LOCAL SPECTRA: CONTINUITY AND RESOLVENT EQUATION

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Abstract

Let $T \in B(X)$ and $x \in X$. In this paper we consider continuity of local spectra like the mappings $T \mapsto \sigma_T(x)$ and $x \mapsto \sigma_T(x)$ started in [2]. Also, we introduce one generalization of the well-known first resolvent equation by means of local resolvent function.

1. Introduction

Throughout this note let X be Banach space, let $B(X)$ denote the set of bounded linear operators on X. If $T \in B(X)$, let $\sigma(T)$ and $\rho(T)$ denote the spectrum and resolvent set of T, respectively. Also, with $r(T)$ we denote spectral radius of T, i.e. $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \inf_{n\in\mathbf{N}} \|T^n\|^{1/n}$ and $k(T)$ denote the lower bound, $k(T) = \inf\{\|Tx\| \ : \ x \in X \text{ with } \|x\| = 1\}$. Let $i(T) = \lim_{n\to\infty} k(T^n)^{1/n} = \sup_{k\in\mathbf{N}} k(T^n)^{1/n}$ (see [4, pg. 77]).

We say that $T \in B(X)$ has the *single valued extension property* (SVEP) if for every open set U of $\mathbf{C}$ the only analytic function $f : U \longrightarrow X$ which satisfies the equation

$$(T-\lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$ on U, and T satisfies (Dunford's) condition (C) if the set

$$\chi_T(F) = \{x \in X : \text{there exists an X-valued analytic function } f : \mathbf{C}\backslash F \to X \ \text{ such that} (T-\lambda)f(\lambda) = x\}$$

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is closed for every subset $F \subset \mathbf{C}$. In this case, for $x \in X$ there is a maximal analytic function $f_x : U_x \to X$ satisfying $(T-\lambda)f_x(\lambda) = x$ on U_x and we call this function *the local resolvent function* with respect of T at $x \in X$. Set $\sigma_T(x) = \mathbf{C} \setminus U_x$ and $\rho_T(x) = \mathbf{C} \setminus \sigma_T(x)$. Then $\sigma_T(x)$ is called the *local spectrum* and $\rho_T(x)$ is called the *local resolvent set* for T at x.

2. Continuity of local spectra

In [2] Dollinger and Oberai give examples that in general the mapping $T \mapsto \sigma_T(x)$ is not neither lower and upper semi-continuous (see Examples 1 and 2 in [2]). Also they gave an example shoing that the mapping $x \mapsto \sigma_T(x)$ is not in general lower semi-continuous and that for $x = 0$ is never upper semi-continuous (see [2, Corollary 3.3]). In this paper we will give conditions for continuity of those mappings and we will show that in some class of operators they are continuous

Theorem 2.1 *Let operators $T_n \in B(X)$, $n = 0, 1, 2, \ldots$, have the SVEP, T_n in norm converge to T_0 and λ_0 be an isolated point in $\sigma(T_0)$. If for some $x \in X$, $\lambda_0 \in \sigma_{T_0}(x)$, then $\lambda_0 \in \liminf \sigma_{T_n}(x)$.*

Proof. Since $\lambda_0 \in iso\ \sigma(T_0)$, then there exist $\epsilon > 0$ and a subset $\sigma_0 \subset \mathbf{C}$ such that $\sigma(T_0) = \{\lambda_0\} \cup \sigma_0$ and $dist(\lambda_0, \sigma_0) \geq 3\epsilon$. By the upper semi-continuity of the spectrum (see [5, Theorem 1]) there exists a positive integer n_0 such that for every $n \geq n_0$ follows that $\sigma(T_n) \subset (\sigma(T_0))_\epsilon = B(\lambda_0, \epsilon) \cup (\sigma_0)_\epsilon$. Let $K = \{\lambda \in \mathbf{C} : |\lambda - \lambda_0| = 2\epsilon\}$. Then K is a simple closed rectifiable curve containing λ_0 in its interior $K \cap \sigma(T_0) = \emptyset$ and for every $n \geq n_0$, $K \cap \sigma(T_n) = \emptyset$.

Suppose that $\lambda_0 \notin \liminf \sigma_{T_n}(x)$. Then there exist a positive integer n_1 such that $\lambda_0 \notin \sigma_{T_n}(x)$, for every $n \geq n_1$. For simplicity of notation we may assume that $\max\{n_0, n_1\} = 1$. For every $\mu \in K$ there $(T_i - \mu)^{-1}$ exists, for all $i = 0, 1, 2, \ldots$. Since K is a compact subset in complex plane, $\|T_0 - \mu\|$ is uniformly bounded on K and $T_n - \mu \to T_0 - \mu$ uniformly on K. Now by [5, Lemma 2] it follows that $(T_n - \mu)^{-1} \to (T_0 - \mu)^{-1}$ uniformly on K and, hence, $(T_n - \mu)^{-1}x \to (T_0 - \mu)^{-1}x$ uniformly on K. Let f_n be local resolvent of T_n respect to x. Then $f_n(\mu) \equiv (T_n - \mu)^{-1}$ on K and $f_n(\mu) \to f_0(\mu) \equiv (T_0 - \mu)^{-1}$ uniformly on K. Since the holomorphic functions f_n converge uniformly on the simple closed rectifiable curve K it follows by [3, Theorem 3.11.6] that the sequence $\{f_n\}$ converges uniformly to a vector valued holomorphic function f_0 in the interior of K. We have for this function f_0

$$x = (T_n - \mu)f_n(\mu) \to (T_0 - \mu)f_0(\mu),$$

for all μ inside the curve K. Hence, $\lambda_0 \notin \sigma_{T_0}(x)$ which proves the theorem. $\square$

Now we can prove Theorem 2.2 from [2] like as easy consequence of the previous theorem.

Corollary 2.2 *Let $T_0 \in B(X)$ has totally disconnected spectrum and let $\{T_n\}$ be a sequence of operators in $B(X)$ with SVEP such that $T_n \to T$. Then for every $x \in X$, $\sigma_{T_0}(x) \subset \liminf \sigma_{T_n}(x)$.*

Definition 2.3 *With $\mathcal{I}(X)$ we denote the set of all operators $T \in B(X)$ such that $\cap_{n\in\mathbf{N}} T^n(X) = \{0\}$ and $i(T) = r(T)$.*

Theorem 2.4 *Let $T \in B(X)$ be a point of spectral continuity. If $\{T_n\}$ is a sequence of operators from $\mathcal{I}(X)$ such that T_n converges in norm to the operator T, then $\sigma_T(x) \subset \liminf \sigma_{T_n}(x)$ for every nonzero $x \in X$.*

Moreover, if $T \in \mathcal{I}(X)$, then $\lim \sigma_{T_n}(x) = \sigma_T(x)$, for every nonzero $x \in X$.

Proof. Let $\{T_n\} \subset \mathcal{I}(X)$. Then by [4, Proposition 1.6.5] it follows that $\sigma_{T_n}(X) = \sigma(T_n)$ for every $x \in X \setminus \{0\}$ and for all n. Now let $\lambda \in \sigma_T(x)$, then we have $\lambda \in \sigma_T(x) \subset \sigma(T) \subset \liminf \sigma(T_n) = \liminf \sigma_{T_n}(x)$.

Consider $T \in \mathcal{I}(X)$. Then the equality holds $\sigma_T(x) = \sigma(T)$, for every nonzero $x \in X$ and

$$\limsup \sigma_{T_n}(x) = \limsup \sigma(T_n) \subset \sigma(T) = \sigma_T(x).$$

Hence, the mapping $T \mapsto \sigma_T(x)$ is continuous on the set $\mathcal{I}(X)$ for every $x \in X \setminus \{0\}$. □

Theorem 2.5 *The mapping $T \mapsto \sigma_T(x)$ is continuous over the class of all non-normal, hyponormal weighted shifts on H, for every nonzero $x \in H$, where H is Hilbert space.*

Proof. By [1, Theorem 4.5] we have that the spectrum is a continuous mapping over the class of non-normal, hyponormal weighted shifts on H, and by [7, Theorem 2.5] it follows that $\sigma_T(X) = \sigma(T)$ for every $x \in X \setminus \{0\}$ and for every non-normal, hyponormal weighted shift T. Hence, form every sequence of non-normal, hyponormal weighted shifts $\{T_n\}$ which in norm converge to an operator T from the same class we have $\lim \sigma(x, T_n) = \lim \sigma(T_n) = \sigma(T) = \sigma(x, T)$, for every $x \in H \setminus \{0\}$. □

Lemma 2.6 *Let $T \in B(X)$ and let $x_0 \in X \setminus \{0\}$ be a vector such that $\sigma_T(x_0) = \sigma(T)$. Then the mapping $x \mapsto \sigma_T(x)$ is upper semi-continuous at x_0.*

Proof. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ in norm. Since $\sigma(T) = \sigma_T(x_0)$ we have $\limsup \sigma_T(x_n) \subset \limsup \sigma(T) \subset \sigma(T) = \sigma_T(x_0)$, i.e. $x \mapsto \sigma_T(x)$ is upper semi-continuous at $x_0 \in X \setminus \{0\}$. □

Corollary 2.7 *If $T \in \mathcal{I}(X)$, then the function $x \mapsto \sigma_T(x)$ is continuous at each $x \in X \setminus \{0\}$.*

Proof. Since every $T \in \mathcal{I}(X)$ satisfies Dunford's condition (C) (see [4, Proposition 1.6.5]), by [2, Theorem 3.1] it follows that the mapping $x \mapsto \sigma_T(x)$ is lower semi-continuous at each $x \in X$. The rest of the proof follows by Lemma 2.6 □

Corollary 2.8 *If $T \in B(X)$ satisfies Dunford's condition (C), then the set*

$$S = \{x \in X \ : \ x \textit{ is the point of continuity of mapping } x \mapsto \sigma(x, T)\}$$

is of the second category in X.

Proof. By [4, Proposition 1.3.2] the set $S = \{x \in X \ : \ \sigma(x, T) = \sigma_{su}(T))\}$ is of the second category in X, then by [2, Theorem 3.1] and Lemma 2.6 it follows that every $x \in S$ is a point of continuity of the mapping $x \mapsto \sigma(x, T)$. □

3. Local resolvent equation

Let $T \in B(X)$. Then it is well known that for every $\lambda, \mu \in \rho(T)$, the first resolvent equation

$$R_T(\lambda) - R_T(\mu) = (\lambda - \mu)(R_T(\lambda) - R_T(\mu)),$$

holds where $R_T(\lambda) = (T - \lambda)^{-1}$ denotes the resolvent function of T at $\lambda \in \rho(T)$.

Trying to find a generalization of this equation in the context of local spectra theory, we can ask if a similar equation holds for the local resolvent function, i.e. if the *first local resolvent equation*

$$\hat{x}_T(\lambda) - \tilde{x}_T(\mu) = (\lambda - \mu)\hat{x}_T(\lambda)\tilde{x}_T(\mu)$$

holds, for every $\lambda, \mu \in \rho_T(x)$, where $\hat{x}_T(\cdot)$ and $\tilde{x}_T(\cdot)$ are the local resolvent functions of T at $x \in X$ in some neighborhood of λ and μ, respectively. We will give a condition for this local resolvent equation to hold, but in general this question is still open.

Theorem 3.1 *Let $T \in B(X)$ has SVEP and $x \in X \setminus \{0\}$. Then first local resolvent equation holds for every $\lambda, \mu \in \rho_T(x)$ satisfying $\lambda, \mu \in \rho(T)$ or $\lambda, \mu \in iso\ \sigma(T)$.*

Proof. If $\lambda, \mu \in \rho(T) \cap \rho_T(x)$, then it is well known that the local resolvents are equal to $\tilde{x}_T(\lambda) = (T - \lambda)^{-1}x$ and $\tilde{x}_T(\mu) = (T - \mu)^{-1}x$, respectively. Now, the local resolvent equation holds by first resolvent equation.

Let $\lambda \in \rho_T(x)$ be an isolated point in $\sigma(T)$ and let $\tilde{x}_T(\cdot)$ be the local resolvent function in some neighborhood U of λ. Then for every $\eta \in U \setminus \{\lambda\}$ we have $\tilde{x}_T(\eta) = (T-\eta)^{-1}x$, but every branch of an analytic function $\tilde{x}_T(\cdot)$ has not necessary an inverse in U. Hence by [6, pg. 254] we have that the analytic function $\tilde{x}_T(\cdot)$ is constant in U, i.e. $\tilde{x}_T(\eta) = y_0$ for every $\eta \in U$, and also $\tilde{x}_T(\lambda) = y_0$. Let $\mu \in \rho_T(x) \cap \rho(T)$ and let $\hat{x}_T(\cdot)$ be the local resolvent function in some neighborhood of μ. Then, for every $\eta \in U \setminus \{\lambda\}$ we have

$$\hat{x}_T(\mu) - \tilde{x}_T(\lambda) = \hat{x}_T(\mu) - y_0 = \hat{x}_T(\mu) - \tilde{x}_T(\eta). \tag{1}$$

By the first part of the proof, for η and μ the local resolvent equation holds, i.e.

$$\hat{x}_T(\mu) - \tilde{x}_T(\eta) = (\mu-\eta)\hat{x}_T(\mu)\cdot\tilde{x}_T(\eta) = (\mu-\eta)\hat{x}_T(\mu)\cdot y_0. \tag{2}$$

By equations (1) and (2) it follows that the local resolvent equation holds in the case when $\lambda \in \rho_T(x) \cap iso\ \sigma(T)$ and $\mu \in \rho_T(x) \cap \rho(T)$. The case when $\lambda, \mu \in \rho_T(x) \cap iso\ \sigma(T)$ is similar to the previous case. $\square$

Example 3.2 Let T be an operator on three-dimensional complex space with matrix representation

$$T = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

with respect to the standard basis. For the vector $x = (0,1,0)$ we have $\sigma_T(x) = \{1\}$ and of course $\sigma(T) = \{1/2, 1, 2\}$. The local resolvent functions for T at $x = (0,1,0)$ in some neighborhood of 1/2 and 2 are respectively constant functions $\hat{x}_T(\cdot) \equiv (0,2,0)$ and $\tilde{x}_T(\cdot) \equiv (0,-1,0)$. It is easy to check that

$$\hat{x}_T(1/2) - \tilde{x}_T(2) = (0,3,0) = (\frac{1}{2} - 2)\cdot\hat{x}_T(1/2)\cdot\tilde{x}_T(2),$$

i.e. the local resolvent equation holds for 1/2 and 2.

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