be more precise.

Let 2α denote the number of fixed points in W of any element of order p in D_p and let k denote an involution in D_p .

If $\alpha \geq 2$, let $\{P_1, \dots , P_\alpha\}$ denote the set of total ramification points of $f_{k,D_p} : W_k \to W_{D_p}$. For each $j \in \{2, \dots , \alpha\}$ we choose $m_j \in \mathrm{Pic}^0(W_{D_p})$ such that $m_j^{\otimes p} = \mathcal{O}_{W_{D_p}}(f_{k,D_p}(P_1) - f_{k,D_p}(P_j))$ and we define

$$\mathcal{F}_j = \mathcal{O}_{W_k}(P_j - P_1) \otimes f^*_{k,D_p}(m_j)\,. \tag{5.7}$$

For p an odd prime, D_p has exactly three irreducible representations over the rational numbers, given by χ_0, χ_1 and $\mathcal{W}_1$, which is the sum of the $(p - 1)/2$ irreducible complex representations of degree two of D_p.

We obtain the next result, which will be proved in Appendix A.

Proposition 5.2. *The morphism*

$$\begin{aligned} P(W_k/W_{D_p}) \times P(W_k/W_{D_p}) &\xrightarrow{\varphi_k} P(W/W_r) \\ (z_1, z_2) &\dashrightarrow f_k^*(z_1) + r f_k^*(z_2) \end{aligned} \tag{5.8}$$

is a D_p-equivariant isogeny whose kernel is isomorphic to

$$K = \{z \in P(W_k/W_{D_p})[p] : f_k^*(z) \text{ is } D_p \text{ invariant}\} \tag{5.9}$$

More precisely, K may be described as follows.

i) $f^*_{k,D_p}(JW_{D_p})[p] \subseteq K$.

ii) *If $\alpha = 0$ or 1 then $K = f^*_{k,D_p}(JW_{D_p})[p]$.*

iii) *If $\alpha \geq 2$ then*

$$K = f^*_{k,D_p}(JW_{D_p})[p] \oplus \bigoplus_{j=2}^{\alpha} \mathcal{F}_j \mathbb{Z}/p\mathbb{Z}$$

where the $\mathcal{F}_j$ are defined by (5.7).

and therefore the cardinality of K is given as follows, if g denotes the genus of W_{D_p} and 2β denotes the number of fixed points of the involution k in D_p.

$$|K| = \begin{cases} p^{2g} & \text{if } \alpha = 0; \\ p^{2g+\alpha-1} & \text{if } \alpha \geq 1\,. \end{cases} \tag{5.10}$$

We obtain as a corollary the result in [Ri].

Corollary 5.3. *Consider a curve W with D_p-action such that r has two or no fixed points in W and such that $W_{D_p} = \mathbb{P}^1$.*

Then $P(W/W_r)$ is isomorphic (as a polarized abelian variety) to $JW_k \times JW_k$.

Proof. Under the hypothesis in the corollary, φ_k is an isomorphism and the polarization induced on $P(W/W_r)$ by the natural principal polarization on JW is twice a principal polarization. ☐

We now refine Theorem 5.1 for this case as follows.

Theorem 5.4. *Let p denote an odd prime and let W be a curve such that $D_p \subseteq \mathrm{Aut}(W)$, with associated Diagram (2.1).*

Also, denote by k an involution in D_p.

Then the isogeny decomposition of JW with respect to D_p is the following.

$$R : JW_{D_p} \times P(W_r/W_{D_p}) \times 2P(W_k/W_{D_p}) \sim JW \tag{5.11}$$

with $R(w_1,w_2,w_3,w_4) = f^*_{D_p}(w_1)+f^*_r(w_2)+f^*_k(w_3)+rf^*_k(w_4)$, *where the* D_p-*action on each factor on the left hand side is given by (appropriate multiples of) the trivial action* χ_0 *on* JW_{D_p} , *the alternating action* χ_1 *on* $P(W_r/W_{D_p})$, *and* $\mathcal{W}_1$ *on* $2P(W_k/W_{D_p})$.

Furthermore, if we denote by 2α *the number of fixed points in* W *of the element* r *(of order* p*) in* D_p , *if* g *denotes the genus of* W_{D_p} *and* 2β *denotes the number of fixed points of an involution in* D_p, *then the isogeny* R *has kernel with cardinality given as follows.*

$$|\ker R| = \begin{cases} 2^{2g-1}p^{6g-3+\alpha} & \text{if } \beta = 0; \\ 2^{2g}p^{6g-3+\alpha+2\beta} & \text{if } \beta \geq 1\,. \end{cases}$$

Proof. The decomposition (5.11) follows immediately from Theorem 5.1 once we observe that d in $\Omega(p)$ and $d \neq p$ imply that $d = 1$ and therefore $\mathcal{W}_1$ acts on $2B_1$, where

$$B_1 = P(W_{D_1}/W_{D_p}) = P(W_k/W_{D_p})\,.$$

To prove the last part of the theorem, first observe that the following diagram is commutative

(5.12)

$$\begin{tikzcd}
JW_{D_p} \times P(W_r/W_{D_p}) \times 2P(W_k/W_{D_p}) \arrow[d, "R"'] \arrow[dr, "(\mathrm{id}_{JW_{D_p}\times P(W_r/W_{D_p})},\, \varphi_k)"] & \\
 & JW_{D_p} \times P(W_r/W_{D_p}) \times P(W/W_r) \arrow[dl, "\psi"] \\
JW &
\end{tikzcd}$$

where φ_k is defined in Proposition 5.2 and

$$\psi(z_1,z_2,z_3) = f^*_{D_p}(z_1) + f^*_r(z_2) + z_3$$

has kernel with cardinality as follows (see [Rec–Ro2]).

$$\begin{aligned} |\ker(\psi)| &= \frac{|JW_r[p]|\,|JW_{D_p}[2]|}{|\ker f^*_r|\,|\ker f^*_{r,D_p}|} \\ &= \begin{cases} 2^{2g-1}p^{4g-3} & \text{if } \alpha = \beta = 0 \\ 2^{2g}p^{4g-3+2\beta} & \text{if } \alpha = 0 \text{ and } \beta > 0 \\ 2^{2g-1}p^{4g-2+2\beta} & \text{if } \alpha > 0 \text{ and } \beta = 0 \\ 2^{2g}p^{4g-2+2\beta} & \text{if } \alpha\,\beta > 0 \end{cases} \end{aligned} \tag{5.13}$$

We can now finish the proof by concluding that

$$|\ker(R)| = |\ker(\varphi_k)|\,|\ker(\psi)|$$

from Diagram (5.12) and then combining (5.13) and (5.10). ∎

6. The case n is a power of 2

In this section we assume $n = 2^\alpha$.

The case $\alpha = 1$ corresponds to the Klein group, which was completely described in [Rec–Ro2], as well as the case $\alpha = 2$. Therefore, we also assume $\alpha \geq 2$ in this section.

In this case $\Omega(n) = \{2^j; 0 \leq j \leq \alpha\}$, and for each $d \in \Omega(n)$ with $d \neq n$ there are two non conjugate subgroups of $D_n = \langle r, s\rangle$ both isomorphic to D_d: $D_d = \langle r^{\frac{n}{d}}, s\rangle$ and $\widetilde{D_d} = \langle r^{\frac{n}{d}}, rs\rangle$.

Furthermore, it follows from Theorem 3.2 that the rational irreducible representations of D_n are the following in this case.

$$\chi_0, \chi_1, \chi_2, \chi_3, \mathcal{W}_{n/4}, \mathcal{W}_{n/8}, \ldots, \mathcal{W}_1$$

where $\deg(\mathcal{W}_d) = \varphi\left(\frac{n}{d}\right) = \frac{n}{2d}$.

The isotypical decomposition in this case is the following.

Theorem 6.1.

$$\begin{aligned} JW \sim JT &\times P(W_r/W_{D_n}) \times P(W_{D_{\frac{n}{2}}}/W_{D_n}) \times P(W_{\widetilde{D_{\frac{n}{2}}}}/W_{D_n}) \\ &\times 2P(W_{D_{\frac{n}{4}}}/W_{D_{\frac{n}{2}}}) \times \ldots \times 2P(W_{D_1}/W_{D_2}) \end{aligned}$$

with respective actions

$$\chi_0 \times \chi_1 \times \chi_2 \times \chi_3 \times \mathcal{W}_{\frac{n}{4}} \times \ldots \times \mathcal{W}_1$$

Furthermore, for each $d \in \Omega(n)$ with $d \neq n, \frac{n}{2}$, we have that

$$P(W_{D_d}/W_{D_{2d}}) \text{ is isogenous to } P(W_{\widetilde{D_d}}/W_{\widetilde{D_{2d}}})$$

$$2P(W_{D_d}/W_{D_{2d}}) \text{ is isogenous to } P(W_{r^{\frac{n}{d}}}/W_{r^{\frac{n}{2d}}})$$

and

$$P(W_{D_{\frac{n}{2}}}/W_{D_n}) \times P(W_{\widetilde{D_{\frac{n}{2}}}}/W_{D_n}) \text{ is isogenous to } P(W_{r^2}/W_r)$$

Proof. For each $d \in \Omega(n)$ with $d \neq n, \frac{n}{2}$ we have

$$\begin{aligned} \rho_{D_d} &= \chi_0 + \chi_2 + \mathcal{W}_{\frac{n}{4}} + \ldots + \mathcal{W}_d = \rho_{D_{\frac{n}{2}}} + \mathcal{W}_{\frac{n}{4}} + \ldots + \mathcal{W}_d \;; \\ \rho_{\widetilde{D_d}} &= \chi_0 + \chi_3 + \mathcal{W}_{\frac{n}{4}} + \ldots + \mathcal{W}_d = \rho_{\widetilde{D_{\frac{n}{2}}}} + \mathcal{W}_{\frac{n}{4}} + \ldots + \mathcal{W}_d \end{aligned}$$

and also

$$\begin{aligned} \rho_{\langle r^{\frac{n}{d}}\rangle} &= \chi_0 + \chi_1 + \chi_2 + \chi_3 + 2\mathcal{W}_{\frac{n}{4}} + \ldots + 2\mathcal{W}_d \\ &= \rho_{\langle r^2\rangle} + 2\mathcal{W}_{\frac{n}{4}} + \ldots + 2W_d \end{aligned}$$

$\square$

7. The case n even and not a power of two

In this section we assume n is even and not a power of two. Let $n = 2^{\alpha_0} \cdot p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ be the prime decomposition of n.

Then $\Omega(n) = \{2^{k_0} \cdot p_1^{\alpha_{k_1}} \cdot \ldots \cdot p_r^{\alpha_{k_r}}; 0 \leq k_j \leq \alpha_j, 0 \leq j \leq r, \}$.

For each d in $\Omega(n)$ such that $\dfrac{n}{d}$ is even (equivalently, for each d not divisible by 2^{α}) there are two non conjugate subgroups of $D_n = \langle r, s\rangle$ both isomorphic to the dihedral group D_d; we choose a representatives $D_d = \langle r^{\frac{n}{d}}, s\rangle$ and $\widetilde{D_d} = \langle r^{\frac{n}{d}}, r^{n-1}s\rangle$, respectively. For each d in $\Omega(n)$ with $\dfrac{n}{d}$ odd, the conjugacy class of D_d in D_n is unique; we choose the representative $D_d = \langle r^{\frac{n}{d}}, s\rangle$.

Furthermore, it follows from Theorem 3.2 that the rational irreducible representations of D_n are given by

$$\chi_0, \chi_1, \chi_2, \chi_3, \{\mathcal{W}_d\}_{\substack{d\in\Omega(n)\\ d\neq n, n/2}}$$

where $\deg(\mathcal{W}_d) = \varphi(\frac{n}{d})$.

The isotypical decomposition in this case is the following.

Theorem 7.1. *Let $n = p_0^{\alpha_0} \cdot p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ be the prime decomposition of the even natural number n, not a power of 2. Assume $p_0 = 2$.*

Let W be a curve with D_n action.

Furthermore, for each $d \in \Omega(n)$ with $d \neq n, \dfrac{n}{p_j}$ for $0 \leq j \leq r$, write $d = \frac{n}{p_0^{i_0} \cdot p_1^{i_1} \cdot \ldots \cdot p_r^{i_r}}$ and let

$$B_d = \bigcap_{\substack{j\in\{0,\ldots,r\}\\ i_j\geq 1}} P(W_{D_d}/W_{D_{p_j\cdot d}}) \ .$$

Then

$$JW \sim JW_{D_n} \times P(W_r/W_{D_n}) \times P(W_{D_{\frac{n}{2}}}/W_{D_n}) \times P(W_{\widetilde{D_{\frac{n}{2}}}}/W_{D_n}) \times \prod_{i=1}^{r} 2P(W_{D_{\frac{n}{p_i}}}/W_{D_n}) \times \prod_{\substack{d\in\Omega(n)\\ d\neq n, \frac{n}{p_j}\\ 0\leq j\leq r}} 2B_d$$

with respective actions

$$\chi_0 \times \chi_1 \times \chi_2 \times \chi_3 \times \prod_{i=1}^{r} \mathcal{W}_{\frac{n}{p_i}} \times \prod_{\substack{d\in\Omega(n)\\ d\neq n, \frac{n}{p_i}}} \mathcal{W}_d \ .$$

Furthermore,

$$B_d \textit{ is isogenous to } \widetilde{B_d} = \bigcap_{\substack{j\in\{0,\dots,r\}\\ i_j\geq 1}} P(W_{\widetilde{D_d}}/W_{\widetilde{D_{p_j\cdot d}}})$$

$$2P(W_{D_d}/W_{D_{2d}}) \textit{ is isogenous to } P(W_{r^{\frac{n}{d}}}/W_{r^{\frac{n}{2d}}})$$

and

$$P(W_{D_{\frac{n}{2}}}/W_{D_n}) \times P(W_{\widetilde{D_{\frac{n}{2}}}}/W_{D_n}) \textit{ is isogenous to } P(W_{r^2}/W_r)$$

Proof. If $d = 2^\alpha c = \dfrac{n}{p_1^{i_1}\cdot \ldots \cdot p_r^{i_r}}$ is a divisor of n of the form multiple of 2^α and different from n then

$$\rho_{D_d} = \chi_0 + \sum_{\substack{k\in\Omega(n), k\neq n\\ k\cdot p_1^{i_1}\cdot\ldots\cdot p_r^{i_r}\equiv 0(n)}} \mathcal{W}_k$$

and (if $i_1 \geq 1$)

$$\rho_{D_{p_1\cdot d}} = \chi_0 + \sum_{\substack{\ell\in\Omega(n), \ell\neq n\\ \ell\cdot p_1^{i_1-1}\cdot\ldots\cdot p_r^{i_r}\equiv 0(n)}} \mathcal{W}_\ell$$

For any other $d = \dfrac{n}{2^{i_0}\cdot p_1^{i_1}\cdot \ldots \cdot p_r^{i_r}}$ divisor of n different from n and from $n/2$ and not a multiple of 2^α we have

$$\rho_{D_d} = \chi_0 + \chi_2 + \sum_{\substack{k\in\Omega(n), k\neq n\\ k\cdot 2^{i_0}\cdot p_1^{i_1}\cdot\ldots\cdot p_r^{i_r}\equiv 0(n)}} \mathcal{W}_k$$

and (if $i_1 \geq 1$)

$$\rho_{D_{p_1\cdot d}} = \chi_0 + \chi_2 + \sum_{\substack{\ell\in\Omega(n), \ell\neq n\\ \ell\cdot 2^{i_0}\cdot p_1^{i_1-1}\cdot\ldots\cdot p_r^{i_r}\equiv 0(n)}} \mathcal{W}_\ell \ .$$

Furthermore, if $2d$ is not a multiple of 2^α

$$\rho_{D_{2\cdot d}} = \chi_0 + \chi_2 + \sum_{\substack{\ell\in\Omega(n),\ell\neq n \\ \ell 2^{i_0-1}\cdot\cdot p_1^{i_1}\cdot\ldots\cdot p_r^{i_r}\equiv 0(n)}} \mathcal{W}_\ell$$

and if $2d$ is a multiple of 2^α

$$\rho_{D_{2\cdot d}} = \chi_0 + \sum_{\substack{\ell\in\Omega(n),\ell\neq n \\ \ell 2^{i_0-1}\cdot\cdot p_1^{i_1}\cdot\ldots\cdot p_r^{i_r}\equiv 0(n)}} \mathcal{W}_\ell$$

Hence, if $d = \dfrac{n}{p_1^{i_1}\cdot\ldots\cdot p_r^{i_r}}$ is a divisor of n and a multiple of 2^α, different from n and with $1 \leq i_1 < \alpha_1$ then

$$\rho_{D_d} - \rho_{D_{p_1\cdot d}} = \sum_{\substack{j\in\{2,\ldots,r\} \\ s_j\in\{0,\ldots,i_j\}}} \mathcal{W}_{d\cdot p_2^{s_2}\cdot\ldots\cdot p_r^{s_r}} \tag{7.14}$$

and therefore (by Corollary 2.2)

$$P(W_{D_d}/W_{D_{p_1\cdot d}}) \sim \prod_{\substack{j\in\{2,\ldots,r\} \\ s_j\in\{0,\ldots,i_j\}}} B_{d\cdot p_2^{s_2}\cdot\ldots\cdot p_r^{s_r}}$$

For any other $d = \dfrac{n}{2^{i_0}\cdot p_1^{i_1}\cdot\ldots\cdot p_r^{i_r}}$ divisor of n, not a multiple of 2^α and different from n and from $n/2$ we have

$$\rho_{D_d} - \rho_{D_{p_1\cdot d}} = \sum_{\substack{j\in\{2,\ldots,r\} \\ s_j\in\{0,\ldots,i_j\}}} \mathcal{W}_{d\cdot p_2^{s_2}\cdot\ldots\cdot p_r^{s_r}} \,. \tag{7.15}$$

Furthermore, if $2d$ is not a multiple of 2^α then

$$\rho_{D_d} - \rho_{D_{2\cdot d}} = \sum_{\substack{j\in\{2,\ldots,r\} \\ s_j\in\{0,\ldots,i_j\}}} \mathcal{W}_{d\cdot p_2^{s_2}\cdot\ldots\cdot p_r^{s_r}} \tag{7.16}$$

and if $2d$ is a multiple of 2^α then

$$\rho_{D_d} - \rho_{D_{2\cdot d}} = \chi_2 + \sum_{\substack{j\in\{2,\ldots,r\} \\ s_j\in\{0,\ldots,i_j\}}} \mathcal{W}_{d\cdot p_2^{s_2}\cdot\ldots\cdot p_r^{s_r}} \tag{7.17}$$

and therefore (by Corollary 2.2)

$$P(W_{D_d}/W_{D_{p_1\cdot d}}) \sim \prod_{\substack{j\in\{2,\dots,r\}\\ s_j\in\{0,\dots,i_j\}}} B_{d\cdot p_2^{s_2}\cdot\ldots\cdot p_r^{s_r}}$$

and

$$P(W_{D_d}/W_{D_{2\cdot d}}) \sim \prod_{\substack{j\in\{2,\dots,r\}\\ s_j\in\{0,\dots,i_j\}}} B_{d\cdot p_2^{s_2}\cdot\ldots\cdot p_r^{s_r}}$$

□

7.1. **The case $n = 2p$, with p an odd prime.** In this case we can be more precise.

There are two conjugacy classes of involutions of the form $r^j s$: one for j even, say $j = 0$, and the other for j odd, say $j = p$.

Also, there are two non conjugate dihedral groups of order $2p$ inside D_{2p}: one generated by r^2 and s and the other by r^2 and $r^p s$. They will be denoted by D_p and $\widetilde{D_p}$ respectively. Note that we may apply the results of section 5.1 to both of these dihedral groups.

There is one conjugacy class of Klein (or D_2) subgroups of D_{2p}, of the form $K_j = \langle r^j s, r^p\rangle = \{1, r^j s, r^p, r^{p+j} s\}$ for $j \in \{0, \dots p-1\}$. The quotient curve W/K_0 will we denoted by W_{D_2}.

Since p is an odd prime, in this case the complete diagram of curves and covers is the following:

(7.18)
$$\begin{tikzcd}
 & W \arrow[rrr, "f_{r^p s}"] \arrow[dl, "f_{r^2}"'] \arrow[ddr, "f_{r^p}"] \arrow[dddd, "f_s"] & & & W_{r^p s} \arrow[dl, "f_{r^p s,\widetilde{D_p}}"] \arrow[dddd, "f_{r^p s,D_2}"] \\
W_{r^2} \arrow[rrr, "f_{r^2,\widetilde{D_p}}"] \arrow[ddr, "f_{r^2,r}"] \arrow[dddd, "f_{r^2,D_p}"'] & & & W_{\widetilde{D_p}} \arrow[dddd, "f_{\widetilde{D_p},D_{2p}}"] & \\
 & & W_{r^p} \arrow[dl] \arrow[ddrr, "f_{r^p,D_2}"] & & \\
 & W_r \arrow[ddrr, "f_{r,D_{2p}}"] & & & \\
 & W_s \arrow[rrr, "f_{s,D_2}"] \arrow[dl, "f_{s,D_p}"] & & & W_{D_2} \arrow[dl, "f_{D_2,D_{2p}}"] \\
W_{D_p} \arrow[rrr, "f_{D_p,D_{2p}}"'] & & & W_{D_{2p}} &
\end{tikzcd}$$

The conditions on the numbers of fixed points are the following: $2\beta_1$ for any involution conjugate to s, $2\beta_2$ for any involution conjugate to rs, 2κ for r, 4γ for the number of points fixed by r^2 but not by r, $2p\delta$ for the number of points fixed by r^p but not by r, where $\beta_1 + \beta_2$,

$\beta_1+\delta+\kappa$ and $\beta_2+\delta+\kappa$ are all divisible by 2; i.e., β_1, β_2 and $\delta+\kappa$ have the same parity.

If we let $g_T = g$, then we have

genus	order of ramification $\|B\|$
$g_W = 4pg+1-4p+p(\beta_1+\beta_2+\delta)+ +(2p-1)\kappa+2(p-1)\gamma$	
$g_{W_s} = 2pg+1-2p+p\,\frac{\beta_2+\delta}{2}+ +\frac{p-1}{2}(\beta_1+2\gamma)+\frac{2p-1}{2}\kappa$	$\|B(W\to W_s)\| = 2\beta_1$
$g_{W_{r^ps}} = 2pg+1-2p+p\,\frac{\beta_1+\delta}{2}+ +\frac{p-1}{2}(\beta_2+2\gamma)+\frac{2p-1}{2}\kappa$	$\|B(W\to W_{r^ps})\| = 2\beta_2$
$g_{W_{r^2}} = 4g-3+\beta_1+\beta_2+\kappa+\delta$	$\|B(W\to W_{r^2})\| = 2(p-1)(2\gamma+\kappa)$
$g_{W_{r^p}} = 2pg+1-2p+p\,\frac{\beta_1+\beta_2}{2}+ +(p-1)(\kappa+\gamma)$	$\|B(W\to W_{r^p})\| = 2p\delta+2\kappa$
$g_{W_{D_p}} = 2g-1+\frac{\beta_2+\kappa+\delta}{2}$	$\|B(W_s\to W_{D_p})\| = (p-1)(\beta_1+2\gamma+\kappa)$ $\|B(W_{r^2}\to W_{D_p})\| = 2\beta_1$
$g_{W_{\widetilde{D_p}}} = 2g-1+\frac{\beta_1+\kappa+\delta}{2}$	$\|B(W_{r^ps}\to W_{\widetilde{D_p}})\| = (p-1)(\beta_2+2\gamma+\kappa)$ $\|B(W_{r^2}\to W_{\widetilde{D_p}})\| = 2\beta_2$
$g_{W_{D_2}} = pg+\frac{p-1}{2}\left(-2+\kappa+\gamma+ +\frac{\beta_1+\beta_2}{2}\right)$	$\|B(W_{r^ps}\to W_0)\| = \beta_1+p\delta+\kappa$ $\|B(W_s\to W_0)\| = \beta_2+p\delta+\kappa$ $\|B(W_{r^p}\to W_0)\| = \beta_1+\beta_2$
$g_{W_r} = 2g-1+\frac{\beta_1+\beta_2}{2}$	$\|B(W_{r^2}\to W_r)\| = 2\delta+2\kappa$ $\|B(W_{r^p}\to W_r)\| = 2(p-1)(\kappa+\gamma)$
	$\|B(W_{D_p}\to W_{D_{2p}})\| = \beta_2+\kappa+\delta$ $\|B(W_r\to W_{D_{2p}})\| = \beta_1+\beta_2$ $\|B(W_{\widetilde{D_p}}\to W_{D_{2p}})\| = \beta_1+\kappa+\delta$ $\|B(W_0\to W_{D_{2p}})\| = (p-1)\left(\frac{\beta_1+\beta_2}{2}+\beta+\gamma\right)$

Remark 7.2. It follows from Theorem 3.2 that there are 6 rational irreducible representations of D_{2p}: χ_0, χ_1, χ_2, χ_3, $\mathcal{W}_1 = \oplus_{i=1}^{(p-1)/2} \mathcal{V}_{2i-1}$ y $\mathcal{W}_2 = \oplus_{i=1}^{(p-1)/2} \mathcal{V}_{2i}$.

In this case, Theorem 7.1 may be stated as follows.

Theorem 7.3. *Let $n = 2p$ with p an odd prime and let W be a curve with D_{2p} action.*

Furthermore, let

$$B_1 = P(W_s/W_{D_p}) \cap P(W_s/W_{D_2}) \ .$$

Then

$$\begin{aligned} JW \sim JW_{D_n} \times P(W_r/W_{D_{2p}}) \times P(W_{D_p}/W_{D_{2p}}) \times P(W_{\widetilde{D_p}}/W_{D_{2p}}) \\ \times 2B_1 \times 2P(W_{D_2}/W_{D_{2p}}) \end{aligned}$$

with respective actions

$$\chi_0 \times \chi_1 \times \chi_2 \times \chi_3 \times \mathcal{W}_1 \times \mathcal{W}_2 \ .$$

Furthermore,

$$B_1XXX \text{ is isogenous to } P(W_{r^ps}/W_{\widetilde{D_p}}) \cap P(W_{r^ps}/W_{D_2})$$

$$2P(W_{D_1}/W_{D_2}) \text{ is isogenous to } P(W/W_{r^p})$$

and

$$P(W_{D_p}/W_{D_{2p}}) \times P(W_{\widetilde{D_p}}/W_{D_{2p}}) \text{ is isogenous to } P(W_{r^2}/W_r) \ .$$

But in this case we can obtain more information.

Proposition 7.4. *By the theory of D_p developed in Section 5.1, there are natural isogenies as follows. (XXXX general even?)*

$$\begin{aligned} 2P(W_s/W_{D_p}) &\to P(W/W_{r^2}) \\ 2P(W_{r^p s}/W_{\widetilde{D_p}}) &\to P(W/W_{r^2}) \\ 2P(W_{D_2}/W_{D_{2p}}) &\to P(W_{r^p}/W_r) \end{aligned}$$

Proposition 7.5. *By the theory of the Klein group developed in* [Rec–Ro2], *there are natural isogenies as follows. (XXXX general even?)*

$$\begin{aligned} P(W_{D_p}/W_{D_{2p}}) \times P(W_{\widetilde{D_p}}/W_{D_{2p}}) &\to P(W_{r^2}/W_r) \\ P(W_r/W_{D_{2p}}) \times P(W_{\widetilde{D_p}}/W_{D_{2p}}) &\to P(W_{r^2}/W_{D_p}) \\ P(W_{D_p}/W_{D_{2p}}) \times P(W_r/W_{D_{2p}}) &\to P(W_{r^2}/W_{\widetilde{D_p}}) \end{aligned}$$

Furthermore, particular to the case D_{2p} is the following result.

Theorem 7.6. *(XXXX general even?)*
There is a natural isogeny

$$h = (\mathrm{Nm}\, f_{rs} \circ f_s^*)|_{P(W_s/W_{D_p})} : P(W_s/W_{D_p}) \to P(W_{rs}/W_{\widetilde{D_p}})$$

whose kernel has cardinality given as follows.

(7.19)

$$|\ker(h)| = \begin{cases} p^{2g(4p-1)-6p+3+4p\beta_1+2p\beta_2+2p\delta+(6p-2)\kappa+(8p-3)\gamma} & \text{if } \gamma > 1 \text{ and } \kappa \geq 2; \\ p^{2g(4p-1)-6p+2+4p\beta_1+2p\beta_2+2p\delta+(6p-2)\kappa+(8p-2)\gamma} & \text{if } \gamma \leq 1 \text{ and } \kappa \geq 2; \\ p^{2g(4p-1)-6p+2+4p\beta_1+2p\beta_2+2p\delta+(6p-1)\kappa+(8p-3)\gamma} & \text{if } \gamma > 1 \text{ and } \kappa \leq 1; \\ p^{2g(4p-1)-2p+2+4p\beta_1+2p\beta_2+2p\delta} & \text{if } \gamma = 0 \text{ and } \kappa = 1; \\ p^{2g(4p-1)-8p+2+4p\beta_1+2p\beta_2+2p\delta} & \text{if } \kappa = 0 \text{ and} \\ & \qquad \gamma = 0 \text{ or } 1. \end{cases}$$

Proof. Denote by

$$h = (\mathrm{Nm}\, f_{rs} \circ f_s^*)|_{P(W_s/W_{D_p})} : P(W_s/W_{D_p}) \to P(W_{rs}/W_{\widetilde{D_p}})$$

and consider the following commutative diagram

(7.20)

$$\begin{tikzcd} & A \arrow[rr, "1+rs"] \arrow[dr, "\mathrm{Nm}\, f_{rs}"] & & B \\ P(W_s/W_{D_p}) \arrow[ur, "f_s^*"] \arrow[rr, "h"'] & & P(W_{rs}/W_{\widetilde{D_p}}) \arrow[ur, "f_{rs}^*"'] & \end{tikzcd}$$

where (see [Rec–Ro2])

$$A = \{z \in JW^{\langle s \rangle} : (1 + r^2 + r^4 + \ldots + r^{2(p-1)})(z) = 0\}^0$$

and

$$B = \{w \in JW^{\langle rs \rangle} : (1 + r^2 + r^4 + \ldots + r^{2(p-1)})(w) = 0\}^0 \,. >$$

Note that

$$(1 + rs)_{|A} = (1 + r)_{|A} \,.$$

Consider the complex isotypical decomposition

$$T_0 JW = \chi_{\mathbf{0}}^{\mathbf{m_0}} \oplus \chi_{\mathbf{1}}^{\mathbf{m_1}} \oplus \chi_{\mathbf{2}}^{\mathbf{m_2}} \oplus \chi_{\mathbf{3}}^{\mathbf{m_3}} \oplus \mathbf{V_1} \oplus \ldots \oplus \mathbf{V_{p-1}}$$

where $\mathbf{V_j} = \mathcal{V}_j{}^{n_j}$ for each complex irreducible representation $\mathcal{V}_j$.
Comparing it with the respective actions one obtains that

$$T_0 A = (df_s^*)_0 (T_0 P(W_s/W_{D_p})) = \bigoplus_{i=1}^{p-1} \mathbf{V_i}^{\langle s \rangle}$$

and

$$T_0B = (df_{rs}^*)_0(T_0P(W_{rs}/W_{\widetilde{D_p}})) = \bigoplus_{i=1}^{p-1} \mathbf{V_i}^{\langle rs\rangle}$$

Furthermore, it is clear that we have

$$(1+rs)(\mathcal{V}_i^{\langle s\rangle}) = \mathcal{V}_i^{\langle rs\rangle} \quad \text{for each } i\,. \tag{7.21}$$

and then it follows that

$$(1+rs)(\mathbf{V_1}^{\langle s\rangle} \oplus \ldots \oplus \mathbf{V_{p-1}}^{\langle s\rangle}) = \mathbf{V_1}^{\langle rs\rangle} \oplus \ldots \oplus \mathbf{V_{p-1}}^{\langle rs\rangle}\,;$$

Therefore h is an isogeny as claimed.

To compute the kernel of h we first look at

$$\ker(1+r)_{|A} = \{z \in A : z = -r(z)\}$$

Claim 7.7. $\ker(1+r)_{|A} \subseteq A[p]$

Proof of the claim: If $z \in \ker(1+r)_{|A}$ then

$$0 = (1 + r^2 + r^4 + \ldots + r^{2(p-1)})(z) = pz$$

where the last equality holds since $r^2(z) = z$. □

We know that

$$f_s^{*-1}(A[p]) = P(W_s/W_{D_p})[p] + \ker f_s^*$$

and

$$\ker f_s^* \subseteq JW_s[2]\,;$$

since p and 2 are relatively prime, it follows that

$$f_s^{*-1}(A[p]) = P(W_s/W_{D_p})[p]$$

and therefore

$$\ker h \subseteq P(W_s/W_{D_p})[p]$$

Similarly, $\ker f_{rs}^* \subseteq JW_{rs}[2]$ implies that

$$\ker(f_{rs}^* \circ h) = \ker h$$

Therefore we have proven that

$$\ker h = \{x \in P(W_s/W_{D_p})[p] : rf_s^*(x) + f_s^*(x) = 0\} \approx K/L \tag{7.22}$$

where

$$K = \{x \in P(W_s/W_{D_p})[p] : r^2 f_s^*(x) = f_s^*(x)\}$$

and

$$L = \{x \in P(W_s/W_{D_p})[p] : rf_s^*(x) = f_s^*(x)\}\,.$$

We may apply the results of Theorem 5.4 to the subgroup D_p (dihedral of order $2p$, with p odd) of D_{2p}, with $\alpha = \kappa + 2\gamma$ and $\beta = \beta_1$, to conclude the following.

$$K = \begin{cases} f^*_{s,D_p} JW_{D_p}[p] & \text{if } \alpha = 0 \text{ or } 1; \\ f^*_{s,D_p} JW_{D_p}[p] \oplus \oplus_{j=2}^{\alpha} \mathcal{F}_j \mathbb{Z}/p\mathbb{Z} & \text{if } \alpha \geq 2. \end{cases}$$

Therefore,

$$|K| = \begin{cases} p^{4pg_{W_{D_p}}+4p\beta+(4p-2)\alpha+2-4p} & \text{if } \alpha = 0 \text{ or } 1 \\ p^{4pg_{W_{D_p}}+4p\beta+(4p-1)\alpha+1-2p} & \text{if } \alpha \geq 2 \end{cases}$$

We will now compute

$$\begin{aligned} L &= \{x \in P(W_s/W_{D_p})[p] : r f_s^*(x) = f_s^*(x)\} \\ &= \{x \in P(W_s/W_{D_p})[p] : f_s^*(x) \text{ is } D_{2p}\text{– invariant}\} \end{aligned}$$

First observe that

$$f^*_{s,D_p} : JW_{D_p} \to JW_s$$

is injective, since the p-fold cover $f_{s,D_p} : W_s \to W_{D_p}$ is not a Galois cover. Hence $\ker \operatorname{Nm} f_{s,D_p}$ has only one connected component; i.e.,

$$P(W_s/W_{D_p}) = \ker(\operatorname{Nm} f_{s,D_p})$$

On the other hand, r descends to an involution

$$\widetilde{r} : W_{D_p} \to W_{D_p}$$

whose quotient gives the morphism

$$f_{D_p,D_{2p}} : W_{D_p} \to W_{D_{2p}}$$

and therefore

$$(f_s^* \circ f^*_{s,D_p}) JW_{D_p}^{\langle \widetilde{r} \rangle} \subseteq JW^{D_{2p}}$$

On the other hand, since $\operatorname{Nm} f_{s,D_p} \circ f^*_{s,D_p} = p_{JW_{D_p}}$, we have that

$$f^*_{s,D_p}(JW_{D_p}^{\langle \widetilde{r} \rangle}[p]) \subseteq \ker(\operatorname{Nm} f_{s,D_p}) = P(W_s/W_{D_p})\,.$$

Therefore

$$L = f^*_{s,D_p}(JW_{D_p}^{\langle \widetilde{r} \rangle}[p]) + \text{ part coming from the ramification of } f_{s,D_p}$$

Since p is an odd prime, it follows from Corollary 3.2 in [Rec–Ro2] that

$$JW_{D_p}^{\langle \widetilde{r} \rangle}[p] = f^*_{D_p,D_{2p}}(JW_{D_{2p}}[p])$$

To compute the part coming from the ramification, note that we obtain p–torsion points from the ramification of $f_{s,D_p} : W_s \to W_{D_p}$ only from points of type κ and γ.

So let $P_1, \ldots, P_\kappa$ denote the κ points in W_s (i.e., each P_i is a total ramification point of f_{s,D_p} such that $\widetilde{r} f_{s,D_p}(P_i) = f_{s,D_p}(P_i)$) and let $Q_1, \dots, Q_{2\gamma}$ denote the γ points in W_s (i.e., Q_j is a total ramification point of f_{s,D_p} such that $\widetilde{r} f_{s,D_p}(Q_j) = f_{s,D_p}(Q_{j+1})$ for each $j = 1, 3, \dots, 2\gamma - 1$).

For each $i = 2, 3, \dots, \kappa$ and for each $j = 3, 5, \dots, 2\gamma - 1$ choose m_i and n_j in $JW_{D_p}^{\langle \widetilde{r} \rangle}$ such that

$$m_i^{\otimes p} = \mathcal{O}_{W_{D_p}}(f_{s,D_p}(P_1) - f_{s,D_p}(P_i))$$

and

$$n_j^{\otimes p} = \mathcal{O}_{W_{D_p}}(f_{s,D_p}(Q_1) + f_{s,D_p}(Q_2) - f_{s,D_p}(Q_j) - f_{s,D_p}(Q_{j+1}))$$

and define

$$\mathcal{F}_i = \mathcal{O}_{W_s}(P_i - P_1) \otimes f_{s,D_p}^*(m_i) \qquad i = 2, 3, \dots, \kappa$$

and

$$\mathcal{G}_j = \mathcal{O}_{W_s}(Q_j + Q_{j+1} - Q_1 - Q_2) \otimes f_{s,D_p}^*(n_j) \quad j = 3, 5, \dots, 2\gamma - 1$$

Then

$$L = f_{s,D_p}^* \circ f_{D_p,D_{2p}}^*(JW_{D_{2p}}[p]) \oplus \oplus_{i=2}^{\kappa} \mathcal{F}_i \, \mathbb{Z}/p\mathbb{Z} \oplus \oplus_{j=1}^{\gamma-1} \mathcal{G}_{2j+1} \, \mathbb{Z}/p\mathbb{Z}$$

Observe that there is no relation among the $\mathcal{F}_i$'s and $\mathcal{G}_j$'s because we have already taken away some of the elements of $P(W_s/W_{D_p})[p]$ coming from the ramification.

It follows that, if we let g denote the genus of $W_{D_{2p}}$,

$$|L| = \begin{cases} p^{2g-2+\kappa+\gamma} & \text{if } \gamma > 1 \text{ and } \kappa \geq 2; \\ p^{2g-1+\kappa} & \text{if } \gamma \leq 1 \text{ and } \kappa \geq 2; \\ p^{2g-1+\gamma} & \text{if } \gamma > 1 \text{ and } \kappa \leq 1; \\ p^{2g} & \text{if } \gamma \leq 1 \text{ and } \kappa \leq 1. \end{cases} \tag{7.23}$$

□

Remark 7.8. Note that B_1 is the orthogonal complement in $P(W_s/W_{D_2})$ of $f_{s,D_p}^*(P(W_{D_p}/W_{D_{2p}}))$ with respect to the polarization induced from the canonical one on JW_s.

This subvariety may be described as follows.

$$f_s^*(B_1) = \{w \in JW^{\langle s \rangle} : (1+r)(w) = 0, \sum_{j=0}^{p-1} r^{2j}(w) = 0\}^0$$

Even though it is not a Prym variety of some intermediate cover, in the sense of being the orthogonal complement of a Jacobian inside another Jacobian, it is an analog, in the sense of being the orthogonal complement of a Prym inside another Prym.

Appendix A.

In this section we prove Proposition 5.2.

Proof. Since (see [Rec–Ro2])

$$P(W/W_r) = \{w \in JW : \left(\sum_{j=0}^{p-1} r^j\right) w = 0\}^0$$

and

$$f_k^*(P(W_k/W_{D_p})) = \{z \in JW^{\langle k\rangle} : \left(\sum_{j=0}^{p-1} r^j\right) z = 0\}^0$$

it is clear that both $A = f_k^*(P(W_k/W_{D_p}))$ and $r(A)$ are contained in $P(W/W_r)$ and therefore

$$\varphi_k(P(W_k/W_{D_p}) \times P(W_k/W_{D_p})) \subseteq P(W/W_r)\,.$$

Since $f_k^*|_{P(W_k/W_{D_p})} : P(W_k/W_{D_p}) \to A$ is an isomorphism (by Corollary 4.4), in order to describe the kernel of φ_k it is enough to study the kernel of

$$\begin{aligned} A \times A &\longrightarrow P(W/W_r) \\ (x_1, x_2) &\dashrightarrow x_1 + r(x_2) \end{aligned}$$

and we do this next.

Given $(x_1, x_2) \in A \times A$ such that

$$x_1 + r(x_2) = 0 \tag{1.24}$$

and using that both x_1 and x_2 are fixed under k, we obtain

$$0 = k(x_1 + r(x_2)) = x_1 + kr(x_2) = x_1 + r^{p-1}(x_2)\,. \tag{1.25}$$

Combining (1.24) and (1.25) it follows that

$$r^2(x_1) = x_1$$

That is, x_1 is invariant under $\langle r^2\rangle = \langle r\rangle$. But then x_1 in $A \subseteq P(W/W_r)$ implies that $0 = (1 + r + \ldots + r^{p-1})(x_1) = p\,x_1$; i.e., x_1 is in $A[p]$ and it is invariant under D_p; furthermore, $x_2 = -r^{p-1}(x_1) = -x_1$. Thus (5.8) and (5.9) are proved.

To prove i) note that f_{k,D_p}^* is injective, by Proposition 4.2 i), hence $\ker \mathrm{Nm}\, f_{k,D_p}$ has only one connected component (see [Rec–Ro2]) and $P(W_k/W_{D_p}) = \ker \mathrm{Nm}\, f_{k,D_p}$. Now $(f_k^* \circ f_{k,D_p}^*)(x)$ is D_p-invariant for every $x \in JW_{D_p}$ and therefore $f_{k,D_p}^*(JW_{D_p}[p])$ is contained in K, since $\mathrm{Nm}\, f_{k,D_p} \circ f_{k,D_p}^* = p_{JW_{D_p}}$ is multiplication by p in JW_{D_p}.

We consider first the case $\alpha = 0$; we will show that then $K = f_{k,D_p}^*(JW_{D_p}[p])$.

Since the Galois cover $W \xrightarrow{f_r} W_r$ is étale, it follows that $JW^{\langle r\rangle} = f_r^*(JW_r)$ or, equivalently,

$$\ker(1-r) = \operatorname{Im}\left(\sum_{j=0}^{p-1} r^j\right)$$

If we observe that

$$\begin{aligned} K \cong f_k^*(K) &= f_k^*(P(W_k/W_{D_p})) \cap \ker(1-r) \\ &= f_k^*(P(W_k/W_{D_p})) \cap \operatorname{Im}\left(\sum_{j=0}^{p-1} r^j\right) \end{aligned}$$

then we obtain that

$$\text{for any } z \in K \text{ there exists } w \in JW \text{ such that } f_k^*(z) = \left(\sum_{j=0}^{p-1} r^j\right)(w)\,.$$

Now let $x = \operatorname{Nm}\gamma(w)$, where $\gamma = f_{k,D_p} \circ f_k$. From $\operatorname{Nm}\gamma(w) = \operatorname{Nm}\gamma(g(w))$ for any $g \in D_p$, we obtain

$$\begin{aligned} p\,x = p\,\operatorname{Nm}\gamma(w) &= \operatorname{Nm}\gamma\left(\sum_{j=0}^{p-1} r^j(w)\right) = \operatorname{Nm}\gamma(f_k^*(z)) \\ &= \operatorname{Nm} f_{k,D_p} \circ \operatorname{Nm} f_k(f_k^*(z)) = \operatorname{Nm} f_{k,D_p}(2z) = 2\operatorname{Nm} f_{k,D_p}(z) = 0 \end{aligned}$$

and hence x is in $JW_{D_p}[p]$.

Therefore $f_{k,D_p}^*\left(\frac{p-1}{2}\,x\right)$ is in K and it follows that

$$z + f_{k,D_p}^*\left(\frac{p-1}{2}\,x\right) \text{ is in } K\,.$$

But then

$$
\begin{aligned}
f_k^*\left(z + f_{k,D_p}^*\left(\frac{p-1}{2}\,x\right)\right) &= f_k^*(z) + \frac{p-1}{2}\, f_k^* \circ f_{k,D_p}^*(x) \\
&= f_k^*(z) + \frac{p-1}{2}\,\gamma^* \circ \operatorname{Nm}\gamma(w) \\
&= f_k^*(z) + \frac{p-1}{2} \sum_{g \in D_p} g(w) \\
&= f_k^*(z) + \frac{p-1}{2}\left(\sum_{j=0}^{p-1} r^j(w) + k\sum_{j=0}^{p-1} r^j(w)\right) \\
&= f_k^*(z) + \frac{p-1}{2}(f_k^*(z) + k f_k^*(z)) \\
&= f_k^*(z) + (p-1) f_k^*(z) \\
&= p\, f_k^*(z) \\
&= 0\,.
\end{aligned}
$$

Therefore

$$
z = -f_{k,D_p}^*\left(\frac{p-1}{2}\,x\right) \quad \text{is in } f_{k,D_p}^*(JW_{D_p})[p]
$$

and we have proven that $K = f_{k,D_p}^*(JW_{D_p})[p]$ for $\alpha = 0$.

Now assume $\alpha \geq 1$ and denote by $\{P_1, \dots , P_\alpha\} \subseteq W_k$ the set of total ramification points of $f_{k,D_p} : W_k \to W_{D_p}$. If we denote

$$
f_k^*(P_j) = P_j^{'} + P_j^{''}
$$

for $j \in \{1, \dots , \alpha\}$ with $P_j^{'}$ and $P_j^{''}$ in W, then $r(P_j^{'}) = P_j^{'}$, $r(P_j^{''}) = P_j^{''}$ and $k(P_j^{'}) = P_j^{''}$.

Let m be such that $2m \geq g_W$ (the genus of W) and consider the Abel-Jacobi map given as follows.

$$
\begin{aligned}
\varphi : W^{(2m)} &\to \operatorname{Pic}^{2m} W \to \operatorname{Pic}^0 W \\
D &\to \mathcal{O}_W(D) \to \mathcal{O}_W(D - m(P_1^{'} + P_1^{''}))
\end{aligned}
$$

Let $z \in K$. Then $f_k^*(z) \in \operatorname{Pic}^0 W$ and (since $2m \geq g_W$) we have that $\varphi^{-1}(f_k^*(z)) = |f_k^*(z)|$ is non empty.

Since $r(f_k^*(z)) = f_k^*(z)$, r induces a projective transformation on $|f_k^*(z)|$ which has a fixed point $\mathcal{D}$ in $|f_k^*(z)|$, because r generates a cyclic group (see [M1]).

As $\mathcal{D}$ is an r-invariant effective divisor, it must be of the form

$$\mathcal{D} = \sum_{j=1}^{\alpha}(n_j P_j^{'} + m_j P_j^{''}) + \sum_{t=1}^{T}(\sum_{j=1}^{p-1} r^j) w_t$$

for some points $w_t \in W$ and we also have

$$f_k^*(z) = \mathcal{O}_W(\mathcal{D} - m(P_1^{'} + P_1^{''}))\,.$$

But $k(f_k^*(z)) = f_k^*(z)$ and hence

$$f_k^*(2z) = \mathcal{O}_W\left(\sum_{j=1}^{\alpha} s_j(P_j^{'} + P_j^{''}) + \sum_{t=1}^{T}(\sum_{g\in D_p} g)w_t - 2m(P_1^{'} + P_1^{''})\right)\,.$$

Applying $\operatorname{Nm} f_k$ to the last equality we obtain

$$4z = \begin{cases} \mathcal{O}_{W_k}\Big(\sum_{j=2}^{\alpha} 2s_j(P_j - P_1) + \\ \qquad + \sum_{t=1}^{T}\Big[2f^*_{k,D_p}(\gamma(w_t)) - 2pP_1\Big]\Big) & \text{if } \alpha \geq 2 \\ \mathcal{O}_{W_k}\Big(\sum_{t=1}^{T}\Big[2f^*_{k,D_p}(\gamma(w_t)) - 2pP_1\Big]\Big) & \text{if } \alpha = 1 \end{cases}$$

which we may rewrite as follows

$$4z = \begin{cases} \mathcal{O}_{W_k}\Big(\sum_{j=2}^{\alpha} b_j(P_j - P_1)\Big) \otimes f^*_{k,D_p}(\mathcal{L}) & \text{if } \alpha \geq 2 \\ f^*_{k,D_p}(\mathcal{L}) & \text{if } \alpha = 1 \end{cases}$$

with $0 \leq b_j \leq p-1$ and $\mathcal{L} \in \operatorname{Pic}^0 W_{D_p}$.

Since $4z$ is a point of order p and since f^*_{k,D_p} is injective, from the last expression for $4z$ we obtain that

$$\mathcal{L}^p = \begin{cases} \mathcal{O}_{W_{D_p}}\Big(\sum_{j=2}^{\alpha} b_j(f_{k,D_p}(P_1) - f_{k,D_p}(P_j))\Big) & \text{if } \alpha \geq 2 \\ \mathcal{O}_{W_{D_p}} & \text{if } \alpha = 1 \end{cases}$$

If $\alpha = 1$ we have just shown that $K = f^*_{k,D_p}(JW_{D_p})[p]$.

If $\alpha \geq 2$ then we have

$$4z \otimes \mathcal{F}_2^{-b_2} \otimes \ldots \otimes \mathcal{F}_\alpha^{-b_\alpha} = f^*_{k,D_p}(\mathcal{L} \otimes m_2^{-b_2} \otimes \ldots \otimes m_\alpha^{-b_\alpha})$$

with

$$(\mathcal{L} \otimes m_2^{-b_2} \otimes \ldots \otimes m_\alpha^{-b_\alpha})^p = \mathcal{O}_{W_{D_p}}\,.$$

Hence $4z = \mathcal{F}_2^{b_2} \otimes \ldots \otimes \mathcal{F}_\alpha^{b_\alpha} \otimes f^*_{k,D_p}(t)$, with $t \in \operatorname{Pic}^0 W_{D_p}$ such that $t^p = \mathcal{O}_{W_{D_p}}$; also, any multiple of $4z$ has a similar expression, including z, since the subgroups of K generated by $4z$ and by z coincide.

We now prove that the b_j are unique.

Assume there is a non-trivial combination of the $\mathcal{F}_j$'s with coefficients in $\mathbb{Z}/p\mathbb{Z}$ that is trivial modulo $f^*_{k,D_p}(JW_{D_p})[p]$.

Then, after reordering the P_i's if needed, we may assume there is $t \in \operatorname{Pic}^0 W_{D_p}$ such that $t^{\otimes p} = \mathcal{O}_{W_{D_p}}$ and such that

$$\otimes_{i=1+s}^{s_1} \mathcal{F}_i \otimes_{i=1+s_1}^{s_2} \mathcal{F}_i^{\otimes 2} \otimes \ldots \otimes_{i=1+s_{p-2}}^{s_{p-1}} \mathcal{F}_i^{\otimes(p-1)} = f_{k,D_p}^*(t)$$

It follows from (5.7) that then

$$\mathcal{O}_{W_k}\left(\sum_{j=s+1}^{s+s_1} P_j + 2\sum_{j=s+s_1+1}^{s+s_1+s_2} P_j + \ldots \right.$$
$$\left. +(p-1)\sum_{j=s+s_1+\ldots+s_{p-2}+1}^{s+s_1+\ldots+s_{p-2}+s_{p-1}} P_j - (s_1 + 2s_2 + \ldots + (p-1)s_{p-1})P_1\right)$$
$$= f_{k,D_p}^*\left(t \otimes \bigotimes_{j=s+1}^{s+s_1} (m_j)^{-1} \otimes \bigotimes_{j=s+s_1+1}^{s+s_1+s_2} (m_j)^{-\otimes 2} \otimes \ldots \right.$$
$$\left. \otimes \bigotimes_{j=s+s_1+\ldots+s_{p-2}+1}^{s+s_1+\ldots+s_{p-2}+s_{p-1}} (m_j)^{-\otimes(p-1)}\right)$$

Therefore, if we let

$$s_1 + 2s_2 + \ldots + (p-1)s_{p-1} = pm - \varepsilon\,, \text{ with } \varepsilon \in \{0, 1, \ldots, p-1\}$$

we obtain

$$\mathcal{O}_{W_k}\left(\sum_{j=s+1}^{s+s_1} P_j + 2\sum_{j=s+s_1+1}^{s+s_1+s_2} P_j + \ldots + (p-1)\sum_{j=s+s_1+\ldots+s_{p-2}+1}^{s+s_1+\ldots+s_{p-2}+s_{p-1}} P_j + \right.$$
$$\left. +\varepsilon P_1\right)$$
$$= f_{k,D_p}^*\left(\mathcal{O}_{W_{D_p}}(m f_{k,D_p}(P_1)) \otimes t \otimes \bigotimes_{j=s+1}^{s+s_1} (m_j)^{-1} \otimes \right.$$
$$\left. \bigotimes_{j=s+s_1+1}^{s+s_1+s_2} (m_j)^{-\otimes 2} \otimes \ldots \otimes \bigotimes_{j=s+s_1+\ldots+s_{p-2}+1}^{s+s_1+\ldots+s_{p-2}+s_{p-1}} (m_j)^{-\otimes(p-1)}\right)$$

Therefore, if we let

$$\mathcal{L} = \mathcal{O}_{W_{D_p}}(m f_{k,D_p}(P_1)) \otimes t \otimes \bigotimes_{j=s+1}^{s+s_1} (m_j)^{-1} \otimes \bigotimes_{j=s+s_1+1}^{s+s_1+s_2} (m_j)^{-\otimes 2} \otimes \ldots$$
$$\ldots \otimes \bigotimes_{j=s+s_1+\ldots+s_{p-2}+1}^{s+s_1+\ldots+s_{p-2}+s_{p-1}} (m_j)^{-\otimes(p-1)}$$

then $\mathcal{L} \in \mathrm{Pic}^m(W_{D_p})$ and we have just seen that

$$f_{k,D_p}^*(\mathcal{L}) = \mathcal{O}_{W_k}(\sum_{j=s+1}^{s+s_1} P_j + 2\sum_{j=s+s_1+1}^{s+s_1+s_2} P_j + \dots$$
$$\dots + (p-1)\sum_{j=s+s_1+\dots+s_{p-2}+1}^{s+s_1+\dots+s_{p-2}+s_{p-1}} P_j + \varepsilon P_1)$$

But we also have

$$\mathcal{L}^{\otimes p} = \mathcal{O}_{W_{D_p}}(\sum_{j=s+1}^{s+s_1} f_{k,D_p}(P_j) + 2\sum_{j=s+s_1+1}^{s+s_1+s_2} f_{k,D_p}(P_j) + \dots$$
$$\dots + (p-1)\sum_{j=s+s_1+\dots+s_{p-2}+1}^{s+s_1+\dots+s_{p-2}+s_{p-1}} f_{k,D_p}(P_j) + \varepsilon f_{k,D_p}(P_1))$$

These conditions on $\mathcal{L}$ imply that $f_{k,D_p} : W_k \to W_{D_p}$ is a Galois cover with group $\mathbb{Z}/p\mathbb{Z}$ (and branched over $\sum_{j=s+1}^{s+s_1} f_{k,D_p}(P_j) + 2\sum_{j=s+s_1+1}^{s+s_1+s_2} f_{k,D_p}(P_j) + \dots + \varepsilon f_{k,D_p}(P_1))$, hence a contradiction. $\square$

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