

# STABLE NORM AND $L^2$ NORM ON HOMOLOGY OF SURFACES

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# STABLE NORM AND $L^2$ NORM ON HOMOLOGY OF SURFACES

EUGENE GUTKIN AND DANIEL MASSART

ABSTRACT. We study the stable norm on the homology of a closed oriented surface endowed with a possibly singular Riemannian metric and compare it with the norm induced by the  $L^2$  norm on one-forms.

## 1. INTRODUCTION

The stable norm on the homology of a manifold depends on the choice of a Riemannian metric [Fe 69, G-L-P 81]. It has been extensively used in geometry and analysis. See [Ba 94, BI94, McS-R 95 I, Mt 97]. For convenience of the reader, we briefly recall the basic definitions.

Let  $(M, g)$  be a Riemannian manifold. We denote by  $\ell(\gamma)$  the length, with respect to the metric  $g$ , of a connected, rectifiable curve. By multicurves we will mean formal linear combinations of curves. If  $\gamma = \sum_i r_i \gamma_i$  is a multicurve, we set  $\|\gamma\| = \sum_i |r_i| \ell(\gamma_i)$ . Let  $h \in H_1(M, \mathbb{Z})$  be a homology class. Its stable norm is  $\|h\| = \min_\gamma \|\gamma\|$ , where  $\gamma$  runs through the geodesic multicurves in the class  $h$ .

An alternative definition of the stable norm is due to H. Federer [Fe 69]. It is based on the notion of the mass of a Lipschitz current. The stable norm of  $h \in H_1(M, \mathbb{R})$  is then the minimal mass of a Lipschitz current in the homology class  $h$ . Federer's approach allows to minimize over more general objects than the multicurves or laminations. However, J. Mather proved that the minimizing currents are supported by geodesic laminations. See [M 91].

Since the stable norm is defined by the length, it makes sense for the singular Riemannian metrics. The latter arise in several contexts. In particular, flat Riemannian surfaces with singular points, and their geodesic flows, are closely related to the billiards in polygons. See [GJ 00] and the bibliography there.

The theme of the present work is the stable norm on possibly singular Riemannian surfaces. Our goal is analyze the stable norm on these surfaces, in general, in relation to their topology and the geometry of their metrics. This continues the study started in the doctoral dissertation of one of the authors. See [Mt 96].

We will now describe the contents of the paper in more detail.

In section 2 we give a brief overview of the basic properties of the stable norm. In subsection 2.1 we obtain an inequality relating the stable norm and the  $L^2$  norm on the homology. See Theorem 1 and equation 3. Recently G. Paternain has extended the upper bound in inequality (3) to higher dimensions. See [Pa

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01]. Since the  $L^2$  norm encodes the analysis, and the stable norm encodes the geometry of the surface, it is useful to be able to compare the two. In subsection 2.2 we study the intersection form on the homology of a topological surface,  $M$ , in relation to a metric on  $M$ . Let  $\mathcal{M}$  be the moduli space of the metrics of curvature  $-1$  on  $M$ . The norm of the intersection form is a function on  $\mathcal{M}$ . We establish geometric lower and upper bounds on the norm, which are valid everywhere on  $\mathcal{M}$ . See Theorem 3.

D. M. thanks Albert Fathi, who suggested to him inequality (3). Some of the work on the paper was done in the Summer of 1998 while the authors were visiting the University of Freiburg. It is a pleasure to thank Victor Bangert for the invitation and the faculty and staff of the Mathematics Institute for their hospitality.

## 2. RELATIONS BETWEEN THE STABLE NORM AND THE $L^2$ -NORM

Let  $M$  be a closed manifold and let  $g$  be a Riemannian metric on it. The complex of Lipschitz currents on  $M$  is dual to the complex of smooth differential forms. It satisfies the Eilenberg-Steenrod axioms, and its homology is naturally isomorphic to the real homology of  $M$ . See [Fe 69, Fe 74] and [F-F 60] for details.

The comass of a differential one-form on  $M$  is given by

$$(1) \quad \text{comass}(\omega) = \sup_{x \in M} \sup_{v \in T_x M} \left\{ \frac{\omega(v)}{\|v\|} \right\}.$$

Equation (1) defines a Banach norm on the space of one-forms on  $M$ . We denote by  $\text{mass}(\eta)$  the dual norm on the space of one-currents. Thus

$$(2) \quad \text{mass}(\eta) = \sup\{\langle \omega, \eta \rangle \mid \text{comass}(\omega) \leq 1\}.$$

Taking the infimum of  $\text{mass}(\eta)$  over the currents in a homology class,  $[\eta] = h$ , yields a Minkowski norm on the homology. It is equal to the stable norm on  $H_1(M, \mathbb{R})$ , defined in Section 1. See [Fe 69]. In what follows we will freely use both definitions of the stable norm,  $\|h\|$ , for  $h \in H_1(M, \mathbb{R})$ . Note that  $\text{comass}(\omega)$  defined by equation (1) implicitly depends on the Riemannian metric on  $M$ . Hence,  $\text{mass}(\eta)$ , and therefore the stable norm on the homology depend on the metric.

From now on we restrict our analysis to the manifolds of two dimensions. More precisely, we will assume that  $(M, g)$  is a closed oriented Riemannian surface. If  $\omega \in H^1(M, \mathbb{R})$ , we denote by the same symbol the unique harmonic form in the cohomology class  $\omega$ . The integral  $\sqrt{\int_M \omega \wedge \star \omega}$  defines an  $L^2$ -norm on the space of differential one-forms on  $M$ , and a Euclidean norm on  $H^1(M, \mathbb{R})$ . The duality between the homology and the cohomology transfers the latter to  $H_1(M, \mathbb{R})$ . We will use the notation  $\|\cdot\|_2$  for all of them. We will denote by  $\langle \cdot, \cdot \rangle$  the pairing between linear spaces, and by  $(\cdot, \cdot)$  the scalar product of two elements in a Euclidean or a pre-Hilbert space. The spaces should be clear from the context. If  $\omega$  is a one-form on  $M$ , we denote by  $X_\omega$  the dual vector field of  $\omega$ . It is determined by the condition that for any  $z \in M$  and any  $v \in T_z M$ ,

we have  $\langle \omega(z), v \rangle = (v, X_\omega(z))$ . Let  $d\text{vol}$  be the volume form induced by the metric.

We will use symbols like  $A(\cdot, \cdot)$  for bilinear forms on a vector space. Let  $\text{Int}(h, k)$  be the intersection form on  $H_1(M, \mathbb{R})$ . Set

$$K = \sup \left\{ \frac{\text{Int}(h, k)}{\|h\| \|k\|} : h, k \in H_1(M, \mathbb{R}) \setminus \{0\} \right\}$$

be the norm of the intersection form with respect to the stable norm on the homology. It implicitly depends on the metric.

Since  $\dim H_1(M, \mathbb{R}) < \infty$ , the stable norm and the  $L^2$ -norm on the homology are equivalent : There exist constants  $c_1, c_2 > 0$  such that

$$c_1 \|h\| \leq \|h\|_2 \leq c_2 \|h\|.$$

The theorem below provides geometrically meaningful constants for this inequality.

**Theorem 1.** *Let  $\text{vol}(M)$  be the total volume of  $M$ , and let  $K$  be the norm of the intersection form on  $H_1(M, \mathbb{R})$ . Then for all  $h \in H_1(M, \mathbb{R})$  we have*

$$(3) \quad \frac{1}{\sqrt{\text{vol}(M)}} \|h\| \leq \|h\|_2 \leq K \sqrt{\text{vol}(M)} \|h\|.$$

*Proof.* For any differential one-form  $\omega$  and for  $z \in M$  we denote by  $\|\omega(z)\|$  the norm of the corresponding linear form on  $T_z M$ . Then  $(\omega \wedge \star \omega)(z) = \|\omega(z)\|^2 (d\text{vol})(z)$ . Hence, by equation (1)

$$(4) \quad (\|\omega\|_2)^2 = \int_M \|\omega(z)\|^2 d\text{vol} \leq \int_M \sup_{z \in M} \|\omega(z)\|^2 d\text{vol} = \text{vol}(M) (\text{comass}(\omega))^2.$$

Thus, the  $L^2$ -norm and the comass-norm on the space of differential forms are related by the inequality

$$\|\omega\|_2 \leq \sqrt{\text{vol}(M)} \text{comass}(\omega).$$

The  $L^2$ -norm on the space of one-forms induces the dual  $L^2$ -norm on the space of Lipschitz one-currents. Dualizing the preceding inequality, we obtain

$$(5) \quad \text{mass}(\eta) \leq \sqrt{\text{vol}(M)} \|\eta\|_2.$$

Minimizing both sides of equation (5) over the currents in a homology class, we obtain the left inequality in equation (3).

Let  $P : H_1(M, \mathbb{R}) \rightarrow H^1(M, \mathbb{R})$  be the operator of the Poincaré duality. It is a linear isomorphism, satisfying the following identity. For any  $\omega \in H^1(M, \mathbb{R})$  and any  $h \in H_1(M, \mathbb{R})$  we have

$$(6) \quad \langle \omega, h \rangle = \text{Int}(P^{-1}\omega, h).$$

In the equation below we tacitly assume that the denominators are not equal to zero. Then

$$(7) \quad \|\omega\| = \sup_{h \in H_1(M, \mathbb{R})} \frac{\langle \omega, h \rangle}{\|h\|} = \sup_{h \in H_1(M, \mathbb{R})} \frac{\text{Int}(P^{-1}\omega, h)}{\|h\|} \leq K \|P^{-1}\omega\|.$$

Rewriting equation (7) as  $\|Px\| \leq K\|x\|$ , where  $x \in H_1(M, \mathbb{R})$ , and taking the supremum of  $\|Px\|/\|x\|$  over  $x \neq 0$ , we obtain that  $K$  is the norm of the Poincare duality, as an operator from  $(H_1(M, \mathbb{R}), \|\cdot\|)$  to  $(H^1(M, \mathbb{R}), \|\cdot\|)$ . Using this and the bound on the  $L^2$ -norm by the comass-norm on  $H^1(M, \mathbb{R})$  (see the inequality following equation (4)), we obtain

$$\|Ph\|_2 \leq \sqrt{\text{vol}(M)} \|Ph\| \leq \sqrt{\text{vol}(M)} K\|h\|.$$

Since the Poincare duality is an  $L^2$ -isometry between the homology and the cohomology, the preceding inequality implies  $\|h\|_2 \leq \sqrt{\text{vol}(M)} K\|h\|$ . This concludes the proof of Theorem 1.  $\square$

The following is immediate from equation (3).

**Corollary 2.** *Let  $M$  be an arbitrary closed, oriented Riemannian surface. Let  $K$  be the norm of the intersection form on  $H_1(M, \mathbb{R})$  with respect to the stable norm on the first homology of  $M$ . Then*

$$(8) \quad K \cdot \text{vol}(M) \geq 1.$$

### 3. BOUNDS ON THE INTERSECTION FORM IN CONSTANT CURVATURE

In this subsection we study more in detail the stable norm on the homology, with respect to a metric of curvature  $-1$ . We fix the surface, and vary the metric. Thus, the stable norm and its attributes become “functions” on the moduli space. When needed, we will notationally indicate the dependence of a quantity on the point in the moduli space. Thus, if  $g$  is a metric of curvature  $-1$ , we denote by  $K(g)$  the norm of the intersection form on the first homology. The following theorem estimates  $K(g)$  in terms of the length of the shortest closed geodesic. Heuristically, it means that the norm of the intersection form is controlled by the “short” closed geodesics.

**Theorem 3.** *Let  $M$  be a closed surface, and let  $g$  be a metric of curvature  $-1$  on it. Let  $\gamma_1$  be the shortest non-separating closed curve in  $M$ , and let  $l_1 = l_1(g)$  be its length.*

*Then there exist positive constants  $a$  and  $b$ , depending only on the genus of the surface, such that*

$$(9) \quad \frac{a}{l_1 |\log(l_1)|} \leq K(g) \leq \frac{b}{l_1 |\log(l_1)|}.$$

Before going to the proof of the inequality 9, we will establish an lower bound on  $K(g)$ , valid for any metric  $g$  on  $M$ . This bound is of independent interest. See the proposition below. We will also formulate and prove a geometric property of the geodesics in a metric of curvature  $-1$ . See the lemma below. We will use the Proposition and the Lemma in the proof of Theorem 3.

**Proposition 4.** *Let  $M$  be a closed surface, and let  $g$  be any metric on it. Let  $\gamma_1$  be the shortest non-separating closed curve in  $M$ , and let  $l_1 = l_1(g)$  be its length. Then*

$$(10) \quad K(g) \geq \frac{9}{l_1^2}.$$

*Proof.* Let  $\alpha$  and  $\beta$  be different, closed, oriented, non-separating curves. We will prove that the arclength between any consecutive intersections is at least  $l_1/2$ . Our argument is modeled on the proof of lemma 2, p. 58 of [F-L-P 79]. Assume the opposite, and let  $P$  and  $Q$  be two consecutive intersections with the same sign. Consider the closed curve formed by the arc of  $\alpha$  joining  $P$  with  $Q$ , and the arc of  $\beta$  joining  $Q$  with  $P$ . Its length is less than  $l_1$ , hence it separates. But it is homotopic to a curve intersecting  $\alpha$  exactly once. Thus, it cannot separate. See Figure 3.

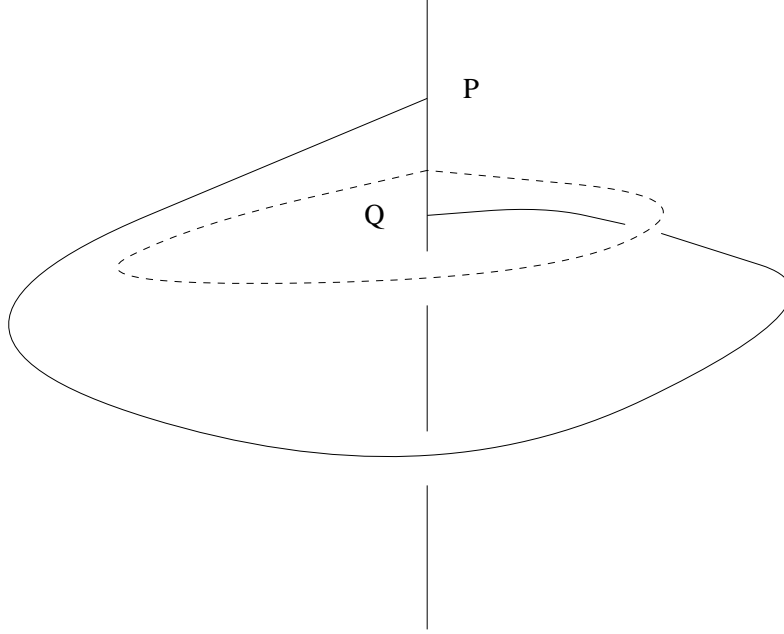


FIGURE 1. two consecutive intersections

Let now  $0 < r < l_1/2$ . We cut  $\alpha$  and  $\beta$  into pieces of the length at most  $r$ . In view of the above, any pair of these segments cannot intersect twice with the same sign. Let  $n_\alpha$  and  $n_\beta$  be the numbers of respective pieces. Then  $|\text{Int}([\alpha], [\beta])| \leq n_\alpha n_\beta$ . Let  $l(\alpha)$  and  $l(\beta)$  be the respective lengths. Then

$$n_\alpha \leq \frac{l(\alpha)}{r} + 1, \quad n_\beta \leq \frac{l(\beta)}{r} + 1.$$

Therefore

$$\frac{\text{Int}([\alpha], [\beta])}{l(\alpha)l(\beta)} \leq \left(\frac{1}{r} + \frac{1}{l_1}\right)^2.$$

Since  $r$  is arbitrarily close to  $l_1/2$ , we obtain

$$\frac{\text{Int}([\alpha], [\beta])}{\|\alpha\|\|\beta\|} \leq \frac{9}{l_1^2}.$$

□

The following corollary is immediate from equation 8 and the proposition above.

**Corollary 5.** *For any metric on  $M$  we have*

$$\text{vol}(M) \geq \frac{l_1^2}{9}.$$

From now on we assume that the metric has curvature  $-1$ . An infinite or a semi-infinite geodesic is asymptotic to a closed geodesic,  $\gamma$ , if either the  $\alpha$ -limit or the  $\omega$ -limit set of the infinite geodesic is equal to  $\gamma$ .

**Lemma 6.** *Let  $\gamma$  be a closed geodesic on  $M$ , of length  $l$ . Set*

$$(11) \quad \epsilon(x) = \text{arccosh} \left( \coth \left( \frac{x}{2} \right) \right).$$

*Then any simple geodesic that enters the  $\epsilon(l)$ -neighborhood of  $\gamma$  either intersects it, or is asymptotic to it.*

*Proof.* Using the action of  $SL(2, \mathbb{R})$ , we may assume without loss of generality that the lift,  $\Gamma$ , of  $\gamma$  to  $\mathbb{H}^2$  is  $i\mathbb{R}_+$ . Let  $\alpha$  be a simple geodesic in  $M$ , and let  $A$  be the lift of  $\alpha$ . If  $A$  is a vertical line, or a circular arc through the origin, then  $\alpha$  is asymptotic to  $\gamma$ . Otherwise,  $A$  is given by the equation  $(x - a)^2 + y^2 = r^2$ . The case  $r = |a|$  has been considered. If  $|a| < r$ , then  $\alpha$  intersects  $\gamma$ , and there is nothing to prove. Thus, we assume from now on that  $|a| > r$ .

Let  $N(\epsilon) \subset \mathbb{H}$  be the  $\epsilon$ -neighborhood of  $\Gamma$ . The boundary of  $N(\epsilon)$  is bounded by the Euclidean straight lines with equation

$$y = \pm \frac{x}{\sinh(\epsilon)}.$$

Since  $\alpha$  enters the  $\epsilon$ -neighborhood of  $\gamma$ , its lift  $A$  intersects  $N(\epsilon)$ . This implies

$$a^2 \geq (a^2 - r^2)(1 + \delta(\epsilon)^2).$$

In what follows we assume that  $a > 0$ . The case  $a < 0$  is disposed of in the same manner. The isometry of  $\mathbb{H}$ , corresponding to  $\gamma$  is  $z \mapsto \exp(l)z$ . Since  $\alpha$  is simple,  $A$  and  $\exp(l)A$  don't intersect. The endpoints of  $\exp(l)A$  are  $\exp(l)(a + r), \exp(l)(a - r)$ . Hence

$$a + r \leq \exp(l)(a - r).$$

Combining the two inequalities above, we get  $2 \cosh(\epsilon) \geq \coth(\frac{l}{2})$ , which implies the claim.  $\square$

*Proof.* We will now prove Theorem 3. Let  $\alpha_1, \dots, \alpha_k$  ( $k \leq 3(g + 1)$ ) be the closed geodesics in  $M$  of length less than  $\text{arcsinh}(1)$ , and let  $l_1, \dots, l_k$  be their respective lengths. For  $1 \leq i \leq k$  let  $T_i \subset M$  be the closed, embedded collar neighborhood of  $\alpha_i$ , of width  $w_i = \epsilon(l_i) \leq \text{argsinh}(1/\sinh(l_i))$ . By the collar lemma (see [Bu]), they exist, and  $T = \cup_{i=1}^k T_i$  is a disjoint union. Set  $S = M \setminus T$ . Then every closed geodesic in  $S$  is longer than  $\text{arcsinh}(1)$ .

Let now  $\gamma_1 \neq \gamma_2$  be any pair of simple closed geodesics in  $M$ . From the decomposition  $M = S \cup T$ , we have

$$\frac{\text{Int}(\gamma_1, \gamma_2)}{l(\gamma_1)l(\gamma_2)} \leq \frac{|\text{Int}(\gamma_1 \cap T, \gamma_2 \cap T)|}{l(\gamma_1 \cap T)l(\gamma_2 \cap T)} + \frac{|\text{Int}(\gamma_1 \cap S, \gamma_2 \cap S)|}{l(\gamma_1 \cap S)l(\gamma_2 \cap S)}$$

By an argument similar to that in the proof of Proposition 4, the second term in the right hand side is bounded on  $\mathcal{M}$  by  $9/\epsilon(\text{arcsinh}^2(1))$ .

The first term in the right hand side of the inequality above satisfies

$$\frac{|\text{Int}(\gamma_1 \cap T, \gamma_2 \cap T)|}{l(\gamma_1 \cap T)l(\gamma_2 \cap T)} \leq \sum_{i=1}^k \frac{|\text{Int}(\gamma_1 \cap T_i, \gamma_2 \cap T_i)|}{l(\gamma_1 \cap T_i)l(\gamma_2 \cap T_i)}.$$

To explain our argument, consider first the case  $k = 1$ . In this case,  $T = T_1$ , and we suppress the corresponding indexation in what follows. The general case is treated similarly, only the constants are modified.

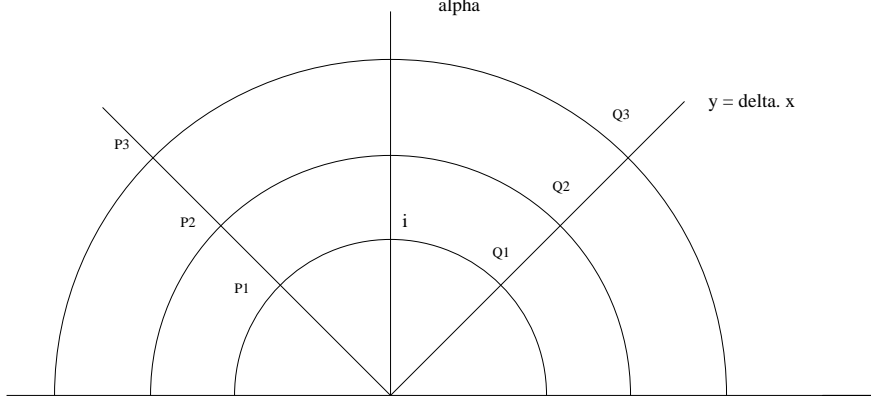


FIGURE 2. the lifted picture

As in the proof of Lemma 6, we assume without loss of generality, that the lift of  $\alpha$  lifts to  $\mathbb{H}$  is  $\mathbb{R}_+ i$ . Then the corresponding lift of  $T$  is bounded by the straight lines  $y = \pm x / \sinh \epsilon$ . Let  $\beta$  be a geodesic arc in  $T$ , perpendicular to  $\alpha$ , that lifts to  $\mathbb{H}$  as the half-circles centered at zero with radius  $\exp(nl(\alpha))$ :  $n \in \mathbb{Z}$ . Call  $P_n$  (resp.  $Q_n$ ) the intersection point of the  $n$ -th half-circle with the line  $y = x / \sinh \epsilon$  (resp.  $y = -x / \sinh \epsilon$ ). See figure 3. We may assume that a lift of one, hence all (since  $\gamma_1$  has no self-intersections) connected component of  $\gamma_1 \cap T$  to  $\mathbb{H}$  has one endpoint between  $P_1$  and  $P_2$ , and the other between  $P_{n-1}$  and  $P_n$ ; and likewise a lift of any connected component of  $\gamma_2 \cap T$  to  $\mathbb{H}$  has one endpoint between  $Q_1$  and  $Q_2$ , and the other between  $P_{k-1}$  and  $P_k$ . Then, since the number of intersections between  $[P_1, Q_n]$  and  $[Q_1, P_k]$  is  $n + k - 1$ ,

$$\frac{\text{Int}(\gamma_1 \cap T, \gamma_2 \cap T)}{l(\gamma_1 \cap T)l(\gamma_2 \cap T)} \leq \frac{n + k - 1}{d(P_1, Q_n)d(Q_1, P_k)}$$

Let us compute  $d(P_1, Q_n)$ . The point  $P_1$  has the coordinates  $((1 + \delta^2)^{-1/2}, \delta(1 + \delta^2)^{-1/2})$ . The hyperbolic isometry  $A$  with matrix

$$\begin{bmatrix} \cosh \frac{\epsilon}{2} & -\sinh \frac{\epsilon}{2} \\ -\sinh \frac{\epsilon}{2} & \cosh \frac{\epsilon}{2} \end{bmatrix}$$

sends  $P_1$  to  $i$  and  $i$  to  $Q_1$ . We compute  $d(i, A(Q_n))$ . Note that  $Q_n = \exp(nl)Q_1 = \exp(nl)A(i)$  whence  $A(Q_n)$  is the image of  $i$  under the hyperbolic isometry with matrix

$$\begin{bmatrix} \exp(\frac{nl}{2})(\cosh \frac{\epsilon}{2})^2 + \exp(-\frac{nl}{2})(\sinh \frac{\epsilon}{2})^2 & -2 \sinh(\frac{\epsilon}{2}) \cosh \frac{\epsilon}{2} \cosh(\frac{nl}{2}) \\ -2 \sinh(\frac{\epsilon}{2}) \cosh \frac{\epsilon}{2} \cosh(\frac{nl}{2}) & \exp(\frac{nl}{2})(\sinh \frac{\epsilon}{2})^2 + \exp(-\frac{nl}{2})(\cosh \frac{\epsilon}{2})^2 \end{bmatrix}$$



and (see [F-L-P 79], p. 151)  $2 \cosh d(i, A(Q_n))$  is the sum of the squares of the coefficient of the above matrix, that is,

$$\begin{aligned} \cosh d(P_1, Q_n) &= \cosh(nl)(\cosh(\epsilon))^2 + (\sinh(\epsilon))^2 \\ &= \cosh(nl)\left(\coth\left(\frac{l}{2}\right)\right)^2 + \frac{1}{(\sinh(\frac{l}{2}))^2}. \end{aligned}$$

We see that  $d(P_1, Q_n) \geq nl$  so, assuming without loss of generality that  $n \geq k$ ,

$$\frac{n+k-1}{d(P_1, Q_n)d(P_k, Q_1)} \leq \frac{1+\frac{k-1}{n}}{ld(P_k, Q_1)} \leq \frac{2}{ld(P_k, Q_1)} \leq \frac{2}{ld(P_1, Q_1)}.$$

Now  $d(P_1, Q_1)$  is equivalent to  $|\log l|$  when  $l$  goes to zero so there exists some universal constant  $c$  such that  $d(P_1, Q_1) \geq c|\log l|$  whence

$$\frac{n+k-1}{d(P_1, Q_n)d(P_k, Q_1)} \leq \frac{2c}{l|\log l|}.$$

This establishes the upper bound in equation 9.

Now for the lower bound. By a theorem of P. Buser ([Bu], p. 125) we can find a collection of geodesics  $\gamma_2, \dots, \gamma_{3\zeta-3}$  of length  $\leq 26(\zeta-1)$ , such that  $\gamma_1, \dots, \gamma_{3\zeta-3}$  cut  $M$  into pair of pants  $P_1, \dots, P_{2\zeta-2}$ . We exhibit a curve  $\alpha$  intersecting  $\gamma_1$  exactly once, and estimate its length  $l$ . Then  $\alpha$  cannot separate, and we have  $K(g) \geq 1/l_1l$ .

Let us assume that  $\gamma_1$  belongs to two pair of pants  $P_1$  and  $P_2$ ; if it belongs to only one we proceed similarly. Begin with the case when  $P_1$  and  $P_2$  have another common boundary component  $\gamma_2$ . Call  $\gamma_3$  the third boundary component of  $P_1$ . According to [F-L-P 79], p. 152, the length  $a$  of the common perpendicular to  $\gamma_1$  and  $\gamma_2$  in  $P_1$  is given by

$$\cosh a = \frac{\cosh(\frac{l_3}{2}) + \cosh(\frac{l_2}{2}) \cosh(\frac{l_1}{2})}{\sinh(\frac{l_2}{2}) \sinh(\frac{l_1}{2})} \leq \frac{\cosh(13\zeta-13)(\cosh(13\zeta-13)+1)}{\sinh(\frac{l_1}{2})^2}$$

since  $l_2, l_3 \leq 26(\zeta-1)$  and the function  $\sinh$  increases strictly. We get the same inequality in  $P_2$ . Now we connect the two common perpendiculars by segments of  $\gamma_1$  and  $\gamma_2$  (see figure 3).

The length of the closed curve  $\alpha$  thus obtained is less than

$$2\operatorname{arccosh}\left(\frac{\cosh(13\zeta-13)(\cosh(13\zeta-13)+1)}{\sinh(\frac{l_1}{2})^2}\right) + \frac{l_1}{2} + 13(\zeta-1).$$

The above function of  $l_1$  grows like  $|\log l_1|$  when  $l_1$  goes to zero, whence the conclusion.

If  $P_1$  and  $P_2$  only have one common boundary, we take a sequence  $P_{i_1}, \dots, P_{i_n}$  of pair of pants such that  $i_1 = 1$ ,  $P_{i_k}$  has a common boundary with  $P_{i_{k+1}}$  for  $k < n$ , and  $\gamma_1$  is a common boundary for  $P_{i_1}$  and  $P_{i_n}$  (see figure 3).

The closed curve  $\alpha$  is constructed as above, by gluing segments of boundaries and common perpendiculars. The length is estimated similarly, the constant is just multiplied by  $2\zeta-2$ . This yields the claim.  $\square$

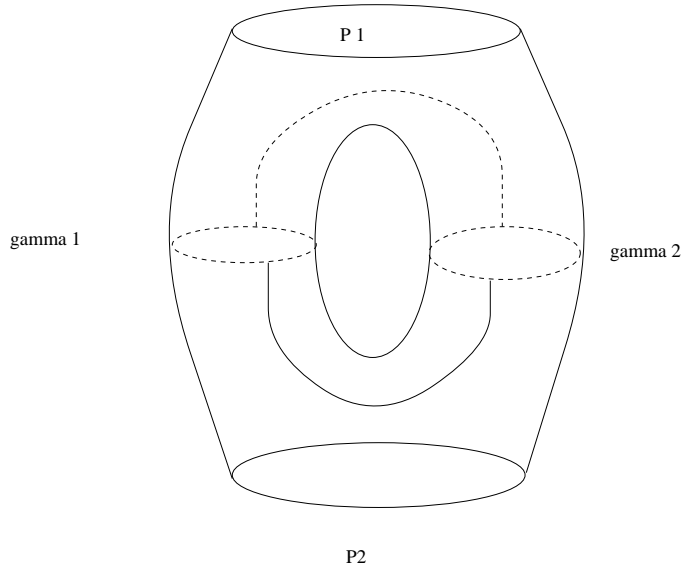


FIGURE 3. connecting  $\gamma_1$  and  $\gamma_2$

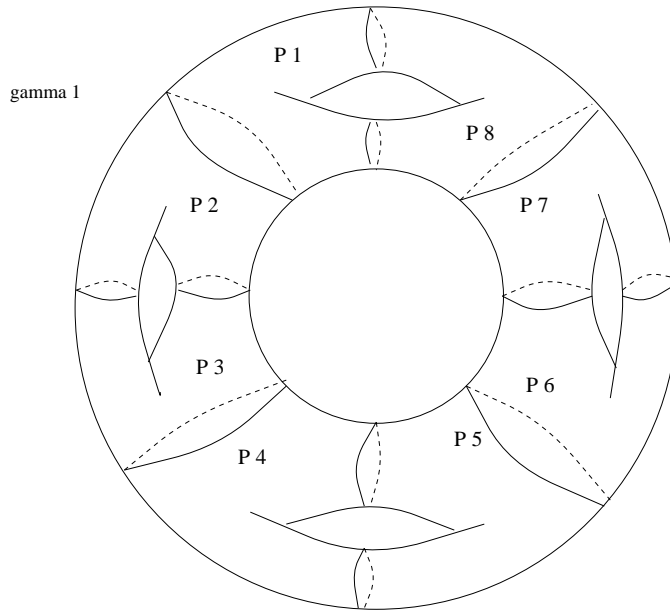


FIGURE 4. a sequence of pair of pants

#### 4. MISCELLANEOUS

We finish with a brief discussion of the behavior of the stable norm, when the metric  $g \in \mathcal{M}$  goes to the boundary,  $\partial\mathcal{M}$ . The set of hyperbolic metrics of curvature  $-1$ , with all lengths of closed curves bounded below by some fixed constant, is compact in moduli space. One may go to infinity in moduli space

by shrinking a separating geodesic or a non-separating one. This induces two different behaviours for the stable norm.

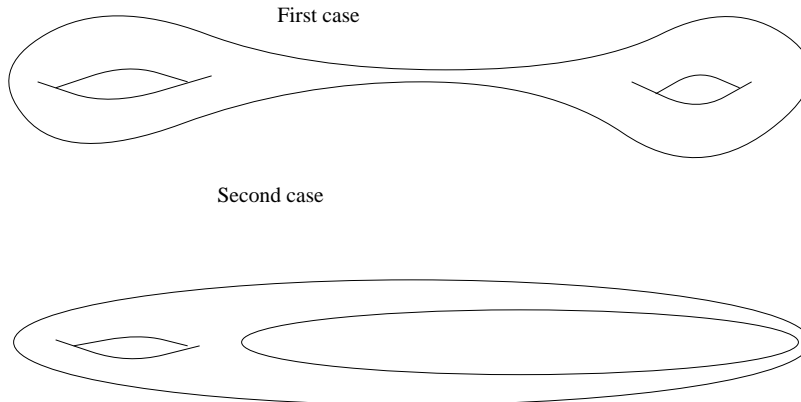


FIGURE 5. degenerating to a cusped surface

**First case** Let  $g_n$  be a sequence of hyperbolic metrics such that there exists a non zero homology class  $h$ , and for each  $n$ , a geodesic  $\gamma_n$  in the class  $h$ , the length  $l_n$  of which goes to zero when  $n$  goes to infinity. Then the stable norm of  $h$  with respect to  $g_n$  goes to zero, and the sequence of stable norms corresponding to  $g_n$  does not converge.

This is reflected in the fact that the surface  $M$  converges to a surface with two cusps. For such a surface the stable norm is not defined, since the boundary of a cusp region is non homologically trivial but can be made arbitrarily short. Note that the stable norm is continuous as a function of the metric.

**Second case** Let  $g_n$  be such that the lengths of all non-separating geodesics are bounded away from zero. Let  $\gamma$  be a free homotopy class of separating geodesics, the length  $l_n$  of which goes to zero. Now the point is, for  $n$  large enough, no geodesic multicurve minimising length in its homotopy class intersects  $\gamma$ . Indeed by the collar lemma (see [Bu]) there exists an embedded collar neighborhood  $C$  of  $\gamma$ , of width  $w_n = \operatorname{argsinh}(1/\sinh(l_n))$ . The boundary components of  $C$  have length  $l_n \cosh(w_n)$ , which is bounded above by some constant  $K$ . Assume that a geodesic multicurve  $\delta$  intersects  $\gamma$ . Since  $\gamma$  separates, the intersection number must be even. We erase its segments contained in  $C$ , and connect the remainder with segments of boundary components. The multicurve thus obtained is homologous to  $\delta$ , and is shorter if  $w_n \geq K/2$ .

Let  $M_1, M_2$  be the two surfaces with boundary obtained by cutting  $M$  along  $\gamma$ , endowed with the metric induced by  $g_n$ . When  $w_n \geq K/2$ , the stable norm on  $M$  splits as the direct sum of the stable norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  for  $M_1$  and  $M_2$ . This is because  $H_1(M, \mathbb{R}) = H_1(M_1, \mathbb{R}) \oplus H_1(M_2, \mathbb{R})$ . Let  $h = (h_1, h_2) \in H_1(M, \mathbb{R})$ . If  $w_n \geq K/2$ , by the above remark, a minimising representative of  $h$  is the union of length-minimizing representatives of  $h_1$  and  $h_2$ . Then  $\|h\| = \|h_1\|_1 + \|h_2\|_2$ .

Note that this implies that the stable norm is not differentiable at any class of the form  $(h_1, 0)$  or  $(0, h_2)$ . Indeed we have  $\|(h_1, 0) + t(0, h_2)\| = \|h_1\|_1 + |t| \|h_2\|_2$  which is not differentiable as a function of  $t$  for  $t = 0$ .

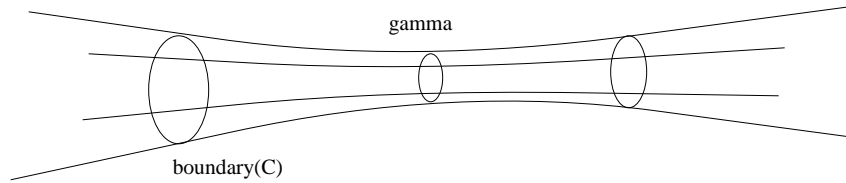


FIGURE 6. taking shortcuts

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