ON MAÑÉ'S LAST CONJECTURE

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Abstract. It is not C^4-generic for Lagrangians to have a minimising periodic orbit.

1. Introduction

Let M be a smooth, closed, connected manifold and L be a Lagrangian on the tangent bundle TM, that is, a $C^r, r \geq 2$ function on TM which is convex and superlinear when restricted to any fiber. The Euler-Lagrange equation then defines a complete flow Φ_t on TM. See [Fa00] and [Mr91] for background and reference.

Call $\mathcal{M}$ the set of compactly supported, Φ_t-invariant probability measures on TM. The objects under scrutiny here are the measures-henceforth called minimising-that minimise the L-action, that is, the integral $\int_{TM}(L)d\mu$, over $\mathcal{M}$.

We say a property is true for a generic Lagrangian if, given a Lagrangian L, there exists a residual (countable intersection of open and dense subsets) subset $\mathcal{O}$ of $C^\infty(M)$ such that the property holds for $L + f, \forall f \in \mathcal{O}$. Mañé ([Mn96]) proved for a generic Lagrangian, there exists a unique minimising measure and in [Mn97] (see also [CDI97]) he made the following

Conjecture 1. *For a generic Lagrangian, there exists a unique minimising measure and it is supported on a periodic orbit.*

Given a Lagrangian L, denote $\mathcal{O}_L$ the set of $f \in C^\infty(M)$ such that $L + f$ has a unique minimising measure, supported on a periodic orbit. To our dismay we are going to disprove the conjecture by proving this

Theorem 2. *There exists a Lagrangian L on the two-torus such that the set $\mathcal{O}_L$ is not dense in the C^4-topology.*

Our Lagrangian is just a flat metric plus a constant 1-form with the ratio of its coefficients irrational and quadratic. The idea is then to use the Diophantine approximation properties of the ratio.

See [Mn96], Problem IV for a slightly weaker conjecture and [Mt02] for a proof thereof when M is a closed, orientable surface.

2. Preliminaries

2.1. **Globally minimising orbits.** Given a C^1 curve γ defined on some compact interval I into M, the L-action of γ is the integral $\int_I L(\gamma, \dot{\gamma})ds$. The curve γ is said to be minimising if it minimises the L-action over all C^1 curves defined over the same interval, with the same endpoints. A C^1

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curve $\gamma\colon \mathbb{R} \longrightarrow M$ is said to be minimising if its restriction to any compact interval is. An orbit $\gamma\colon \mathbb{R} \longrightarrow TM$ is said to be minimising if its projection to M is. We denote by $\mathcal{G}(L)$ the union in TM of all minimising orbits. Note that the support of a minimising measure is always contained in $\mathcal{G}(L)$(see [Fa00]).

We choose an auxiliary Riemann metric on M and denote by d_{TM} the associated distance function on TM.

Proposition 3. *The set $\mathcal{G}(L)$ is upper semi-continuous with respect to the Lagrangian.*

Proof. Here the topology on the $\mathcal{G}$'s is the Hausdorff topology on compact sets, and the topology on Lagrangians is that induced by uniform convergence on compact sets of the Lagrangian and its derivatives up to second order. Then convergence of the Lagrangian ensures uniform convergence of the time t map of the flow.

Let L_n be a sequence of Lagrangians converging to some L and denote by ϕ_t^n the associated flow on TM. First note that by *a priori Compactness* ([Fa00], Corollary 4.1.2) the sets $\mathcal{G}(L_n)$ remain in a bounded region K of TM. So all we have to show is that if a subsequence of the sets $\mathcal{G}(L_n)$ converges, in the Hausdorff topology, to some Λ, then Λ consists of orbits of Φ_t, and those orbits are minimising.

Assume Λ is not Φ_t-invariant. Then there exists a point (x, v) in Λ and $t \in \mathbb{R}$ such that $\epsilon := d_{TM}(\Lambda, \phi_t(x, v)) > 0$. Choose δ such that

$$d_{TM}((x, v), (y, w)) \leq \delta \text{ implies } d_{TM}(\phi_s(x, v), \phi_s^n(y, w)) \leq \epsilon/3 \;\; \forall s \in [0, t].$$

Let N be such that $d_{TM}(\Lambda, \mathcal{G}_n) \leq \min(\delta, \epsilon/3)$ for all $n \geq N$. Take (x_N, v_N) in $\mathcal{G}_N$ such that $d_{TM}((x, v), (x_N, v_N)) \leq \delta$. Then by definition of δ we have $d_{TM}(\phi_t(x, v), \phi_t^N(x_N, v_N)) \leq \epsilon/3$ which implies , $\mathcal{G}_N$ being Φ_t^N-invariant, $d_{TM}(\mathcal{G}_N, \phi_t(x, v)) \leq \epsilon/3$. Then since $d_{TM}(\Lambda, \mathcal{G}_N) \leq \epsilon/3$ we have

$$d_{TM}(\Lambda, \phi_t(x, v)) \leq 2\epsilon/3 < \epsilon,$$

a contradiction.

Assume an orbit δ in Λ is not minimising. Then there exists $s, t \in \mathbb{R}$, $\epsilon > 0$ and a C^1 curve $\gamma\colon [s, t] \longrightarrow M$ such that $\gamma(s) = \delta(s)$, $\gamma(t) = \delta(t)$ and

$$\int_s^t L(\gamma(\tau), \dot{\gamma}(\tau))d\tau + \epsilon \leq \int_s^t L(\delta(\tau), \dot{\delta}(\tau))d\tau.$$

Let N_1 be such that

$$|L_n(x, v) - L(x, v))| \leq \frac{\epsilon}{6|t - s|} \; \forall n \geq N_1, \; (x, v) \in K$$

and let α be such that

$$|L(x, v) - L(y, w))| \leq \frac{\epsilon}{6|t - s|} \; \forall (x, v), (y, w) \in K, d_{TM}((x, v), (y, w)) \leq \alpha.$$

There exists N_2 such that $d_{TM}(\Lambda, \mathcal{G}_n) \leq \alpha$ for all $n \geq N_2$. Then there exists $n \geq \max(N_1, N_2)$ and an orbit δ_n in $\mathcal{G}_n$ such that $d_{TM}(\delta_n(\tau), \delta(\tau)) \leq \alpha$ for all $\tau \in [s, t]$. By adding short segments at the ends of $\gamma([s, t])$ and reparametrising we construct a curve $\gamma_n\colon [s,t] \longrightarrow M$ such that $\gamma_n(s) = \delta_n(s)$, $\gamma_n(t) = \delta_n(t)$. Posibly taking a larger n, we may assume that

$$\int_s^t L_n(\gamma_n(\tau), \dot\gamma_n(\tau))d\tau \le \int_s^t L(\gamma(\tau), \dot\gamma(\tau))d\tau + \frac{\epsilon}{3}.$$

Then we have

$$\begin{aligned}\int_s^t L_n(\delta_n(\tau), \dot\delta_n(\tau))d\tau &\ge \int_s^t L_n(\delta(\tau), \dot\delta(\tau))d\tau - \frac{\epsilon}{6}\\ &\ge \int_s^t L(\delta(\tau), \dot\delta(\tau))d\tau - \frac{\epsilon}{3}\\ &\ge \int_s^t L(\gamma(\tau), \dot\gamma(\tau))d\tau + \frac{2\epsilon}{3}\\ &\ge \int_s^t L_n(\gamma_n(\tau), \dot\gamma_n(\tau))d\tau + \frac{\epsilon}{3}\end{aligned}$$

which is impossible since all orbits in $\mathcal{G}_n$ are minimising. □

2.2. **The Lagrangian.** Let $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ be the standard two-torus, let (x, y, u, v) be coordinates in $T\mathbb{T}^2 = \mathbb{T}^2 \times \mathbb{R}^2$, and let r be a quadratic irrational real number. Let p, q be real numbers such that $p/q = r$ and $p^2 + q^2 = 1$. Let ω be the closed one-form on $\mathbb{T}^2$ defined by $\omega_{(x,y)}(u, v) = pu + qv$. Now let L be the Lagrangian on $\mathbb{T}^2$ defined by

$$L(x, y, u, v) = \frac{1}{2}(u^2 + v^2) - \omega_{(x,y)}(u, v).$$

Since the one-form ω is closed, the orbits of the Euler-Lagrange flow of L are geodesics parametrized with constant speed.

Lemma 4. *The critical value of the Lagrangian L is $1/2$.*

Proof. Let us prove that $c(L) \le 1/2$, that is, $\int L d\mu \ge -1/2$ for all $\mu \in \mathcal{M}$. Since every invariant measure is a convex combination of ergodic measures, it is enough to prove it when μ is ergodic. For such a μ there exists a geodesic with velocity (u, v) such that

$$\int L d\mu = \lim_{T\to\infty} \frac{1}{T}\int_0^T (\frac{1}{2}(u^2 + v^2) - pu - qv)dt = \frac{1}{2}(u^2 + v^2) - pu - qv.$$

Now since $p^2 + q^2 = 1$ we have $pu + qv \le \sqrt{u^2 + v^2}$ so

$$(1/2)(u^2 + v^2) - pu - qv \ge -1/2.$$

Let us prove that $c(L) \ge 1/2$. Consider the probability measure μ in $T\mathbb{T}^2$ evenly distributed on the torus

$$\{(x, y, u, v)\colon\ u^2 + v^2 = 1,\ pu + qv = 1\} = \{(x, y, p, q)\colon\ (x, y) \in \mathbb{T}^2\}.$$

We have $\int L d\mu = -1/2$ so $c(L) = 1/2$ and μ is minimising. □

If f is a function on $\mathbb{T}^2$, it may be viewed as a function on $T\mathbb{T}^2$ by setting $f(x, v) = f(x)$. Then by the Riesz Representation Theorem the map that takes any continuous function f on M to $\int_{T\mathbb{T}^2} f d\mu$ may be represented by a Borel probablity measure Λ on M. Since μ is invariant under the Euler Lagrange flow of L, the measure Λ is invariant under the translation by

(p,q). This translation is ergodic since p/q is irrational so Λ is the normalised Lebesgue measure on $\mathbb{T}^2$.

Lemma 5. *For the Lagrangian L we have $\mathcal{G}(L) = supp(\mu)$.*

Proof. This amount to showing that if $(\gamma, \dot\gamma)$ is a minimising orbit, we have $\dot\gamma(t) = (p,q)$ for all t. Note that an orbit is a geodesic of a flat metric so it has constant velocity. Let (p_1, q_1) be that velocity and $1/2(p_1^2+q_1^2)+pp_1+qq_1 := \epsilon > 0$. Then we have $\int_0^t (L+c(L))(\gamma,\dot\gamma)ds = t\epsilon$ so for t large enough we may connect a long segment of an orbit of velocity (p,q) with a short geodesic segment to get a curve with the same endpoints as $\gamma([0,t])$, same parameter interval and smaller action, showing that γ is not minimising. □

Now assume there exists a sequence of C^2 functions f_n on $\mathbb{T}^2$, and closed curves $\gamma_n\colon [0,T_n] \longrightarrow \mathbb{T}^2$ such that the uniformly distributed probablility measure μ_n on the set $(\gamma_n, \dot\gamma_n)([0,T_n])$ in $T\mathbb{T}^2$ is $L+f_n$-minimising. Then in particular

$$\int_{T\mathbb{T}^2} (L+f_n)d\mu_n \leq \int_{T\mathbb{T}^2} (L+f_n)d\mu = -c(L) + \int_{\mathbb{T}^2} f_n d\Lambda \text{ so}$$

$$\int_{T\mathbb{T}^2} (L+c(L))d\mu_n \leq \int_{\mathbb{T}^2} f_n d\Lambda - \int f_n d\mu_n. \tag{1}$$

2.3. **Evaluation of the left-hand side in Equation 1.** Let $\mathbb{Z}^2$ be identified with $H_1(\mathbb{T}^2, \mathbb{Z})$. Let (p_n, q_n) be the homology class of γ_n. The left-hand side of Equation 1 is minimal when γ_n is a geodesic of the flat metric $\|(u,v)\| = \sqrt{u^2+v^2}$ parametrised with energy $c(L)$, or unit speed. Such a geodesic is given by

$$\begin{array}{rcl} \gamma_n\colon [0, \sqrt{p_n^2+q_n^2}] & \longrightarrow & \mathbb{T}^2 \\ t & \mapsto & \frac{t}{\sqrt{p_n^2+q_n^2}}(p_n,q_n)\mathrm{mod}\mathbb{Z}^2 \end{array}$$

Let r_n be p_n/q_n and ϵ_n be $|r_n - r|$. Since p_n, q_n are integers and $r = p/q$ is a quadratic irrational number, there exists a constant $C_1 > 0$ such that $\epsilon_n \geq C_1/q_n^2$. Then

$$\begin{aligned} \int_{T\mathbb{T}^2} (L+c(L))d\mu_n &= \frac{1}{\sqrt{p_n^2+q_n^2}} \int_0^{\sqrt{p_n^2+q_n^2}} \left(\frac{1}{2} - \frac{pp_n+qq_n}{\sqrt{p_n^2+q_n^2}} + \frac{1}{2} \right) dt \\ &= 1 - \frac{pp_n+qq_n}{\sqrt{p_n^2+q_n^2}} \\ &= q\sqrt{1+r^2} - \frac{rqr_nq_n+qq_n}{\sqrt{r_n^2q_n^2+q_n^2}} \\ &= q\frac{\sqrt{1+r^2}\sqrt{1+r_n^2}-(1+rr_n)}{\sqrt{1+r^2}\sqrt{1+r_n^2}} \\ &= q\frac{\sqrt{1+r^2}\sqrt{1+(r+\epsilon_n)^2}-(1+r(r+\epsilon_n)}{\sqrt{1+r^2}\sqrt{1+(r+\epsilon_n)^2}} \\ &= C_3\epsilon_n^2 + o(\epsilon_n^2) \geq \frac{C_2}{q_n^4} \end{aligned}$$

for some positive constants C_2, C_3.

2.4. Foliation of $\mathbb{R}^2$ by translates of lifts of γ_n.

Lemma 6. *Let $\gamma \colon [0,T] \longrightarrow \mathbb{T}^2$ be a C^1 simple closed curve. If there exists a direction $v \in \mathbb{R}^2$ such that $\dot\gamma(t) \notin \mathbb{R}v$ for all t in $[0,T]$, then $\mathbb{R}^2$ is foliated by translates of a given lift of γ in the direction v.*

Proof. Lift the situation to the universal cover $\mathbb{R}^2$ of $\mathbb{T}^2$, choose a lift of γ and denote it γ. All we have to prove is that $\gamma + s_1 v$ and $\gamma + s_2 v$ are disjoint or equal for all $s_1, s_2 \in \mathbb{R}$. Assume for some $s_1, s_2 \in \mathbb{R}$, $t_1, t_2 \in [0,T]$, we have $\gamma(t_1) + s_1 v = \gamma(t_2) + s_2 v$. Then $\gamma(t_1) - \gamma(t_2) = (s_2 - s_1)v$ so if $s_1 \neq s_2$, by Rolle's theorem there exists $t_3 \in [t_1, t_2]$ such that $\dot\gamma(t_3)$ is colinear to v. □

Now let us prove that for n large enough, γ_n satisfies the sufficient condition of the lemma. By lower semicontinuity of $\mathcal{G}$, we see that any limit point of $\operatorname{supp}(\mu_n)$, in the Hausdorff topology on compact subsets of $T\mathbb{T}^2$, is contained in $\mathcal{G}(L) = \operatorname{supp}(\mu)$. Hence the velocity vectors of γ_n converge uniformly to (p, q) so for large n the direction of the tangent vector $\partial/\partial y$ is missed by $\dot\gamma_n$ for all $t \in [0, T_n]$.

3. Proof of Theorem 2

From now on we work in $\mathbb{R}^2$, still denoting f_n the composition $f_n \circ \pi$, where π is the projection of the universal cover of $\mathbb{T}^2$. So f_n is now a $\mathbb{Z}^2$-periodic function and its mean value on $\mathbb{R}^2$ is well-defined. The mean value of f_n on γ_n and its translates is also well-defined since f_n is periodic on γ_n and its translates.

Define on the line $\{x = 0\}$ the function $\phi_n(y)$ as the mean value of f_n on the leaf of the foliation going through y. Then ϕ_n is C^k if f_n is C^k and $\phi_n^{(k)}(y)$ is the mean value of $\partial^{(k)} f_n/\partial y^{(k)}$ on the leaf of the foliation going through y, that is

$$\forall k \in \mathbb{N},\ \phi_n^{(k)}(y) = \frac{1}{T_n}\int_0^{T_n} \frac{\partial^k f_n}{\partial y^k}(x,y)dx.$$

Note that the C^4-norm of f_n is greater than or equal to that of ϕ_n. Indeed so if for some y we have $\phi_n^{(4)}(y) \geq K$ for some K, then there exists x such that $\partial^k f_n/\partial y^k(x,y) \geq K$. Besides the mean value of ϕ_n over $\{x = 0\}$ equals the mean value of f_n over $\mathbb{T}^2$. Since ϕ_n is 1-periodic and C^∞, for any k, $\phi_n^{(k)}$ vanishes at least once in $[0,1]$.

Assume for definiteness that γ_n crosses $\{x = 0\}$ at $y = 0$. Since γ_n is minimising in particular it minimises among its translates so we may assume, up to adding a constant, that $\phi_n \geq 0 = \phi_n(0)$. Since γ_n crosses $\{x = 0\}$ q_n times, there exists at least one interval in $\{x = 0\}$ of length $a_n \leq 1/q_n$ which is crossed exactly once by all leaves of the foliation. Assume for simplicity that this interval is $[0, a_n]$. So every value of ϕ_n and its derivatives is taken at least once in $[0, a_n]$. Thus for every k in $\mathbb{N}$, there exists x_k in $[0, a_n]$ such that $\phi_n^{(k)}(x_k) = 0$.

The following Proposition shows that ϕ_n, hence f_n, does not go to zero in the C^4-topology.

Proposition 7. *For all k in $\mathbb{N}$, there exists y_k in $[0, a_n]$ such that $\phi_n^{(k)}(y_k) \geq C_2 q_n^{k-4}$.*

Proof. By induction : for $k = 0$, Equation 1 and the Mean Value Theorem yield existence of an y_0 in $[0, 1]$ such that $\phi_n(y_0) \geq C_2/q_n^4$. Then, every value of ϕ_n being achieved in $[0, a_n]$, we may assume $y_0 \in [0, a_n]$.

Now assume we have proved the Proposition up to some k. Then by the Fundamental Theorem of Calculus we have

$$\begin{aligned} \phi_n^{(k)}(y_k) = |\phi_n^{(k)}(y_k) - \phi_n^{(k)}(x_k)| &\leq \sup_{x\in[0,1]} |\phi_n^{(k)}(x)|.|y_k - x_k| \\ &\leq \sup_{x\in[0,1]} |\phi_n^{(k)}(x)|\frac{1}{q_n} \end{aligned}$$

so there exists $y_{k+1} \in [0, 1]$ such that $\phi'_n(y_{k+1}) \geq C_2 q_n^{k-4+1}$ and as previously we may assume $y_{k+1} \in [0, a_n]$. □

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