ASYMPTOTIC HOMOLOGY OF BROWNIAN MOTION ON A RIEMANNIAN MANIFOLD

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Abstract. We construct the analogue of Schwartzman's asymptotic cycle for Brownian motion on a Riemannian manifold, and we prove that it vanishes in real homology. We generalize Itô's construction of parallel transport along a Brownian path in a Riemannian manifold to a principal bundle with connection, and define the holonomy for Brownian motion and its expected value. We also define and prove the existence of the average linking number of the asymptotic cycle associated to Brownian motion in any homology 3-sphere in analogy with V. I. Arnold's definition of asymptotic linking number for flows.

1. Introduction

The purpose of this note is to construct the homological representative of a random path for any compact Riemannian manifold. We use the idea of Schwartzman [Sc], to construct the asymptotic homological object that corresponds to a random Brownian path thought of as a diffuse cycle. The construction is based on an approximation to Brownian motion by piecewise smooth curves, following Itô [I]. Of course, there are other possible ways to do this. For example, Pinsky [P] constructs Brownian motion by isotropic transport process, and this construction also gives a nice way to approximate Brownian motion by a piecewise smooth path. To understand better the interpretation of the asymptotic cycle associated with Brownian motion in the first homology group with real coefficients, we review the Eilenberg-Bruschlinsky homology theory as in [Sc]. Using this together with some results of harmonic maps, we prove that the asymptotic cycle associated with Brownian motion is trivial in homology, and compare it with a result of Elworthy for Brownian motion on a manifold of bounded negative curvature with ends. An interesting example is provided [VV1], where we compare it with results in harmonic maps for hyperbolic manifolds.

Additionally, given a principal bundle with fibre a compact Lie group on a Riemannian manifold, we construct the parallel transport along a random path, as in Itô [I], where

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he constructs it for the parallel transport of the Levi-Civita connection of a Riemannian manifold.

In the last section we extend the definition of V. I. Arnold ([A] and [AK]) of asymptotic linking number to the orbits of two Brownian paths on a Riemannian homology 3-sphere.

We remark that all the constructions depend on the choice of a Riemannian metric on the manifold, since the construction of Brownian motion depends on the latter.

For more geometrical properties of Brownian motion on a Riemannian manifold, and as an introduction to the topic, we refer to the beautiful survey by Kendall [K] on stochastic differential geometry.

Finally we want to thank Luis Gorostiza for helpful suggestions and references.

2. Brownian motion on a Riemannian manifold (a review).

The existence of Brownian motion is equivalent to the existence of Wiener measure on the space of continuous paths. We review the properties of Wiener measure. We first describe the Wiener measure on $\mathbb{R}^N$. We will denote the *Wiener measure* associated to the Brownian motion starting at $0 \in \mathbb{R}^N$ by W_0. It is a probability measure on

$$\mathcal{C}_0(\mathbb{R}^N) = \{\alpha : I \longrightarrow \mathbb{R}^N,\ \alpha(0) = 0,\ \alpha \textit{ continuous}\},$$

(where I is an interval starting at 0), constructed in the following way.

Let Δ be the Laplace operator on $\mathbb{R}^N$, then the *heat operator* $L = \Delta - \partial/\partial t$. It has a fundamental solution $h(x, y, t) = h_t(x, y)$ with the following properties:

(1) $h : (\mathbb{R}^N \times \mathbb{R}^N - Diag) \times \mathbb{R}^{(\geq 0)} \mapsto \mathbb{R}^{(\geq 0)}$ is smooth.
(2) $h(x, y, t) = h(y, x, t)$.
(3) $L_x h = 0 = L_y h$ where L_x is L operating on the first variable of L.
(4) The Chapman-Kolmogoroff identity

$$h_{t+s}(x, y) = \int_{\mathbb{R}^N} h_s(x, z) h_t(z, y) dz.$$

(5) $\int_{\mathbb{R}^N} h_t(x, y) dy = 1$ for all $x \in \mathbb{R}^N$ and $t > 0$.

The Borel σ-algebra is generated by the sets $\rho_\tau^{-1}(B) \subset \mathcal{C}_0(\mathbb{R}^N)$ where $\tau = (t_1, \dots, t_n)$ with $0 < t_1 < \cdots < t_n < 1$, and $\rho_\tau : \mathcal{C}_0(\mathbb{R}^N) \mapsto \mathbb{R}^N \times \cdots \times \mathbb{R}^N$ (n-copies of $\mathbb{R}^N$) is the evaluation map $\rho_\tau(\alpha) = (\alpha(t_1), \dots \alpha(t_n))$, and B is a Borel subset of $\mathbb{R}^N \times \cdots \times \mathbb{R}^N$ (n-times, where n is an arbitrarily changing integer). We have

$$W_0(\rho_\tau^{-1}(B)) = \int_B h_{t_1}(0, x_1) h_{t_2 - t_1}(x_1, x_2) \cdots h_{t_n - t_{n-1}}(x_{n-1}, x_n) dx_1 dx_2 \cdots dx_n$$

It is known that W_0 determines a Borel measure on $\mathcal{C}_0(\mathbb{R}^N)$ with $W_0(\mathcal{C}_0(\mathbb{R}^N)) = 1$, (a construction esentially due to Wiener, see [EE1] or [EE2] and the references therein)

If (M, g) is a compact Riemannian manifold, we can construct Brownian motion on M by "rolling" M along a Brownian path in its tangent space. This construction is formalized below. We remark that the existence of Brownian motion is equivalent to the existence of the Wiener measure on the space of continuous paths with a fixed starting point $\mathcal{C}_p = \{\alpha : I \mapsto M,\ \alpha(0) = p,\ \alpha \textit{ continous}\}$, and I is an interval starting at 0 (in all our constructions below the interval I will be $[0, \infty)$).

Assume that we have a compact Riemannian manifold (M, g), and we have Brownian motion $B_p(t)$ starting at a point p. Following Itô [I], this is constructed by solving a stochastic differential equation. We assume as given Brownian motion $\beta(t)$ on $\mathbb{R}^n$ starting at 0, identify the tangent space $T_p(M)$ isometrically with $\mathbb{R}^n$ using the Riemannian metric g_p on $T_p(M)$ and the usual Euclidean metric on $\mathbb{R}^n$. Then we obtain the Brownian motion on the manifold M starting at a point p by the *development* map at the point. Formally, one solves a stochastic differential equation as follows (see [I]). In local coordinates on M, we denote by g_{ij} the components of the Riemannian tensor, $g^{ik}g_{kj} = \delta^i_j$ and Γ^k_{ij} the Christoffel symbols of the Riemannian connection. Let

$$\Delta = \frac{1}{2} g^{ij} \nabla_i \nabla_j$$

$$= \frac{1}{2} g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} - \frac{1}{2} g^{ij} \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

be the Laplace-Beltrami operator.

In the same coordinates, we define σ^i_k and m^k by

$$\sum_k \sigma^i_k \sigma^j_k = g^{ij}$$

$$m^k = -\frac{1}{2} g^{ij} \Gamma^k_{ij}$$

and solve the stochastic differential equation:

$$dB^i(t) = \sigma^i_k(B(t)) d\beta^k(t) + m^i(B(t)) dt$$

with $B^i(0) = p$. We obtain the paths of Brownian motion starting at $p \in M$. (See the nice intuitive explanation in [S] or the more formal one in [K]).

Let $\mathcal{C} = \{\alpha : [0, \infty) \longrightarrow M,\ \ \alpha \textit{ continuous}\}$. Let $\Pi : \mathcal{C} \longrightarrow M$ be defined by $\Pi(\alpha) = \alpha(0)$. Then, as shown by [EE1], C is a Frechet manifold and Π is a differentiable locally trivial fibration with fibre $\mathcal{C}_p = \Pi^{-1}(p)$, for each $p \in M$.

Let us define the measure W on $\mathcal{C}$ as the measure whose restriction to a local trivialization $U \times \mathcal{C}_p \simeq \Pi^{-1}(U)$ is the volume measure of M restricted to the open set $U \subset M$ times W_p. More precisely, if $\mathcal{U} \in \mathcal{C}(M)$ is a Borel subset, we have

$$W(\mathcal{U}) = \int_M W_p[\mathcal{U} \cap \mathcal{C}_p(M)] dv_g(p),$$

where dv_g denotes the volume measure on M induced by the Riemannian metric g. In other words, W is a measure on $\mathcal{C}$ which desintegrates to $h_1 dv_g$, where again we have that $h_t(x,y) : M \times M \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is the fundamental solution of the heat equation. The measure W is the *Wiener measure* on $\mathcal{C}$.

If we define the *time shift* semigroup $F_s : \mathcal{C} \longrightarrow \mathcal{C},\ s \geq 0$, as $F_s(\alpha)(t) = \alpha(s+t)$, then this semigroup preserves the measure W. Therefore the triple $(\mathcal{C}, W, F_s)$ is a measure preserving non reversible dynamical system.

In this paper, we will follow Ito's construction [I]. We review the way he approximates the Brownian path by a piecewise smooth curve. Given a Brownian path $B(t)$ on M, consider a partition $\mathcal{P} = \{0 = t_0, t_1, t_2, \dots, t_n = T\}$ of the interval $[0,T]$. Join each pair of points $B(t_i)$, $B(t_{i+1})$ with the unique geodesic path joining these two points, obtaining a piecewise smooth curve which we denote by $B(t,\mathcal{P})$. Itô shows that as the mesh of the partition tends to zero, the piecewise curve obtained tends to the given Brownian path. We will use several times below this approximation. We will also consider from now on, Brownian motion on the interval $[0,\infty)$, even if in the constructions below, we only use for each path $B(t)$, the trajectory up to a finite time T

2.1 Remark. Since we will need an approximation of the Brownian curve by piecewise smooth curves, it is also instructive to describe Pinsky's method [P] to obtain a random path on a manifold using the so called *isotropic transport process*. The idea is as follows: we give a partition $\mathcal{P} = \{0 = t_0, t_1, \dots, t_n = T\}$ of the interval $[0,T]$ (n arbitrary). With initial point $\gamma(0)$, choose successively an arbitrary (random) direction in the unit sphere of each $T_{\gamma(t_i)}M$, with the uniform distribution in the unit sphere defined using the Riemannian metric in the corresponding tangent space. Then consider the geodesic path $\gamma : [t_i, t_{i+1}] \to M$ starting at the base point $\gamma(t_i)$ in the direction of the unit tangent vector chosen. Pinsky proves that when the manifold M is (i) compact or (ii) it has bounded negative curvature, the union of all these geodesics will define a piecewise smooth curve that converges to a random curve as the size of the partition $\mathcal{P}$ tends to zero at an appropriate rate. This is another way to construct Brownian motion on M.

3. The asymptotic cycle associated to Brownian motion

In this section we use the existence of Brownian motion on a Riemannian manifold to construct the homological cycle associated to it, and then we will give some applications to Riemannian geometry and harmonic maps.

We will assume we have chosen *a priori*, for every pair of points q_1 and q_2 of M, a collection of smooth *short curves* $\{\gamma_{q_1,q_2}\}$ of uniformly bounded length joining the points. Such a collection of curves exists, and depends in a measurable way of q_1 and q_2. (See Arnold and Khesin [AK], page 145. We give a more precise definition of a collection of short curves in section 4 below). Our construction will be independent of the choice of this collection of short curves.

Let us define the following map:

$$[\ \cdot\ ] : \mathcal{C}_p \times \mathbb{R}^+ \to H_1(M, \mathbb{R})$$

by

$$(B,T) \mapsto [\Gamma(B,T)],$$

where $\Gamma(B,T)$ is the closed curve obtained by joining the end points of a Brownian trajectory $B(t) : [0,T] \to M$ with the curve $\gamma_{B(T),B(0)}$ of our collection of *short curves*. This defines a cycle $[\Gamma(B,T)]$ in homology, which is the value of our map. To see its meaning, we use the approximation of the random path and Poincaré duality as follows. Given a partition $\mathcal{P}$ of the interval $[0,T]$, we denote by $B(T,\mathcal{P})$ the piecewise smooth curve as constructed above that approximates $B(t)$. By joining the endpoints in exactly the same way with $\gamma_{B(T),B(0)}$, we get a cycle that depends on the random path $B(t)$, the length of the interval T, and the partition $\mathcal{P}$. We denote it by $\Gamma^{\mathcal{P}}(B,T)$. This is a piecewise smooth cycle, over which we can integrate a 1-form w:

$$\int_{\Gamma^{\mathcal{P}}(B,T)} w\,.$$

Then the following limit is well defined

$$\int_{\Gamma(B,T)} w = \lim_{max|\Delta t|\to 0} \int_{\Gamma^{\mathcal{P}}(B,T)} w\,.$$

Poincaré duality permits us to interpret the cycle $[\Gamma(B,T)]$ as follows. We represent $[\Gamma(B,T)]$ by the linear map $A_T : H^1(M,\mathbb{R}) \to \mathbb{R}$ given by:

$$A_T([w]) = \int_{\Gamma(B,T)} w \in \mathbb{R}.$$

Now consider the *time shift* map F_s acting in the image of our map $[\,\cdot\,]$, which is just a translation by time s on our Brownian path. For $s \geq 0$, it acts as follows. We will first describe it for the piecewise smooth approximation. Given the path $B(t)$, consider the curve given by the composition $F_s(B(t)) = B(s+t)$. Now, construct in the same way, the piecewise smooth approximation to this new random path (which starts in $B(s)$), and the cycle $(F_s)_*[\Gamma^{\mathcal{P}}(B,T)]$ as the cycle obtained with the piecewise approximation of the curve $F_s \cdot B : I \mapsto M$, closed with a curve $\gamma_{B(T+s),B(s)}$ from our fixed collection of short curves. Of course the map behaves well after taking limits, giving a time translation on the cycle constructed above, $(F_s)_*[\Gamma(B,T)]$.

3.1 Lemma. *The map $[\cdot]$ is measurable, and subadditive with respect to F_s.*

Proof. Suppose we denote by $\mathcal{P}$ a partition of the interval $[0,T+S]$ such that both S and T are on the partition. Since we will need to consider the cycle $\Gamma^{\mathcal{P}}(B,T+S)$ in two pieces, first going to time S, then from S to time $T+S$. Let us denote by $\Gamma^{\mathcal{P}}(B,S)$, and consider the curve B under the shift F_S by S units, denoted by B_{F_S}. Then it defines a cycle $\Gamma^{\mathcal{P}}(B_{F_S},T)$. (To simplify notation we are using the same notation $\mathcal{P}$ for the part of the partition on each of the corresponding subintervals). We have:

$$\int_{\Gamma^{\mathcal{P}}(B,S+T)} w = \int_{\Gamma^{\mathcal{P}}(B,S)} w + \int_{\Gamma^{\mathcal{P}}(B_{F_S},T)} w - \int_{\gamma_{B(S),B(0)}} w$$

Consequently,

$$\left\| \int_{\Gamma^{\mathcal{P}}(B,S+T)} w \right\| \leq \left\| \int_{\Gamma^{\mathcal{P}}(B,S)} w \right\| + \left\| \int_{\Gamma^{\mathcal{P}}(B_{F_S},T)} w \right\| + c$$

Since the curves $\gamma_{p,q}$ are uniformly bounded, c is a constant, depending only on the 1-form w. Now take limits as the mesh of the partition $\mathcal{P}$ tends to zero. We obtain an analogue inequality, omitting $\mathcal{P}$ in the formulae. This is the subadditive property. Measurability is a direct consequence of the properties of Brownian motion.

Now we construct the *Schwartzman cycle* associated with a random path. Given $B \in \mathcal{C}$, consider the limits

$$\Gamma(B) = \lim_{T\to\infty} \frac{1}{T}[\Gamma(B,T)]$$

$$A = \lim_{T\to\infty} \frac{1}{T} A_T.$$

3.2 Lemma. *These limits exist and define the same homology class. Furthermore, this homology class is $\{F_s\}$ invariant, i.e.*

$$(F_s)_*[\Gamma(B,T)] = [\Gamma(B,T)].$$

Proof. We need to show, by Poincaré duality, that the map A is well defined as a limit. But this is a consequence of the fact that the map $[\,\cdot\,]$ is measurable and subadditive as proved in the previous lemma. On the other side, Poincaré duality gives an equivalence for each time T of the two cycles: $[\Gamma(B,T)]$ and A_T. Then the lemma follows in the same way as the proof of the Subadditive Ergodic Theorem (Thm. 10.1 in [W]).

3.3 Definition. *The cycle $\Gamma(B)$ (or its equivalent A) will be called the random asymptotic cycle. The Schwartzman asymptotic cycle associated with Brownian motion is defined as*

$$\mathbb{E}[\Gamma] = \int_{\mathcal{C}} \Gamma(\alpha)dW(\alpha),$$

where W denotes Wiener measure on the space of continuous paths in M.

Schwartzman [Sc] gives another way to understand the asymptotic cycle in terms of the Eilenberg-Bruschlinsky homology theory, that we will use for the cycle associated to Brownian movement.

We will consider the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and the group of continuous mappings $\mathcal{C}(M, S^1)$ with pointwise multiplication. We denote by $\mathcal{R}(M)$ the closed subgroup of $\mathcal{C}(M, S^1)$ consisting of

$$\mathcal{R}(M) = \{\phi \in \mathcal{C}(M,S^1) \,|\, \exists\, \alpha : M \to \mathbb{R} \text{ } with \text{ } \phi(x) = exp(2\pi i\alpha(x))\}.$$

3.4 Theorem (Eilenberg-Bruschlinsky [Sc]**).** *$\mathcal{C}(M,S^1)/\mathcal{R}(M)$ is algebraically isomorphic to $H_1(M,\mathbb{Z})$. The isomorphism is given explicitly as follows. Let $\phi \in \mathcal{C}(M,S^1)$ and consider for any cycle C the restriction of ϕ to the cycle C. This is a map from the cycle to S^1, and the degree is well defined. Then the pairing:*

$$\mathcal{C}(M,S^1)/\mathcal{R}(M) \times H_1(M,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$(\phi, C) \mapsto deg(\phi|_C)$$

is a duality pairing.

In the case of the Asymptotic cycle defined by a Brownian path $\Gamma(B,T)$ we can always take the limit

$$\phi \mapsto \lim_{T\to\infty} \frac{1}{T}[deg(\phi|_{\Gamma(B,T)})]$$

defining an element of $H_1(M,\mathbb{R})$ by duality.

3.5 Theorem. *With probability one, the random asymptotic cycle associated with Brownian motion in a compact manifold is trivial in homology. Consequently, the Schwartzman asymptotic cycle is also zero*

Proof. Note that as a special case of the Hodge theorem, a map $\phi : M \to S^1$ can always be deformed to a harmonic map (see [EL1], sect. (7.1)). In fact, it is a harmonic morphism. Recall that a harmonic morphism is a map ϕ that is harmonic and horizontally conformal, that is, at any point $p \in M$ at which $d\phi_p \neq 0$, the restriction of $d\phi_p$ to the orthogonal complement of $Ker(d\phi_p)$ in $T_p(M)$ is conformal and surjective. In our case, the image is of dimension one therefore, it is immediately conformal.

Now, we know also that since ϕ is a harmonic morphism, the image of Brownian motion on M is Brownian motion on S^1 up to a random bounded time change. ([EL2] Pag. 395, sect. 2.44). The collection of closing curves goes to the collection of closing curves, so that $\phi|_{\Gamma(B,T)}$ is a cycle in S^1 corresponding to Brownian motion. That is, the image of the cycle associated to a Brownian path is a cycle corresponding to a Brownian path in S^1. But with probability one, Brownian motion in S^1 is recurrent, and its asymptotic cycle is zero.

To see this, recall that Brownian motion on a manifold is constructed from Brownian motion in the tangent space and then by developing the manifold on the tangent space along the Brownian path, we obtain the Brownian path on the manifold. Namely, if one takes a random Brownian path on S^1, then it is induced by a random path on $\mathbb{R}$. But Brownian motion on $\mathbb{R}$ is recurrent, and this means that with probability one, given a random continuous path $\beta(t)$ in $\mathbb{R}$, there is a sequence of times $t_1, t_2, \ldots, t_n, \ldots$ such that $t_n \to \infty$, and $\beta(t_n) = \beta(0)$. So that all the curves in S^1 defined by closing the trajectory at the points t_n are homotopic to zero, hence also zero in homology. This gives a subsequence of cycles that converges to zero in homology, but we have proved that the cycle converges, so it must converge to the trivial cycle.

We have the following interesting application.

We define the space of *Ends* of a manifold M, $End(M)$, as the inverse limit

$$End(M) = \varprojlim\{C_K : K \subset M\}$$

where C_K denotes the set of connected components of $M \setminus K$, and we have the partial order given by inclusion.

3.6 Corollary. *If M is a* **compact** *Riemannian manifold that admits an infinite cyclic covering $\widetilde{M}$, then the lift of Brownian motion from M to $\widetilde{M}$ cannot escape through any of the two ends of $\widetilde{M}$.*

We recall that Elworthy and Rosenberg proved the following.

3.7 Theorem (Elworthy [ER]**).** *If M is a Riemannian manifold such that its sectional curvature is bounded above and below by negative constants, and its Ricci curvature $Ricc(M)$ is bounded below by $C > -\lambda_0$ were λ_0 is the first eigenvalue of the Laplacian. Then M has at most one end through which Brownian motion can escape to infinity with positive probability.*

4. An example of hyperbolic manifolds with ends

We will take as model for the hyperbolic spaces $\mathbb{H}^n$, the disc model *Interior* $(\mathbb{D}^n) \subset \mathbb{R}^n$ with the Poincaré metric. The sphere at infinity, $S^{n-1} = \partial\mathbb{D}^n$, will be denoted by S^{n-1}_∞. The hyperbolic metric of $\mathbb{H}^n$ will be denoted by $\mathfrak{h}_n$. We construct a harmonic map $h : \mathbb{H}^2 \to \mathbb{H}^3$ ($\mathbb{H}^2$ and $\mathbb{H}^3$ with their corresponding hyperbolic metrics) such that:

(1) The function h extends to the circle at infinity as a continuous function with values in $\mathbb{D}^3$ and we obtain the continuous function $g : S^1_\infty = \partial\mathbb{H}^2 \to \partial\mathbb{H}^3 = S^2_\infty$
(2) The function g is a Peano curve *i.e.* $g(S^1_\infty) = S^2_\infty$.

Let $\Sigma := \Sigma_g$ be a compact and orientable surface of genus $g \geq 2$. Let $f : \Sigma \to \Sigma$ be a diffeomorphism which is isotopic to a pseudo-Anosov diffeomorphism. Then, by a celebrated theorem by Thurston, the mapping torus is a hyperbolic 3-manifold. Namely, the 3-manifold obtained from $\Sigma \times \mathbb{R}$ by taking the orbit space by the proper action of the diffeomorphism

$$F : \Sigma \times \mathbb{R} \to \Sigma \times \mathbb{R}, \quad (\sigma, t) \mapsto (f(\sigma), t+1)$$

is a compact hyperbolic 3-manifold. We will denote this manifold by $M := M_f$ and its hyperbolic metric by $\mathfrak{m}$. The manifold M fibres over the circle $\pi : M \to S^1$ with fibre Σ and monodromy f. The fundamental group of M is a semidirect product of the fundamental group of Σ and the integers $\mathbb{Z}$, *i.e.* there exists an exact sequence:

$$1 \to G \to \pi_1(M) \to \mathbb{Z} \to 0,$$

where G is isomorphic to $\pi_1(\Sigma)$. Therefore, the fundamental group of Σ can be considered as a normal subgroup of the fundamental group of M. We have that $\Sigma \times \mathbb{R}$ has the structure of a hyperbolic manifold *with two ends*, which is an infinite cyclic covering of the compact manifold M.

Let $p : \mathbb{H}^3 \to M$ the universal covering projection which is taken as a local isometry of Riemannian manifolds $(H^3, \mathfrak{h}_3)$ to $(M, \mathfrak{m})$.

Let $Iso_+\mathbb{H}^3 \cong PSL(2, \mathbb{C})$ be the group of orientation-preserving isometries of $\mathbb{H}^3$, then, M can be described as the quotient $\mathbb{H}^3/\Gamma$, where Γ is a discrete subgroup of $PSL(2, \mathbb{C})$, $\Gamma \subset Iso_+\mathbb{H}^3$, and Γ is isomorphic to $\pi_1(M)$.

The 2-dimensional foliation $\mathcal{F}$ in M, given by the fibres of π lifts to a 2-dimensional foliation $\widetilde{\mathcal{F}}$ in $\mathbb{H}^3$. Since each fibre of π injects its fundamental group into the fundamental group of M we have that the leaves of foliation $\widetilde{\mathcal{F}}$ are diffeomorphic to $\mathit{Interior}(\mathbb{D}^2)$ (or to $\mathbb{R}^2$). Since the foliation $\mathcal{F}$ comes from a fibration onto the circle, we obtain that $\widetilde{\mathcal{F}}$ is a product fibration, *i.e.* there exists a diffeomorphism $\phi : \mathit{Interior}(\mathbb{D}^2) \times \mathbb{R} \to \mathbb{H}^3$ such that the leaves of $\widetilde{\mathcal{F}}$ are the images of $\mathit{Interior}(\mathbb{D}^2) \times \{t\},\ t \in \mathbb{R}$.

A theorem by Thurston and Cannon (unpublished) is the following:

4.1 Theorem (Cannon, Thurston [CT]**).** *Let $\psi : \Sigma \to M$ be a continuous map such that ψ induces an isomorphism from the fundamental group of Σ onto $G \subset \Gamma$. Let $\Pi : \mathbb{H}^2 \to \Sigma$ be the universal covering projection of Σ. Let $\widetilde{\psi} : \mathbb{H}^2 \to \mathbb{H}^3$ be the lifting of ψ so that we have the following commutative diagram:*

$$\begin{array}{ccc} \mathbb{H}^2 & \xrightarrow{\Pi} & \Sigma \\ {\scriptstyle\widetilde{\psi}}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathbb{H}^3 & \xrightarrow{p} & M \end{array}$$

Then,

(1) *$\widetilde{\psi}$ extends to the circle at infinity as a continuous map from $\mathbb{D}^2$ to $\mathbb{D}^3$.*
(2) *The restriction of this map to the circle at infinity is a map*

$$g : S^1_\infty = \partial\mathbb{H}^2 \to \partial\mathbb{H}^3 = S^2_\infty$$

which is a Peano curve.
(3) *This Peano curve is geometrical, i.e. only depends on the choice of the isomorphism from $\pi_1(\Sigma)$ to G.*
(4) *The map g is equivariant under the actions of the fundamental groups of Σ and M on $\mathbb{H}^2$ and $\mathbb{H}^3$, respectively, under covering transformations.*

The theorem of Cannon and Thurston implies that although the foliation $\widetilde{\mathcal{F}}$ is a product foliation, it is embedded in H^3 by the function ϕ above in a very twisted way as the leaves approach the boundary of $\mathbb{H}^3$. In fact we have the following corollary to the theorem of Cannon and Thurston:

4.2 Corollary. *The function $\phi : \mathit{Interior}(\mathbb{D}^2) \times \mathbb{R} \to \mathbb{H}^3$ extends to a continuous function $\widehat{\phi} : \mathbb{D}^2 \times \mathbb{R} \to \mathbb{D}^3$. Furthermore:*

(1) *The restriction of $\widehat{\phi}$ to $S^1_\infty \times \{t\}$ is a Peano curve for each $t \in \mathbb{R}$.*
(2) *For each $t,\ s \in \mathbb{R}$ and $x \in S^1_\infty$ we have $\widehat{\phi}(x, t) = \widehat{\phi}(x, s)$.*
(3) *$\widehat{\phi}(\cdot, t)$ is an equivariant Peano curve.*

Let $\mathcal{T}_g$ be the Teichmüller space of hyperbolic metrics on Σ modulo diffeomorphisms isotopic to the identity. A point in $\mathcal{T}_g$ will be, therefore, a choice of a hyperbolic metric μ in Σ and a marking of the fundamental group of Σ. We can think of the marking as an explicit isomorphism $\alpha : \pi_1(\Sigma) \to G$ from $\pi_1(\Sigma)$ onto G. The elements of $\mathcal{T}_g$ will be denoted by (μ, α), where μ is a hyperbolic metric. Let $(\mu, \alpha) \in \mathcal{T}_g$. Since both M and Σ are Eilenberg-MacLane spaces of type $K(\pi, 1)$, then for each α there exists a homotopy class of maps from Σ into M.

Now we recall the following theorem of Eells and Sampson:

4.3 Theorem (Eells-Sampson [EL2]**).** *Let $(M, \mathfrak{m})$ and $(N, \mathfrak{n})$ be compact Riemannian manifolds with metrics $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Suppose that $(N, \mathfrak{n})$ is of strictly negative sectional curvature. Then, for each homotopy class of maps from M to N there exists one and only one harmonic map in this homotopy class.*

From all of the above we have the following proposition proved by Pereira do Vale and Verjovsky [VV1]:

4.4 Proposition ([VV1]**).** *Let $\alpha : \pi_1(\Sigma) \to G$ be an isomorphism. Let μ be a hyperbolic metric on Σ. Then there exists one and only one harmonic map $h : (\Sigma, \mu) \to (M, \mathfrak{m})$ such that if $\tilde{h} : \mathbb{H}^2 \to \mathbb{H}^3$ is the lifting that makes the following diagram commutative*

$$\begin{array}{ccc} \mathbb{H}^2 & \xrightarrow{\Pi} & \Sigma \\ {\scriptstyle \tilde{h}}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \mathbb{H}^3 & \xrightarrow{p} & M \end{array}$$

Then, $\tilde{h}$ is a harmonic and equivariant map from $(\mathbb{H}^2, \mathfrak{h}_2)$ into $(\mathbb{H}^3, \mathfrak{h}_3)$ which admits a continuous extension as a map from $\mathbb{D}^2$ onto $\mathbb{D}^3$ which at infinity has an equivariant Peano curve $g : S^1_\infty = \partial\mathbb{H}^2 \to \partial\mathbb{H}^3 = S^2_\infty$ as boundary value at infinity.

The standard theorems of harmonic maps from compact Riemannian surfaces into compact Riemannian 3-manifolds imply that the map h is actually an embedding from Σ into M. Also, h depends real analytically on the point in Teichmüler space [EL2]. Using this real analytic dependence, it is possible to introduce real analytic coordinates in Teichmüller space and, in fact, using the second fundamental form of the image of Σ in M, it is possible to introduce complex coordinates as well (as in the Bers theory).

If $\widetilde{M}$ is the covering manifold of M which fibres over the circle with fibre the non-compact surface of genus zero and a Cantor set of ends, in other words, homeomorphic to $S^2 \setminus \{Cantor\ set\}$, that is, the surface obtained from infinitely many pairs of pants according to the infinite binary tree, then, h lifts to a harmonic map $\hat{h} : \mathbb{H}^2 \to \widetilde{M}$ which accumulates to the ends of $\widetilde{M}$. In this fashion we obtain examples of *escaping* harmonic maps.

4.5 Remark. Compare this construction with the following. We use the same notation as in the previous example. Furthermore, denote by $D^2 = Interior(\mathbb{D}^2) = \{(x, y) |\, x^2 + y^2 < 1\}$ contained in $D^3 = Interior(\mathbb{D}^3) = \{(x, y, z) |\, x^2 + y^2 + z^3 < 1\}$ with the inclusion $(x, y) \mapsto (x, y, 0)$. With this convention, $D^2 \subset D^3$ will be considered as hyperbolic metric spaces with the Poincaré metric. Let $G \subset Iso_+(\mathbb{D}^2)$ be a Kleinian group. For instance, $G = \pi(\Sigma)$, the fundamental group of a compact Riemann surface Σ of genus $g \geq 2$. Its fundamental region is a $4g$-sided hyperbolic polygon P_{4g} in D^2. We now extend the action of G to D^3 in the standard way. A fundamental region P for this action is homeomorphic to $P_{4g} \times \mathbb{R}$, but the transversal part to the polygon is formed by orthogonal hyperbolic geodesics to $P_{4g} \times \{0\}$ in D^3. The intersection of the closure $\bar{P}$ of P with the boundary $S^2 = \partial(D^3)$ gives two copies of a $4g$ sided polygon on the top and bottom hemispheres of S^2. Then the quotient $M = \mathbb{H}^3/G$ is a hyperbolic 3-manifold diffeomorphic to $\Sigma \times \mathbb{R}$ in a trivial way. Notice that this construction an that of the previous section, endow $\Sigma \times \mathbb{R}$ with two different hyperbolic metric structures, obtained from two non conjugate embeddings of $G = \pi(\Sigma)$ into $Iso_+(\mathbb{D}^2)$. Furthermore, the hyperbolic manifold M constructed in this paragraph *is not* the infinite cyclic covering of a hyperbolic *compact* 3-manifold. Brownian motion in this *non compact* manifold M is dissipative, since it is dissipative in the hyperbolic disc D^3, and it follows that it is also dissipative in the fundamental region $P = P_{4g} \times \mathbb{R}$ that projects to $M = \Sigma \times \mathbb{R}$. On $P_{4g} \times \mathbb{R}$ we have the isometric involution $(x, y, z) \mapsto (x, y, -z)$, consequently Brownian motion will have probability one half to escape to one end and one half to escape to the other end, implying that the same holds on $M = \Sigma \times \mathbb{R}$.

5. Lifting to Holonomy

In this section we review the construction of parallel transport along a Brownian path for a principal bundle $P \to M$ with structure group a compact Lie group G. We will follow closely Itô's construction [I] of parallel transport for the Riemannian connection on M.

Assume we have a principal bundle $P \to M$ with structure group G a compact connected Lie group. We suppose we have a connection, represented by a Lie algebra valued 1-form $A \in \mathcal{A}^1(M, \mathfrak{g})$, the 1-forms on P with values in the Lie algebra $\mathfrak{g}$ of G. Given any closed, C^1 curve $\gamma : [0, T] \mapsto M$ on M starting and ending at a point $p \in M$, we have the holonomy or parallel transport along γ with respect to the connection A given by the *time ordered* (or path ordered) exponential integral as:

$$P\,exp \int_\gamma A =$$

$$I + \int\limits_{0 \leq s \leq T} A(\gamma(s))\gamma'(s)ds + \iint\limits_{0 \leq s_1 \leq s_2 \leq T} A(\gamma(s_1))A(\gamma(s_2))\gamma'(s_2)\gamma'(s_1)ds_2 ds_1 + \cdots$$

We observe that the ordering is important because in general the evaluation of the connection form at different places of the curve give matrices that do not commute in the multiple integrands, so we must take an ordering of the multiplications of the connection forms evaluated under the multiple integrals, and that causes the factorial constants to disappear (see Nelson [N] pp. 15-16). In the case of an abelian Lie group it is not necessary to specify an ordering and the path ordered exponential is an honest exponential.

We follow Itô's construction closely. Consider a Brownian trajectory $B(t)$. For each time T, we use the collection of closing curves to obtain a closed cycle as above. Again, with a partition $\mathcal{P}$ of the interval $[0,T]$, we approximate the closed cycle by a piecewise smooth closed curve, obtaining the cycle $\Gamma^{\mathcal{P}}(B,T)$ as in the previous section. Given any connection on the principal bundle, it is possible to construct the holonomy along this cycle.

5.1 Theorem. *With the data as in the above paragraph, one can construct the holonomy along the asymptotic cycle defined by Brownian motion. We denote it by $Hol_A(B)$. It is a measurable function $Hol_A : \mathcal{C}_p \longrightarrow G$.*

Proof. The proof is the same as in Itô [I], except that we now use any connection on a principal bundle, instead of the Riemannian connection. Another way to prove it is as follows. We need to take the time ordered exponential integral of the connection form A along the cycle, so we use the approximations of the asymptotic cycle $\Gamma(B)$ by the piecewise smooth cycle $\Gamma^{\mathcal{P}}(B,T)$ as described above. Now consider the path ordered exponential integral of the connection form A over the piecewise smooth cycle,

$$Pexp \int\limits_{\Gamma^{\mathcal{P}}(B,T)} A.$$

Each term of the path ordered exponential integral can be uniformly bounded by a bound of the value of the connection form on the compact manifold M. Dividing the cycle by T, we get:

$$Pexp \int_{\frac{1}{T}\Gamma^{\mathcal{P}}(B,T)} A$$

where this is interpreted as dividing the k^{th} term of the series of integrals written above by $1/T^k$ (counting k from 0).

Take the limit as the mesh of $\mathcal{P}$ goes to zero, and the limit as $T \to \infty$

$$\lim_{T\to\infty} \left(\lim_{mesh(\mathcal{P})\to 0} Pexp \int_{\frac{1}{T}\Gamma^{\mathcal{P}}(B,T)} A \right).$$

The limits exists for each summand in the expression of the path ordered exponential integral with probability one in $\mathcal{C}$, by the lemmas of the previous section. Note that we need the subadditivity of the cycles (in the argument T), implying that each summand of the path ordered exponential defines a subadditive function in T also. Since the convergence of the series is uniform, this proves the theorem. It is obvious that this limit coincides with the limit of the parallel transport on the closed cycles as considered in [I].

We recall that the *Wilson loop* associated to a closed curve γ in M, is the trace of the holonomy determined by the curve, $Tr_\rho(Hol_A(\gamma))$, where the trace Tr_ρ depends on a representation ρ of the structure group G of the principal bundle. It is independent of a choice of base point. Then we have:

5.2 Corollary. *For a Brownian trajectory on a compact Riemannian manifold, the asymptotic Wilson loop for Brownian paths $Tr_\rho(Hol_A(B))$, is well defined.*

5.3 Remarks

(1) We believe that this concept might be useful in Quantum Gravity.

(2) We note that J. Eells has given a construction of Brownian motion on the fundamental group of a Riemannian manifold and as extension, a construction of Brownian motion in the holonomy group corresponding to the Riemannian connection [E].

6. The asymptotic linking number

V. I. Arnold [A] defined the asymptotic linking number of the asymptotic cycle associated to a flow of a homologically trivial vector field on a compact, simply connected three manifold M. We will prove in this section one can also define the linking number of two Brownian paths.

We follow the paper by V. I. Arnold [A] for the following construction. We assume from now on that we have a Riemannian homology 3-sphere M^3. For instance, the standard three sphere S^3 or the Brieskorn manifold $M_{p,q,r}$ defined as the intersection $H \cap S^5$ of the complex hypersurface

$$H = \left\{ (z_1, z_2, z_3) |\, z_1^p + z_2^q + z_3^r = 0;\, p,\, q,\, r \text{ } coprime, \text{ and} \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 \right\}$$

with the unit 5-sphere S^5. Milnor and Dolgachev independently, have shown that $M_{p,q,r}$ is a homogeneous space $\widetilde{SL(2,\mathbb{R})}/\Gamma$, where Γ is a discrete, cocompact subgroup of the universal covering of $SL(2,\mathbb{R})$ with its natural homogeneous Riemannian metric (see [M], [D]).

Let σ be a 1-cycle. Now, by a collection of *short curves* $\{\gamma_{q_1,q_2}\}$ in M^3 we will mean a set of smooth curves joining q_1 with q_2 such that

(1) they have uniformly bounded length with respect to the metric of M^3, and
(2) they do not intersect the cycle σ.

This collection depends in a measurable way on q_1 and q_2, and even piecewise smoothly (we could choose, for instance, a minimizing geodesic joining each pair q_1 with q_2, see [AK] pag. 145 or for more details, see also the paper by T. Vogel [V], where he shows that the probability they will intersect a given cycle σ is zero).

Recall that if we have two non intersecting closed curves σ_1 and σ_2 on M^3 . Then the *linking number* $L(\sigma_1, \sigma_2)$ is defined as the oriented intersection number of σ_1 with a 2-cycle whose boundary is precisely σ_2.

6.1 Lemma. *Suppose given the asymptotic Brownian cycle $\Gamma(B)$ associated with a Brownian curve B. Then the linking number $L(\Gamma(B), \sigma)$ is well defined and is independent of the collection $\{\gamma_{q_1,q_2}\}$.*

Proof. Given a time T and a partition $\mathcal{P}$ of the interval $[0, T]$, consider the approximating cycle $\Gamma^{\mathcal{P}}(B, T)$. The probability that the path $B(t)$ crosses the cycle σ up to time T is zero, because the cycle is a set of Borel measure zero in M^3. So with probability one in $\mathcal{C}$, the function $L(\Gamma^{\mathcal{P}}(B, T), \sigma)$ is well defined, it is measurable and subadditive in T. So the limit

$$\lim_{T\to\infty} \frac{1}{T} \left(\lim_{mesh\mathcal{P}\to 0} L(\Gamma^{\mathcal{P}}(B, T), \sigma) \right)$$

exists. The independence of the limit on the collection of short curves follows precisely by the fact they have uniformly bounded length, and then, their contribution to the linking number tends to zero (see [A]).

With this we can construct the Linking number of two Brownian paths in the following way. Given B_1 and B_2 two random paths on M^3, by the construction above, we can consider for each time T and a partition $\mathcal{P}$ of $[0,T]$, the linking number $L(\Gamma^{\mathcal{P}}(B_1,T),\Gamma(B_2))$.

Recall that we defined the *time shift* F_s of the cycle constructed from Brownian motion (in section 2), giving the cycle associated to $F_s(B(t)) = B(t+s)$ as $(F_s)_*[\Gamma(B)]$.

Then the same argument as in Lemma 2.1 gives:

6.2 Lemma. *With probability one in $\mathcal{C}$ with respect to the Wiener measure, the linking number $L(\Gamma^{\mathcal{P}}(B_1,T),\Gamma(B_2))$ is a measurable and subadditive function on T and the limit*

$$\lim_{T\to\infty}\frac{1}{T}\left(\lim_{mesh\mathcal{P}\to 0} L(\Gamma^{\mathcal{P}}(B_1,T),\Gamma(B_2))\right)$$

exists.

In a similar way, we also have:

6.3 Lemma. *The asymptotic linking number $L(\Gamma(B_1),\Gamma(B_2))$ is invariant under time shift acting on either B_1 or B_2, namely,*

$$L(\Gamma(F_{s_1}(B_1)),\Gamma(F_{s_2}(B_2))) = L(\Gamma(B_1),\Gamma(B_2))$$

for any $s_1,\ s_2 \in [0,\infty)$.

Proof. As in previous arguments, when taking the limit as $T\to\infty$, the path from 0 to s_i ($i = 1\,or\,2$) does not contribute to the linking number, namely, the contribution to the limit (as well as the contribution of the short curves γ_{q_1,q_2}) is zero after dividing by T and taking the limit.

This implies the following.

6.4 Theorem. *On a Riemannian homology 3-sphere, given two random curves B_1 and B_2, the linking number $L(\Gamma(B_1),\Gamma(B_2))$ exists almost always. This number is called the* **asymptotic linking number** *of the Brownian motion. Also the following average exists:*

$$\int_{\mathcal{C}\times\mathcal{C}} L(\Gamma(B_1),\Gamma(B_2))dW dW$$

and it is called the **average asymptotic linking number** *of the Brownian motion.*

Since Brownian motion, and consequently all the constructions we have done, depend on the choice of the Riemannian metric g on M, we will denote by

$$\mathbb{L}(g) = \int_{\mathcal{C}\times\mathcal{C}} L(\Gamma(B_1),\Gamma(B_2))dW dW.$$

6.5 Theorem. *The number $\mathbb{L} = \inf_g \mathbb{L}(g)$, where the infimum is taken over all Riemannian metrics on M, is a differentiable invariant and hence a topological invariant of the homology three sphere M.*

6.6 Remarks.

(1) We do not know how to compute $L(\Gamma(B_1), \Gamma(B_2))$ for Brownian motion on the standard 3-sphere nor on a Brieskorn manifold $M_{p,q,r}$, although we could compute the average asymptotic linking number for a volume preserving flow in [VV2].
(2) We do not know if the number $\mathbb{L}$ might be always zero, or how it relates to other invariants of homology three spheres.
(3) Since the linking number, as invented by Gauss, is related to Ampère's law, our random asymptotic linking number is related to Maxwell's theory of electromagnetism.

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