PARTIALLY HYPERBOLIC GEODESIC FLOWS ARE ANOSOV

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ABSTRACT. We prove that if a $\mathbb{Z}$ or $\mathbb{R}$-action by symplectic linear maps on a symplectic vector bundle E has a weakly dominated invariant splitting $E = S \oplus U$ with $\dim U = \dim S$, then the action is hyperbolic. In particular, contact and geodesic flows with a dominated splitting with $\dim S = \dim U$ are Anosov.

Celebrating the 60th birthday of Jacob Palis.

Les flots géodésiques partiellement hyperboliques sont Anosov.

RÉSUMÉ. Considérons une action de $\mathbb{R}$ ou $\mathbb{Z}$ sur un fibré vectoriel muni d'une struture symplectique par des applications linéaires préservant cette structure symplectique, et supposons que cette action possède une décomposition invariante faiblement dominée $E = S \oplus U$ avec $\dim U = \dim S$. On montre alors cette action est nécessairement hyperbolique.

En l'honneur du 60^{eme} *anniversaire de J. Palis.*

Version française abrégée.

Un fibré vectoriel symplectique $\pi : \mathbf{E} \to B$ est un fibré vectoriel dont les changements de cartes $(U \cap V) \times \mathbb{R}^{2N} \hookleftarrow$ préservent la structure symplectique standard de $\mathbb{R}^{2N}$. Un tel fibré est donc équipé d'une forme symplectique continue dont la valeur sur chaque fibre est induite par la forme symplectique standard de $\mathbb{R}^{2N}$.

Soit $Sp(\mathbf{E})$ l'espace des isomorphismes du fibré vectoriel symplectique $\mathbf{E}$. Considérons une $\mathbb{R}$-action $\Psi : \mathbb{R} \to Sp(\mathbf{E})$ et pour tout $t \in \mathbb{R}$, notons Ψ_t l'isomorphisme de fibré associé. Cette action induit un flot continu $\psi_t : B \hookleftarrow$ qui vérifie $\psi_t \circ \pi = \pi \circ \Psi_t$.

On dit que l'action Ψ est *faiblement partiellement hyperbolique*[1] s'il existe une décomposition invariante du fibré $\mathbf{E} = S \oplus U$, non nécessairement continue, telle que pour tout $b \in B$,

(1) $\{0\} \neq S(b) \neq \mathbf{E}(b)$.
(2) $\inf_{t \geq 0} \left\| \Psi_t|_{S(b)} \right\| \cdot \left\| \Psi_{-t}|_{U(\psi_t b)} \right\| = 0$.
(3) $\inf_{t \geq 0} \left\| \Psi_t|_{S(\psi_{-t} b)} \right\| \cdot \left\| \Psi_{-t}|_{U(b)} \right\| = 0$.

Une action Ψ est dite *hyperbolique* s'il existe une décomposition invariante du fibré $\mathbf{E} = E^s \oplus E^u$ et des constantes $C > 0$, $\lambda > 0$, telles que:

(i) $|\Psi_t(\xi)| \leq C\, e^{-\lambda t}\, |\xi|$ pour tout $t > 0$, $\xi \in E^s$;
(ii) $|\Psi_{-t}(\xi)| \leq C\, e^{-\lambda t}\, |\xi|$ pour tout $t > 0$, $\xi \in E^u$.

Dans ce cas il est facile de montrer que la décomposition est continue et que les sous-fibrés E^s et E^u sont lagrangiens.

Enfin rappelons que l'*ensemble non errant* $\Omega(\psi)$ de ψ est l'ensemble des points $b \in B$ tels que pour tout voisinage U de b il existe $T > 0$ vérifiant $\psi_T(U) \cap U \neq 0$.

Dans cette note, nous montrons le résultat suivant:

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[1] La définition classique de l'hyperbolicité partielle demande la condition plus forte qu'il existe $t > 0$ et $0 < \lambda < 1$ tels que: $\left\| \Psi_t|_{S(b)} \right\| \cdot \left\| \Psi_{-t}|_{U(\psi_t b)} \right\| < \lambda$ pour tout $b \in B$. Cette condition supplémentaire implique la continuité de la décomposition.

Théorème A. *Soit* $\pi : \mathbf{E} \to B$ *un fibré vectoriel symplectique de base* B *compacte et* $\Psi : \mathbb{R} \to Sp(\mathbf{E})$ *une* $\mathbb{R}$*-action continue. Si* Ψ *est faiblement partiellement hyperbolique,* $\dim S = \dim U$ *et* $\Psi|_{\pi^{-1}(\Omega)}$ *est la restriction à l'ensemble non-errant* $\Omega = \Omega(\psi|_B)$ *du flot induit* ψ*, alors* $\Psi|_{\pi^{-1}(\Omega)}$ *est hyperbolique.*

En fait, la restriction à l'ensemble non-errant n'est pas nécessaire si on sait a priori que la décomposition est continue:

Théorème B. *Soit* Ψ *une* $\mathbb{R}$*-action symplectique et continue sur une base compacte* B*. S'il existe deux sous-fibrés continus* S *et* U *tels que* $\dim S = \dim U$ *et tels que* $S \oplus U$ *est une décomposition invariante faiblement partiellement hyperbolique, alors* Ψ *est hyperbolique sur toute la base* B*.*

Il se trouve que lorsque la décomposition $S \oplus U$ est continue, elle coincide avec la décompostion hyperbolique: $S = E^s$ et $U = E^u$. En particulier les fibrés S et U sont lagrangiens.

Des preuves analogues permettent d'énoncer les deux théorèmes ci-dessus dans le cadre des Z-actions: $\mathbb{Z} \to Sp(\mathbf{E})$, et en particulier pour la différentielle de symplectomorphismes.

On peut appliquer le théorème A lorsque $\Psi_t = d\phi_t$, où ϕ_t est un flot de contact sur une variété compacte $\mathcal{N}$ préservant une 1-forme non dégénérée Θ et $\mathbf{E}(x) = \ker \Theta_x$. Puisque ϕ_t préserve la forme volume $\Theta \wedge (d\Theta)^n$, $\dim \mathcal{N} = 2n+1$, nous savons par le théorème de récurrence de Poincaré que $\Omega(\phi) = \mathcal{N}$. Nous avons alors:

Corollaire. *Un flot de contact faiblement partiellement hyperbolique vérifiant* $\dim S = \dim U$ *est Anosov.*

Un exemple de flot de contact est donné par le flot géodésique sur une variété Riemannienne compacte (M, g) avec la 1-forme $\Theta_v(\xi) = \langle v, d\pi(\xi)\rangle_g$, $\xi \in T_v SM$, $SM = \{v \in TM |\, \|v\|_g = 1\}$; et la projection $\pi : SM \to M$. Ceci répond à une question de M. Herman.

Les deux théorèmes ci-dessus peuvent s'appliquer au cas d'un niveau d'energie régulier d'un champ Hamiltonien qui ne possède pas de fibré transverse continu. Voir version en anglais.

1. STATEMENTS

A symplectic vector bundle $\pi : \mathbf{E} \to B$ is a vector bundle whose transition maps $(U \cap V) \times \mathbb{R}^{2N} \hookleftarrow$ preserve the canonical symplectic structure of $\mathbb{R}^{2N}$ on the fibers. Such bundle carries a continuous symplectic form on each fiber induced by the symplectic form on $\mathbb{R}^{2N}$.

Consider a continuous $\mathbb{R}$-action $\Psi : \mathbb{R} \to Sp(\mathbf{E})$, where $\Psi_t : \mathbf{E} \to \mathbf{E}$ is a bundle map which is a symplectic linear isomorphism on each fiber and $\Psi_{s+t} = \Psi_s \circ \Psi_t$. The action Ψ induces a continuous flow $\psi_t : B \hookleftarrow$ such that $\psi_t \circ \pi = \pi \circ \Psi_t$.

We say that the action Ψ is *weakly partially hyperbolic*[2] if there exists an invariant splitting $\mathbf{E} = S \oplus U$ (not necessarily continuous) such that for each $b \in B$,

(1) $\{0\} \neq S(b) \neq \mathbf{E}(b)$.
(2) $\inf_{t \geq 0} \left\| \Psi_t|_{S(b)} \right\| \cdot \left\| \Psi_{-t}|_{U(\psi_t b)} \right\| = 0$.
(3) $\inf_{t \geq 0} \left\| \Psi_t|_{S(\psi_{-t} b)} \right\| \cdot \left\| \Psi_{-t}|_{U(b)} \right\| = 0$.

We say that Ψ is *hyperbolic* if there exists a invariant splitting $\mathbf{E} = E^s \oplus E^u$ and constants $C > 0$, $\lambda > 0$, such that

(i) $|\Psi_t(\xi)| \leq C\, e^{-\lambda t}\, |\xi|$ for all $t > 0$, $\xi \in E^s$;
(ii) $|\Psi_{-t}(\xi)| \leq C\, e^{-\lambda t}\, |\xi|$ for all $t > 0$, $\xi \in E^u$.

It follows that the hyperbolic splitting $E^s \oplus E^u$ is necessarily continuous and that the subspaces E^s and E^u are lagrangian.

Define the *non-wandering* set $\Omega(\psi)$ of ψ as the set of points $b \in B$ such that for every neighbourhood U of b there exists $T > 0$ such that $\psi_T(U) \cap U \neq 0$.

Here we prove:

[2]The usual partial hyperbolicity requires the stronger condition that there exist $t > 0$ and $0 < \lambda < 1$ such that $\left\| \Psi_t|_{S(b)} \right\| \cdot \left\| \Psi_{-t}|_{U(\psi_t b)} \right\| < \lambda$, for all $b \in B$; which implies the continuity of the splitting $S \oplus U$.

Theorem A. *Let $\pi : \mathbf{E} \to B$ be a continuous symplectic vector bundle with B compact and $\Psi : \mathbb{R} \to Sp(\mathbf{E})$ a continuous $\mathbb{R}$-action with induced flow ψ_t: $\pi \circ \Psi_t = \psi_t \circ \pi$.*

If Ψ is weakly partially hyperbolic with $\dim S = \dim U$ and $\Omega = \Omega(\psi|_B)$ is the non-wandering set then the restricted action $\Psi|_{\pi^{-1}(\Omega)}$ is hyperbolic.

The restriction to the non-wandering set is not needed if we know a priori that the splitting is continuous:

Theorem B. *Let Ψ is a continuous symplectic $\mathbb{R}$-action on a compact base B. Suppose that $S \oplus U$ is a weakly partially hyperbolic splitting with $\dim S = \dim U$. If furthermore, S and U are continuous subbundles, then Ψ is hyperbolic over all B.*

It turns out that when the splitting $S \oplus U$ is continuous, necessarily $S = E^s$ and $U = E^u$ in the hyperbolic splitting. In particular they are lagrangian subbundles.

The theorems above, with the same proofs, hold for $\mathbb{Z}$-actions: $\mathbb{Z} \to Sp(\mathbf{E})$. In particular, for the derivatives of symplectic diffeomorphisms.

We say that $\Psi : \mathbb{R} \to Sp(\mathbf{E})$ is *partially hyperbolic* if there is an invariant splitting $\mathbf{E} = S \oplus U$ and $\tau > 0$, $0 < \lambda < 1$ such that

$$\left\| \Psi_t|_{S(b)} \right\| \cdot \left\| \Psi_{-t}|_{U(\psi_t b)} \right\| < \lambda, \qquad \text{for all } b \in B.$$

A partially hyperbolic action is also weakly partially hyperbolic and its splitting $S \oplus U$ is necessarily continuous. Thus we get

Corollary 1.
A partially hyperbolic symplectic action with $\dim S = \dim U$ on a compact base B is hyperbolic.

We can apply theorem A to the case when $\Psi_t = d\phi_t$, where ϕ_t is a contact flow on a compact manifold $\mathcal{N}$ preserving a non-degenerate 1-form Θ and $\mathbf{E}(x) = \ker \Theta_x$, $x \in \mathcal{N}$. Since ϕ_t preserves the volume form $\Theta \wedge (d\Theta)^n$, $\dim \mathcal{N} = 2n + 1$, by Poincaré recurrence, $\Omega(\phi) = \mathcal{N}$. Hence we obtain

Corollary 2. *A weakly partially hyperbolic contact flow with $\dim S = \dim U$ is Anosov.*

An example of contact flow is the geodesic flow of a compact riemannian manifold (M, g), with the 1-form $\Theta_v(\xi) = \langle v, d\pi(\xi)\rangle_g$, $\xi \in T_v SM$; where $SM = \{v \in TM | \, \|v\|_g = 1\}$ and $\pi : SM \to M$ is the projection. This answers a question posed by M. Herman: is a partially hyperbolic geodesic flow Anosov?[3]

The theorems above can be applied to a regular energy level of a non-contact hamiltonian flow which has no obvious continuous invariant transversal bundle. The problem here is that the tangent space to the energy level is odd-dimensional and the bundle $\mathbf{E} = S \oplus U$ can not contain the direction of the hamiltonian vectorfield. One avoids this problem by projecting the derivative of the hamiltonian flow along a continuous transversal bundle as follows. Let (M, ω) be a symplectic manifold, $H : M \to \mathbb{R}$ and e a regular value of H. Suppose that $\mathcal{N} = H^{-1}(\{e\})$ is compact. Let X be the hamiltonian vector field for H on $\mathcal{N}$, $\omega(X, \cdot) = dH$, and let ϕ be its flow. Let $\mathbf{E}$ be a continuous (non-invariant) subbundle of $T\mathcal{N}$ which is transversal to X (e.g. endow M with a riemannian metric and let $\mathbf{E} = \{\, v \in T\mathcal{N} \,|\, \omega(\nabla H, v) = 0 \,\}$). Then $(\mathbf{E}, \omega|_{\mathbf{E}})$ is a symplectic bundle. Let $\Lambda : T\mathcal{N} = \mathbf{E} \oplus \langle X \rangle \to \mathbf{E}$ be the projection in the splitting. Then we ask that the symplectic action $\Psi = \Lambda \circ d\phi$ on $\mathbf{E}$ is weakly partially hyperbolic. Apply theorem A to show that the action Ψ is hyperbolic. Since the growth of $d\phi_t$ in the direction of the hamiltonian vectorfield is subexponential, then standard methods using a graph transformation (cf. Hirsch-Pugh-Shub [4], or [1, p. 930]) show that if Ψ is hyperbolic then ϕ is Anosov.

[3] If $\dim S \neq \dim U$, a product of two manifolds of negative curvature would be a counterexample.

2. Proofs.

We say that Ψ is *quasi-hyperbolic* if $\sup_{t\in\mathbb{R}}|\Psi_t(\xi)|=+\infty$ for all $\xi\in\mathbf{E}$, $\xi\neq 0$. We shall use:

2.1. **Proposition.** *Let $\pi:\mathbf{E}\to B$ is a continuous vector bundle, and $\Psi:\mathbb{R}\to GL(\mathbf{E})$ a continuous $\mathbb{R}$-action of linear isomorphisms with induced flow ψ_t: $\pi\circ\Psi_t=\psi_t\circ\pi$.*

If B is compact and Ψ is quasi-hyperbolic then the restriction $\Psi|_{\pi^{-1}(\Omega)}$ to the lift of the nonwandering set $\Omega=\Omega(\psi|_B)$ is hyperbolic.

The proof is similar to that of [1, §3 theorem 0.2, p. 926–929] and its origins can be traced back to Eberlein [2], Sacker and Sell [8], [9] and Selgrade [10].

Define

$$\begin{aligned}\mathfrak{S}(b):&=\big\{\,\mathfrak{s}\in S(b)\;\big|\;\forall\mathfrak{u}\in U(b),\ \Omega_b(\mathfrak{s},\mathfrak{u})=0\,\big\},\\ \mathfrak{U}(b):&=\big\{\,\mathfrak{u}\in U(b)\;\big|\;\forall\mathfrak{s}\in S(b),\ \Omega_b(\mathfrak{u},\mathfrak{s})=0\,\big\};\end{aligned}$$

Proof of Theorem A. Fix $b\in B$ and fix sequences $\tau_n\to+\infty$ and $\sigma_n\to+\infty$ such that

$$\lim_n \big\|\Psi_{\tau_n}|_{S(b)}\big\|\cdot\big\|\Psi_{-\tau_n}|_{U(\psi_{\tau_n}b)}\big\|=0, \tag{4}$$

$$\lim_n \big\|\Psi_{\sigma_n}|_{S(\psi_{-\sigma_n}b)}\big\|\cdot\big\|\Psi_{-\sigma_n}|_{U(b)}\big\|=0. \tag{5}$$

Write $\mathfrak{B}(b):=\big\{v\in\mathbf{E}(b)\mid \sup_{t\in\mathbb{R}}|\Psi_t v|<+\infty\,\big\}$. By proposition 2.1, we have to prove that $\mathfrak{B}(b)=\{0\}$.

2.2. **Lemma.**

(a) *If* $\liminf_n \big\|\Psi_{\sigma_n}|_{S(\psi_{-\sigma_n}b)}\big\|=0$ *then*

$$\forall\mathfrak{s}\in S(b)\setminus\{0\},\qquad \limsup_n|\Psi_{-\sigma_n}(\mathfrak{s})|=+\infty. \tag{6}$$

(b) *If* $\liminf_n \big\|\Psi_{-\tau_n}|_{U(\psi_{\tau_n}b)}\big\|=0$ *then*

$$\forall\mathfrak{u}\in U(b)\setminus\{0\},\qquad \limsup_n|\Psi_{\tau_n}(\mathfrak{u})|=+\infty. \tag{7}$$

(c) *If both conditions* (6) *and* (7) *hold, then* $\mathfrak{B}(b)=\{0\}$.

Proof: (a). If $\mathfrak{s}\in S(b)\setminus\{0\}$, then

$$\liminf_n\frac{|\mathfrak{s}|}{|\Psi_{-\sigma_n}(\mathfrak{s})|}=\liminf_n\frac{|\Psi_{\sigma_n}(\Psi_{-\sigma_n}\mathfrak{s})|}{|\Psi_{-\sigma_n}\mathfrak{s}|}\le\liminf_n\big\|\Psi_{\sigma_n}|_{S(\psi_{-\sigma_n}b)}\big\|=0.$$

(b). If $\mathfrak{u}\in U(b)\setminus\{0\}$, then

$$\liminf_n\frac{|\mathfrak{u}|}{|\Psi_{\tau_n}(\mathfrak{u})|}=\liminf_n\frac{|\Psi_{-\tau_n}(\Psi_{\tau_n}\mathfrak{u})|}{|\Psi_{\tau_n}\mathfrak{u}|}\le\liminf_n\big\|\Psi_{-\tau_n}|_{U(\psi_{\tau_n}b)}\big\|=0.$$

(c). Conditions (6) and (7) imply that $\mathfrak{B}(b)\cap S(b)=\{0\}$ and $\mathfrak{B}(b)\cap U(b)=\{0\}$. Let $v=\mathfrak{s}\oplus\mathfrak{u}\in S\oplus U=\mathbf{E}$ be such that $\mathfrak{s}\neq0$ and $\mathfrak{u}\neq0$. From (4) we have that

$$\frac{|\Psi_{\tau_n}\mathfrak{s}\,|}{|\mathfrak{s}|}\cdot\frac{|\mathfrak{u}|}{|\Psi_{\tau_n}\mathfrak{u}\,|}=\frac{|\Psi_{\tau_n}\mathfrak{s}\,|}{|\mathfrak{s}|}\cdot\frac{|\Psi_{-\tau_n}(\Psi_{\tau_n}\mathfrak{u})|}{|\Psi_{\tau_n}\mathfrak{u}\,|}\ \xrightarrow{\ n\ }0.$$

Therefore

$$\lim_n|\Psi_{\tau_n}(\mathfrak{s})|/|\Psi_{\tau_n}(\mathfrak{u})|=0.$$

Using (7), we have that

$$\limsup_n|\Psi_{\tau_n}(v)|\ge\limsup_n\big(|\Psi_{\tau_n}(\mathfrak{u})|-|\Psi_{\tau_n}(\mathfrak{s})|\big)=+\infty. \tag{8}$$

Then $v\notin\mathfrak{B}(b)$. ☐

Suppose first that $\mathfrak{S}(b)=\{0\}$ and $\mathfrak{U}(b)=\{0\}$. Let $\mathfrak{u}\in U(b)$, $\mathfrak{u}\neq 0$. Since $\mathfrak{U}(b)=\{0\}$, there exists $\mathfrak{s}\in S(b)$ such that $\Omega(\mathfrak{s},\mathfrak{u})=1$. Since $|\Psi_{\tau_n}\mathfrak{s}\,|\cdot|\Psi_{\tau_n}\mathfrak{u}\,|\geq\Omega(\Psi_{\tau_n}\mathfrak{s},\Psi_{\tau_n}\mathfrak{u})=\Omega(\mathfrak{s},\mathfrak{u})=1$, then

$$\liminf_n|\Psi_{\tau_n}\mathfrak{s}\,|=0 \quad\Longrightarrow\quad \limsup_n|\Psi_{\tau_n}\mathfrak{u}\,|=+\infty. \tag{9}$$

From (4),

$$\frac{|\Psi_{\tau_n}\mathfrak{s}\,|}{|\mathfrak{s}|}\cdot\frac{|\mathfrak{u}|}{|\Psi_{\tau_n}\mathfrak{u}\,|}=\frac{|\Psi_{\tau_n}\mathfrak{s}\,|}{|\mathfrak{s}|}\cdot\frac{|\Psi_{-\tau_n}(\Psi_{\tau_n}\mathfrak{u})|}{|\Psi_{\tau_n}\mathfrak{u}\,|}\ \xrightarrow{\ n\ }\ 0. \tag{10}$$

So that

$$\liminf_n|\Psi_{\tau_n}\mathfrak{s}\,|>0 \quad\Longrightarrow\quad \limsup_n|\Psi_{\tau_n}\mathfrak{u}\,|=+\infty. \tag{11}$$

From (9) and (11), we obtain that

$$\limsup_n|\Psi_{\tau_n}\mathfrak{u}\,|=+\infty,\quad \forall\mathfrak{u}\in U(b)\setminus\{0\}. \tag{12}$$

Similar arguments to (9) and (11) using (5), show that

$$\limsup_n|\Psi_{-\sigma_n}\mathfrak{s}\,|=+\infty,\quad \forall\mathfrak{s}\in S(b)\setminus\{0\}. \tag{13}$$

Then (12), (13) and lemma 2.2(c) imply that $\mathfrak{B}(b)=\{0\}$.

The hypothesis on S and U are symmetric, so, using the inverse action Ψ_{-t} if necessary, it is enough to assume from now on that $\mathfrak{S}(b)\neq\{0\}$.

Let $v\in\mathfrak{S}(b)\setminus\{0\}$. There exists $w=\mathfrak{s}+\mathfrak{u}\in S\oplus U=\mathbf{E}$ such that $\Omega(v,w)=1$. Since by definition of $\mathfrak{S}(b)$, we have that $\Omega(v,\mathfrak{u})=0$, then

$$1=\Omega(v,\mathfrak{s})=\Omega(\Psi_{\tau_n}v,\Psi_{\tau_n}\mathfrak{s})\leq\left\|\Psi_{\tau_n}|_{S(b)}\right\|^2\ |v|\ |\mathfrak{s}|.$$

Therefore $\liminf_n\left\|\Psi_{\tau_n}|_{S(b)}\right\|>0$. From (4) we have that

$$\liminf_n\left\|\Psi_{-\tau_n}|_{U(\psi_{\tau_n}b)}\right\|=0. \tag{14}$$

Suppose that

$$\liminf_n\left\|\Psi_{-\sigma_n}|_{U(b)}\right\|=0.$$

If $\mathfrak{u}_1,\,\mathfrak{u}_2\in U(b)$, then

$$|\Omega(\mathfrak{u}_1,\mathfrak{u}_2)|=\lim_n|\Omega(\Psi_{-\sigma_n}\mathfrak{u}_1,\Psi_{-\sigma_n}\mathfrak{u}_2)|\leq\liminf_n\left\|\Psi_{-\sigma_n}|_{U(b)}\right\|^2=0.$$

Therefore the subspace $U(b)$ is isotropic. If $0\neq\mathfrak{s}\in\mathfrak{S}(b)$ then, by the definition of $\mathfrak{S}(b)$, the subspace $U(b)\oplus\langle\mathfrak{s}\rangle$ would be isotropic. This contradicts the hypothesis $\dim U=\dim S=\frac{1}{2}\dim\mathbf{E}$. Hence

$$\liminf_n\left\|\Psi_{-\sigma_n}|_{U(b)}\right\|>0. \tag{15}$$

Now (15) and (5) imply that

$$\liminf_n\left\|\Psi_{\sigma_n}|_{S(\psi_{-\sigma_n}b)}\right\|=0. \tag{16}$$

But (14), (16) and lemma 2.2 imply that $\mathfrak{B}(b)=\{0\}$.

□

Proof of theorem B. Define

$$E^s := \{\, \mathfrak{v} \in \mathbf{E} \mid \sup_{t\geq 0} |\Psi_t \mathfrak{v}\,| < +\infty \,\}, \qquad E^u := \{\, \mathfrak{v} \in \mathbf{E} \mid \sup_{t\geq 0} |\Psi_{-t} \mathfrak{v}\,| < +\infty \,\}.$$

2.3. **Lemma.** *If S and U are continuous, then $E^s = S$ and $E^u = U$.*

Proof: The limits (12), [(14), 2.2(b)], (8) and (13), [(16), 2.2(a)], (8) imply that $E^s \subseteq S$ and $E^u \subseteq U$.

We only prove that $E^s = S$. Let $v \in S$ and suppose that $v \notin E^s$. Then there exists $b_n \to +\infty$ such that $\lim_n |\Psi_{b_n} v| = +\infty$. Define $a_n \in [0, b_n]$ by

$$|\Psi_{a_n} v\,| = \max_{0\leq t\leq b_n} |\Psi_t\, v\,|.$$

Then $|\Psi_{a_n} v\,| \geq |\Psi_{b_n} v\,| \xrightarrow{n} +\infty$, so that $\lim_n a_n = +\infty$.

Let $u_n := \frac{\Psi_{a_n} v}{|\Psi_{a_n} v\,|}$, $x_n := \pi\, u_n$. For $-a_n \leq t \leq 0$, we have that

$$|\Psi_t u_n| \leq \frac{|\Psi_{t+a_n} v\,|}{|\Psi_{a_n} v\,|} \leq 1.$$

Since B is compact and $|\mathfrak{u}_n| \equiv 1$, taking a subsequence we can assume that $x_n \to x \in B$ and $u_n \to u \in \mathbf{E}(x)$. Then $|\Psi_t(u)| \leq 1$ for all $t \leq 0$. Using the continuity of $x \mapsto S(x)$, we have that

$$u \in E^u(x) \cap \lim_n S(x_n) = E^u(x) \cap S(x) \subseteq U(x) \cap S(x) = \{0\}.$$

Since $|u| = 1$, this is a contradiction. $\square$

The following lemma completes the proof of theorem B. A proof of the lemma can be found in [1, lemma 3.2, p. 927].

2.4. **Lemma.** *Let Ψ be a quasi-hyperbolic action with B compact. Then there exists $\tau > 0$ such that*

$$\big\| \Psi_\tau|_{E^s(b)} \big\| < \tfrac{1}{2} \quad \textit{and} \quad \big\| \Psi_{-\tau}|_{E^u(b)} \big\| < \tfrac{1}{2} \qquad \textit{for all } b \in B.$$

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