

# **GENERICITY OF GEODESIC FLOWS WITH POSITIVE TOPOLOGICAL ENTROPY ON $S^2$**

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# GENERICITY OF GEODESIC FLOWS WITH POSITIVE TOPOLOGICAL ENTROPY ON $S^2$

GONZALO CONTRERAS-BARANDIARÁN AND GABRIEL P. PATERNAIN

ABSTRACT. We show that the set of  $C^\infty$  riemannian metrics on  $S^2$  or  $\mathbb{R}P^2$  whose geodesic flow has positive topological entropy is open and dense in the  $C^2$  topology. The proof is partially based on an analogue of Franks' lemma for geodesic flows on surfaces.

*To the memory of Michel Herman*

## 1. Introduction

Let  $M$  be a closed surface endowed with a  $C^\infty$  riemannian metric  $g$  and let  $\phi_t^g$  be the geodesic flow of  $g$ . One of the most important dynamical invariants that one can associate to  $\phi_t^g$  to roughly measure its orbit structure complexity is the *topological entropy*, which we shall denote by  $h_{top}(g)$ . The first question one asks about  $h_{top}(g)$  is whether it vanishes or not. If  $h_{top}(g) > 0$  a well known result of A. Katok [27] states that the dynamics of  $\phi_t^g$  presents transverse homoclinic intersections and as a consequence the number of periodic hyperbolic geodesics grows exponentially with length. Moreover, other conclusions of a more geometrical nature can be drawn. Given  $p$  and  $q$  in  $M$  and  $T > 0$ , define  $n_T(p, q)$  as the number of geodesic arcs joining  $p$  and  $q$  with length  $\leq T$ . R. Mañé showed in [36] that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \int_{M \times M} n_T(p, q) dp dq = h_{top}(g),$$

and therefore if  $h_{top}(g) > 0$ , we have that in average the number of arcs between two points grows exponentially with length. Even better, K. Burns and G.P. Paternain showed in [13] that there exists a set of positive area in  $M$  such that for any pair of points  $p$  and  $q$  in that set,  $n_T(p, q)$  grows exponentially with exponent  $h_{top}(g)$ .

When the Euler characteristic of  $M$  is negative a result of E.I. Dinaburg [15] ensures that  $h_{top}(g) > 0$  for *any* metric  $g$ . Moreover, Katok in [28] showed that  $h_{top}(g)$  is greater than or equal to the topological entropy of a metric of constant negative curvature and equal area as  $g$  and with equality if and only if  $g$  itself has constant curvature. Therefore one is left with the problem of describing the behavior of the functional  $g \mapsto h_{top}(g)$  on the two-sphere (projective space) and the two-torus (Klein bottle). It is well known that these surfaces admit various completely integrable metrics with zero topological entropy: flat surfaces, surfaces of revolution, ellipsoids and Poisson spheres. On the other hand V. Donnay [17] and Petroll [42] showed how to perturb an homoclinic or heteroclinic connection to create

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transverse intersections. Applying these type of perturbations to the case of an ellipsoid with three distinct axes one obtains convex surfaces with positive topological entropy. Examples of these type were first given by G. Knieper and H. Weiss in [31]. Explicit real analytic convex metrics arising from rigid body dynamics were given by Paternain in [40].

We would like to point out that Katok's theorem mentioned above about the existence of transverse homoclinic intersections when the topological entropy is positive, together with the structural stability of horseshoes implies that the set of  $C^\infty$  metrics for which  $h_{top}(g) > 0$  is *open* in the  $C^r$  topology for all  $2 \leq r \leq \infty$ . Therefore, the relevant question about topological entropy for surfaces with non-negative Euler characteristic is the following: when is the set of  $C^\infty$  metrics with positive topological entropy dense?

Let us recall that a riemannian metric is said to be *bumpy* if all closed geodesics are non-degenerate, that is, if the linearized Poincaré map of every closed geodesic does not admit a root of unity as an eigenvalue. An important tool for proving generic properties of geodesic flows is the bumpy metric theorem which asserts that the set of  $C^r$  bumpy metrics is a residual subset of the set of all  $C^r$  metrics endowed with the  $C^r$  topology for all  $2 \leq r \leq \infty$ . The bumpy metric theorem is traditionally attributed to R. Abraham [1], but see also Anosov's paper [3]. Recall that a closed geodesic is said to be *hyperbolic* if its linearized Poincaré map has no eigenvalue of norm one and it is said to be *elliptic* if its linearized Poincaré map has all the eigenvalues with norm one and are not roots of unity. For a surface with a bumpy metric the closed geodesics are all elliptic or hyperbolic.

Let us recall some facts on heteroclinic orbits. Let  $SM$  be the unit sphere bundle of  $(M, g)$ . Given two hyperbolic periodic orbits  $\gamma, \eta$ , of the geodesic flow  $\phi_t$ , a *heteroclinic orbit* from  $\gamma$  to  $\eta$  is an orbit  $\phi_{\mathbb{R}}(\theta)$  such that

$$\lim_{t \rightarrow -\infty} d(\phi_t(\theta), \gamma) = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} d(\phi_t(\theta), \eta) = 0.$$

The orbit  $\phi_{\mathbb{R}}(\theta)$  is said to be *homoclinic* to  $\gamma$  if  $\eta = \gamma$ . The *weak stable* and *weak unstable* manifolds of the hyperbolic periodic orbit  $\gamma$  are defined as

$$W^s(\gamma) := \left\{ \theta \in SM \mid \lim_{t \rightarrow +\infty} d(\phi_t(\theta), \gamma) = 0 \right\},$$

$$W^u(\gamma) := \left\{ \theta \in SM \mid \lim_{t \rightarrow -\infty} d_{SM}(\phi_t(\theta), \gamma) = 0 \right\}.$$

The sets  $W^s(\gamma)$  and  $W^u(\gamma)$  are  $n$ -dimensional  $\phi_t$ -invariant immersed submanifolds of the unit tangent bundle  $SM$ , where  $n = \dim M$ . Then a heteroclinic orbit is an orbit in the intersection  $W^u(\gamma) \cap W^s(\eta)$ . If  $W^u(\gamma)$  and  $W^s(\eta)$  are transversal at  $\phi_{\mathbb{R}}(\theta)$  we say that the heteroclinic orbit is transverse. A standard argument in dynamical systems (see [23, §6.5.d] for diffeomorphisms) shows that if a flow contains a transversal homoclinic orbit then it has positive topological entropy<sup>1</sup>. (Note that for geodesic flows the closed orbits never reduce to fixed points.) Moreover, if there is a loop of transversal heteroclinic orbits  $W^u(\gamma_i) \cap W^s(\gamma_{i+1}) \neq \emptyset$  with  $\gamma_N = \gamma_0$ , then  $\gamma_0$  has a homoclinic orbit and, in particular,  $\phi_t$  has positive entropy.

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<sup>1</sup>In fact it contains a hyperbolic basic set (see below).

For the case of the two-torus classical results of G.A. Hedlund [24] and H.M. Morse [38] ensure that for a bumpy metric there are always heteroclinic geodesics. In fact, minimal periodic geodesics are always hyperbolic (for bumpy surfaces) and if we choose in  $\mathbb{R}^2$  a strip bounded by two periodic minimal geodesics  $c^+$  and  $c^-$  such that it does not contain other periodic minimal geodesics, then there exist minimal geodesics  $c$  and  $c^*$  such that  $c$  is  $\alpha$ -asymptotic to  $c^-$  and  $\omega$ -asymptotic to  $c^+$  and viceversa for  $c^*$  (cf. [4, theorem 6.8]). If these heteroclinic connections are not transverse, they can be perturbed using Donnay's theorem to easily obtain  $C^r$  density of metrics with positive topological entropy for all  $2 \leq r \leq \infty$ , for the two-torus. Similar arguments can be used for the Klein bottle. Clearly no argument like the one just described can be applied to surfaces with no real homology.

One of the main goals of this paper is to show:

**Theorem A.**

*The set of  $C^\infty$  riemannian metrics  $g$  on  $S^2$  or  $\mathbb{R}P^2$  for which  $h_{top}(g) > 0$  is dense in the  $C^2$  topology.*

**Corollary B.**

*The set of  $C^\infty$  riemannian metrics  $g$  on  $S^2$  or  $\mathbb{R}P^2$  for which  $h_{top}(g) > 0$  is open and dense in the  $C^2$  topology. In particular, if  $g$  belongs to this open and dense set, then the number of hyperbolic prime closed geodesics of length  $\leq T$  grows exponentially with  $T$ .*

From the previous discussion and the theorem we obtain the following corollary which answers a question that Detlef Gromoll posed the second author in 1988.

**Corollary C.**

*There exists a  $C^2$  open and dense set  $\mathcal{U}$  of  $C^\infty$  metrics on  $S^2$  such that for any  $g \in \mathcal{U}$  there exists a set  $G$  of positive  $g$ -area such that for any  $p$  and  $q$  in  $G$  we have*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log n_T(p, q) = h_{top}(g) > 0.$$

The last corollary is sharp in the sense that the sets  $G$  with positive  $g$ -area cannot be taken to have full area. In [12], Burns and Paternain constructed an open set of  $C^\infty$  metrics on  $S^2$  for which there exists a positive area set  $U \subset M$ , such that for all  $(p, q) \in U \times U$ ,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log n_T(p, q) < h_{top}.$$

It seems quite reasonable to conjecture that on any closed manifold the set of  $C^\infty$  riemannian metrics whose geodesic flow exhibits a transverse homoclinic interesection is open and dense in the  $C^r$ -topology for any  $r$  with  $2 \leq r \leq \infty$ . Besides its intrinsic interest there is another motivation for looking at this conjecture. Quite recently, A. Delshams, R. de la Llave and T. Seara proved the existence of orbits of unbounded energy (Arnold diffusion type of phenomenon) for pertubations of geodesic flows with a transverse homoclinic interesection by generic quasiperiodic potentials on any closed manifold [16]. Hopefully our methods here can be further developed to the point in which they yield Theorem A for any closed manifold.

Let us describe the main elements that go into the proof of theorem A. This will clarify at the same time why we can only obtain density in the  $C^2$  topology. Another important tool for proving generic properties for geodesic flows is a local perturbation result of W. Klingenberg and F. Takens [30]. We shall recall its precise statement in section 2. We shall also see in section 2 that the bumpy metric theorem, together with Klingenberg-Takens and a new perturbation lemma (cf. lemma 2.6) imply the analogue of the Kupka-Smale theorem for geodesic flows. Namely, that  $C^r$ -generic riemannian metrics on a manifold of any dimension have closed geodesics whose Poincaré maps have generic  $(r - 1)$ -jets and the heteroclinic intersections of their hyperbolic closed geodesics are transversal.

If there exists an elliptic closed geodesic in our Kupka-Smale surface, using the local perturbation result of Klingenberg and Takens we can approximate our metric by one such that the Poincaré map of the elliptic closed geodesic becomes a generic exact twist map in a small neighborhood of the elliptic fixed point. Then a result of Le Calvez [32] implies that the twist map has positive topological entropy and therefore a metric of class  $C^4$  with a nonhyperbolic closed geodesic can be approximated by one that has positive topological entropy. Details of this argument are given in section 3. Now we are faced with the following question: how can we proceed if all the closed geodesics are hyperbolic and this situation persists in a neighborhood?

It is not known if the two-sphere (or projective space) admits a metric all of whose closed geodesics are hyperbolic. Moreover, it is not known if this can happen for an open set of metrics (see [6] for a thorough discussion about the existence of a nonhyperbolic closed geodesic).

Let  $M$  be a closed surface and let  $\mathcal{R}^1(M)$  be the set of  $C^r$  riemannian metrics,  $r \geq 4$  on  $M$  all of whose closed geodesics are hyperbolic, endowed with the  $C^2$  topology and let  $\mathcal{F}^1(M) = \text{int}(\mathcal{R}^1(M))$  be the interior of  $\mathcal{R}^1(M)$  in the  $C^2$  topology. Given a metric  $g$  let  $\text{Per}(g)$  be the union of the hyperbolic (prime) periodic orbits of  $g$ .

Using Mañé's techniques on dominated splittings in his celebrated paper [35] and an analogue of Franks' lemma for geodesic flows we will show:

**Theorem D.**

*If  $g \in \mathcal{F}^1(M)$ , then the closure  $\overline{\text{Per}(g)}$  is a hyperbolic set.*

Theorem D together with results of N. Hingston and H.-B Rademacher (cf. [26, 46, 45]), will show (cf. section 5):

**1.1. Theorem.**

*A  $C^4$  metric on a surface can be  $C^2$ -approximated by one having an elliptic periodic orbit or by one with a non-trivial hyperbolic basic set.*

This theorem together with the previous discussion will allow us to prove theorem A.

A *hyperbolic set* is a compact invariant subset  $\Lambda$  of a dynamical system  $f^t$  such that there is a splitting of the tangent bundle of the phase space  $T_\Lambda \mathcal{N} = E^s \oplus E^u$  which is invariant under the derivative of  $f$ :  $f_*^t(E^{s,u}) = E^{s,u}$ , and such that there exist  $0 < \lambda < 1$  and  $N > 0$  such that

$$\|f_*^N|_{E^s}\| < \lambda^N, \quad \|f_*^{-N}|_{E^u}\| < \lambda^N.$$

A hyperbolic set  $\Lambda$  is said *locally maximal* if there exists an open neighbourhood  $U$  of  $\Lambda$  such that

$$\Lambda = \bigcap_{t \in \mathbb{R}} f^t(U).$$

A hyperbolic *basic set* is a locally maximal hyperbolic set which has a dense orbit. It is said *non-trivial* if it is not reduced to a closed orbit.

The symbolic dynamics models for hyperbolic basic sets show that they are semiconjugated to suspensions of topological Markov chains (see [23, §18.7] for diffeomorphisms, [9, 10] for flows), which have positive topological entropy [23, §3.2.d] and then the hyperbolic set has positive topological entropy. Moreover, the exponential growth rate of the number of periodic orbits in the hyperbolic set is given by the topological entropy ([23, Th. 18.5.1] for diffeomorphisms, [8, 10] for flows):

$$h_{top}(f|_{\Lambda}) = \lim_{T \rightarrow +\infty} \frac{1}{T} \log \#\{\gamma \mid \gamma \text{ periodic orbit of period } \leq T\}.$$

Mañé's theory on dominated splittings is based on theorem 5.1 on families of periodic sequences of linear maps: if when perturbing each linear map of such a family, the return linear maps remain hyperbolic, then their stable and unstable subspaces satisfy a uniform bound

$$(1) \quad \|T^N|_{E^s}\| \cdot \|T^{-N}|_{E^u}\| < \lambda_1 < 1,$$

for a fixed iterate  $N$  (eventually smaller than the periods), where  $T$  is the differential of our dynamical system. A splitting satisfying the uniform bound (1) is called a *dominated splitting*. The uniform bound (1) implies the continuity of the splitting, i.e. a dominated splitting on an invariant subset  $A$  of a dynamical system extends continuously to the closure  $\overline{A}$ .

The family of (symplectic) linear maps in our situation will be the family of time, say  $\tau > 0$ , of linearized Poincaré maps of the periodic orbits of the geodesic flow. In order to apply theorem 5.1, we first have to change “linear map” by “symplectic linear map” (cf. lemma 5.4). Then we have to be able to modify independently each linearized Poincaré map of time  $\tau$  on the periodic orbits, covering a neighborhood of fixed radius of the original linearized Poincaré map. This is done with the analogue of Franks' lemma for geodesic flows (cf. section 4). Thus we obtain a dominated splitting on the closure of the set of  $C^2$  persistently hyperbolic closed geodesics.

In Contreras [14] it is shown that a dominated invariant splitting  $E \oplus F$  on a non-wandering ( $\Omega(\Lambda) = \Lambda$ ) compact invariant set  $\Lambda$  of a symplectic diffeomorphism is hyperbolic. This is also proved in Ruggiero [49], when the subspaces  $E$  and  $F$  are assumed to be lagrangian.

A result of N. Hingston [26] (cf. also Rademacher [26, 46, 45]) states that if all the periodic orbits of a metric in  $S^2$  are hyperbolic, then they are infinite. Assuming that they are  $C^2$ -persistently hyperbolic, the theory above and Smale's spectral decomposition theorem imply that their closure contains a non-trivial basic set. (Alternatively, we could

have also used for the 2-sphere the stronger results of Franks and Bangert [5, 20] which assert that any metric on  $S^2$  has infinitely many geometrically distinct closed geodesics.

Unfortunately, Mañé's techniques only work in the  $C^2$  topology and that is why in theorem A we can prove density of positive topological entropy on the two-sphere or projective space only for the  $C^2$  topology. We remark that the lack of a closing lemma for geodesic flows does not allow us to conclude that the geodesic flow of a metric nearby  $g$  is Anosov as one would expect.

At this point it seems important to remark that if instead of considering riemannian metrics we were considering Finsler metrics or hamiltonians, then theorem A would have been a corollary of well known results for hamiltonians (cf. [39, 47, 48, 52]). However, as it is well known, perturbation results within the set of riemannian metrics are much harder, basically due to the fact that when we change the metric in a neighborhood of a point of the manifold we affect all the geodesics leaving from those points; in other words, even if the size of our neighborhood in the manifold is small, the effect of the perturbation in the unit sphere bundle could be large. This is the main reason why the closing lemma is not known for geodesic flows (cf. [44]), even though there is a closing lemma for Finsler metrics.

Another remark concerns the degree of differentiability of our metrics. Theorem A holds if instead of requiring our metrics  $g$  to be  $C^\infty$  we require them to be  $C^r$  for  $r \geq 2$ . Given a  $C^2$  metric  $g_0$ , we can approximate it by a  $C^\infty$  metric  $g_1$  in the  $C^2$  topology. Afterwards we  $C^2$ -approximate  $g_1$  by a  $C^\infty$  metric  $g_2$  with a basic set. Then the structural stability theorem works for an open  $C^2$ -neighborhood of  $g_2$  of  $C^2$  metrics. We need  $g_1$  to be at least  $C^4$  in both: Franks' lemma 4.1 to prove theorem 1.1 and to make the Poincaré map of an elliptic closed geodesic a twist map. Observe that we actually find a hyperbolic basic set and not just  $h_{top}(g) > 0$ . Katok's theorem which is based on Pesin theory requires the riemannian metrics to be of class at least  $C^{2+\alpha}$ . This restriction is overcome in our case by the use of the structural stability theorem. On the other hand for Corollary C a  $C^\infty$  hypothesis on the metrics is essential because, as in Mañé's formula [36], Yomdin's theorem [56] is used.

*Related Work.* There is an unpublished preprint by H. Weiss [55] that proves that within the set of positively curved 1/4-pinched metrics, those with positive topological entropy are  $C^r$ -dense. Weiss uses a result of G. Thorbergsson [53] which asserts that any positively curved 1/4-pinched metric on  $S^2$  has a nonhyperbolic closed geodesic and similar arguments to the ones we give in section 3, although the Kupka-Smale theorem for geodesic flows is not proven.

Michel Herman gave a wonderful lecture at IMPA [25] in which he outlined a proof of the following theorem: within the set of  $C^\infty$  positively curved metrics on  $S^2$  those with an elliptic closed geodesic are  $C^2$ -generic. Among other tools, he used an analogue of Franks' lemma just like the one we prove in the present paper. As a matter of fact, not only he pointed out a mistake in a draft of the manuscript we gave him, but he also explained us how solve the self-intersection problem that appears in the proof. This paper is dedicated to his memory.

It is worth mentioning that Herman's motivation was a claim by H. Poincaré [43] that said that any convex surface has a nonhyperbolic closed geodesic without self-intersections. This claim was proved wrong by A.I. Grjuntal [21].

## 2. Bumpy metrics and the Kupka-Smale theorem.

In this section  $M$  is a closed manifold of dimension  $n$ . We begin by recalling some elementary facts. Let  $\phi_t^g$  be the geodesic flow of the riemannian metric  $g$  acting on  $SM$ , the unit sphere bundle of  $M$ . Let  $\pi : SM \rightarrow M$  be the canonical projection. Non-trivial closed geodesics on  $M$  are in one-to-one correspondence to the periodic orbits of  $\phi_t^g$ . For a closed orbit  $\gamma = \{\phi_t^g(z) : t \in [0, a]\}$  of period  $a > 0$  we can define the *Poincaré map*  $\mathcal{P}_g(\Sigma, \gamma)$  as follows: one can choose a local hypersurface  $\Sigma$  in  $SM$  through  $v$  and transversal to  $\gamma$  such that there are open neighborhoods  $\Sigma_0, \Sigma_a$  of  $v$  in  $\Sigma$  and a differentiable mapping  $\delta : \Sigma_0 \rightarrow \Sigma_a$  with  $\delta(v) = a$  such that the map  $\mathcal{P}_g(\Sigma, \gamma) : \Sigma_0 \rightarrow \Sigma_a$  given by

$$u \mapsto \phi_{\delta(u)}^g(u),$$

is a diffeomorphism.

Given a closed geodesic  $c : \mathbb{R}/\mathbb{Z} \rightarrow M$ , all iterates  $c^m : \mathbb{R}/\mathbb{Z} \rightarrow M$ ;  $c(t) = c(mt)$  for a positive integer  $m$  are closed geodesics too. We shall call a closed geodesic *prime* if it is not the iterate of a shorter curve. Analogously a closed orbit of  $\phi_t^g$  of period  $a$  is called prime if  $a$  is the minimal period. A closed orbit  $\gamma$  (or the corresponding closed geodesic  $c$ ) is called *non-degenerate* if 1 is not an eigenvalue of the linearized Poincaré map  $P_c := d_{\gamma'(0)}\mathcal{P}_g(\Sigma, \gamma)$ . In that case,  $\gamma$  is an isolated closed orbit and  $\pi \circ \gamma$  an isolated closed geodesic. Moreover, one can apply the implicit function theorem to obtain fixed points of the Poincaré map  $\mathcal{P}_g$ . Thus, for a nearby metric  $\bar{g}$  there is a unique closed orbit  $\gamma^{\bar{g}}$  for  $\phi^{\bar{g}}$  near  $\gamma$ , given by the implicit function theorem, that we call the *continuation* of  $c$ .

A riemannian metric  $g$  is called *bumpy* if all the closed orbits of the geodesic flow are non-degenerate. Since  $P_{c^m} = P_c^m$  this is equivalent to saying that if  $\exp(2\pi i\lambda)$  is an eigenvalue of  $P_c$ , then  $\lambda$  is irrational. Let us denote by  $\mathcal{G}^r$  the set of metrics of class  $C^r$  endowed with the  $C^r$  topology for  $r \geq 2$ . We state the bumpy metric theorem [1, 3]:

### 2.1. Theorem.

*For  $2 \leq r \leq \infty$ , the set of bumpy metrics of class  $C^r$  is a residual subset of  $\mathcal{G}^r$ .*

The bumpy metric theorem 2.1 clearly implies the following:

**2.2. Corollary.** *There exists a residual set  $\mathcal{O}$  in  $\mathcal{G}^r$  such that if  $g \in \mathcal{O}$  then for all  $T > 0$ , the set of periodic orbits of  $\phi^g$  with period  $\leq T$  is finite.*

The canonical symplectic form  $\omega$  induces a symplectic form on  $\Sigma$  and  $\mathcal{P}_g(\Sigma, \gamma)$  becomes a symplectic diffeomorphism. Periodic points of  $\mathcal{P}_g(\Sigma, \gamma)$  correspond to periodic orbits nearby  $\gamma$ . Let  $N$  denote the orthogonal complement of  $v = \gamma'(0)$  in the tangent space  $T_{\pi(v)}M$ . Recall that  $N \oplus N$  can be identified with the kernel of the canonical contact form and therefore it is a symplectic vector space with respect to  $\omega$ . One can choose  $\Sigma$  such that the *linearized Poincaré map*  $P_g(\gamma) := d_v\mathcal{P}_g(\Sigma, \gamma)$  is a linear symplectic map of  $N \oplus N$  and

$$P_g(\gamma)(J(0), \dot{J}(0)) = (J(a), \dot{J}(a)),$$



where  $J$  is a normal Jacobi field along the geodesic  $\pi \circ \gamma$  and  $\dot{J}$  denotes the covariant derivative along the geodesic. After choosing a symplectic basis for  $N \oplus N$  we can identify the group of symplectic linear maps of  $N \oplus N$  with the symplectic linear group  $Sp(n-1)$  of  $\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}$ .

Let  $J_s^r(n-1)$  be the set of  $r$ -jets of symplectic automorphisms of  $\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}$  that fix the origin. Clearly one can identify  $J_s^1(n-1)$  with  $Sp(n-1)$ . A set  $Q \subset J_s^r(n-1)$  is said to be *invariant* if for all  $\sigma \in J_s^r(n-1)$ ,  $\sigma Q \sigma^{-1} = Q$ . The property that says that the  $r$ -jet of a Poincaré map  $\mathcal{P}_g(\Sigma, \gamma)$  belongs to  $Q$  is independent of the section  $\Sigma$ .

Let  $\gamma = \{\phi_t^{g_0}(v)\}$  be a periodic orbit of period  $a$  of the geodesic flow  $\phi_t^{g_0}$  of the metric  $g_0 \in \mathcal{G}^r$ . Let  $W$  be an open neighborhood of  $\pi(v) \in M$ . We choose  $W$  so that the geodesic  $\pi \circ \gamma$  does not have any self intersection in  $W$ . Denote by  $\mathcal{G}^r(\gamma, g_0, W)$  the set of metrics  $g \in \mathcal{G}^r$  for which  $\gamma$  is a periodic orbit of period  $a$  and such that the support of  $g - g_0$  lies in  $W$ .

We now state the local perturbation result of Klingenberg and Takens [30, theorem 2].

**2.3. Theorem.** *If  $Q$  is an open and dense invariant subset of  $J_s^{r-1}(n-1)$  then there is for every neighborhood  $\mathcal{V}$  of  $g_0$  in  $\mathcal{G}^r$  a metric  $g \in \mathcal{V} \cap \mathcal{G}^r(\gamma, g_0, W)$  such that the  $(r-1)$ -jet of  $\mathcal{P}_g(\Sigma, \gamma)$  belongs to  $Q$ .*

As pointed out by Anosov [3], once that theorem 2.1 is proved, combining corollary 2.2 and theorem 2.3 one gets:

**2.4. Theorem.** *Let  $Q \subset J_s^{r-1}(n-1)$  be open, dense and invariant. Then there exists a residual subset  $\mathcal{O} \subset \mathcal{G}^r$  such that for all  $g \in \mathcal{O}$ , the  $(r-1)$ -jet of the Poincaré map of every closed geodesic of  $g$  belongs to  $Q$ .*

A closed geodesic is said to be *hyperbolic* if its linearized Poincaré map has no eigenvalues of modulus 1. If  $\gamma$  is a hyperbolic closed geodesic and  $\theta = (\gamma(0), \dot{\gamma}(0))$ , define the strong stable and strong unstable manifolds of  $\gamma$  at  $\theta$  by

$$W^{ss}(\theta) = \{v \in SM \mid \lim_{t \rightarrow +\infty} d(\phi_t^g(v), \phi_t^g(\theta)) = 0\},$$

$$W^{su}(\theta) = \{v \in SM \mid \lim_{t \rightarrow -\infty} d(\phi_t^g(v), \phi_t^g(\theta)) = 0\}.$$

Define the weak stable and weak unstable manifolds by

$$W^s(\gamma) := \bigcup_{t \in \mathbb{R}} \phi_t(W^{ss}(\theta)), \quad W^u(\gamma) := \bigcup_{t \in \mathbb{R}} \phi_t(W^{su}(\theta)).$$

It turns out that they are immersed submanifolds of dimension

$$\dim W^{sc}(\gamma) = \dim W^{uc}(\gamma) = \dim M.$$

An heteroclinic point is a point in the intersection  $W^s(\gamma) \cap W^u(\eta)$  for two hyperbolic closed geodesics  $\gamma$  and  $\eta$ . We say that  $\theta \in SM$  is a transversal heteroclinic point if  $\theta \in W^s(\gamma) \cap W^u(\eta)$ , and  $T_\theta W^s(\gamma) + T_\theta W^u(\eta) = T_\theta SM$ .

In [17], Donnay showed for surfaces how to perturb an heteroclinic point of a metric on a surface to make it transversal. In fact a similar method has been used by Petroll [42]

for higher dimensional manifolds and this method actually gives  $C^r$  perturbations. The reference [42] is difficult to find, but there is a sketch of proof in [11].

However, without further analysis, these perturbations do not give a control on the size of the subsets where the stable and unstable manifolds are made transversal, as it is needed for the proof of the Kupka-Smale theorem. Available proofs of the Kupka-Smale theorem [47, 48] for general hamiltonians do not apply to geodesic flows without further arguments.

Using corollary 2.2 and theorem 2.4 we show here how to extend the proof of the Kupka-Smale theorem for hamiltonian flows to the case of geodesic flows, provided that the perturbations used are local. The perturbations in [48] are not local. The perturbation in claim  $a$  in [47] are local, they are written for volume preserving flows but they can be adapted to the hamiltonian case. We choose to present in appendix A another kind of perturbation, suitable for our proof of theorem 2.5 and that could be useful for other types of problems.

**2.5. Theorem.** *Let  $Q \subset J_s^{r-1}(n-1)$  be open, dense and invariant. Then there exists a residual subset  $\mathcal{O} \subset \mathcal{G}^r$  such that for all  $g \in \mathcal{O}$ :*

- *The  $(r-1)$ -jet of the Poincaré map of every closed geodesic of  $g$  belongs to  $Q$ .*
- *All heteroclinic points of hyperbolic closed geodesics of  $g$  are transversal.*

**Proof:** We are going to modify the proof of the Kupka-Smale theorem for general hamiltonians to fit our geodesic flow setting. Let  $\mathcal{H}^r(N)$  be the set of  $C^r$  riemannian  $g$  metrics such that the Poincaré map of every closed geodesic of  $g$  with period  $\leq N$  belongs to  $Q$ . If necessary intersect  $Q$  with the set  $A \subset J^r(n-1)$  of jets of symplectic maps whose derivative at the origin has no eigenvalue equal to 1. Then  $Q$  is still open, dense and invariant. Since the periodic orbits of period  $\leq N$  for such  $g$  are generic, then there is a finite number of them. Since  $Q$  is open and the Poincaré map depends continuously on the riemannian metric, then  $\mathcal{H}^r(N)$  is an open subset of  $\mathcal{G}^r$ . By theorem 2.4,  $\mathcal{H}^r(N)$  is a dense subset of  $\mathcal{G}^r$ .

Let  $\mathcal{K}^r(N)$  be the subset of  $\mathcal{H}^r(N)$  of those metrics  $g$  such that for any hyperbolic periodic orbit  $\gamma$  of  $g$  with period  $\leq N$  the submanifolds  $W^s(\gamma) \cap B(\gamma, N)$  and  $W^u(\gamma) \cap B(\gamma, N)$  are transversal, where

$$B(\gamma, N) := \{ \theta \in SM \mid d(\theta, \gamma) < N \}.$$

Since the stable and unstable manifolds of a hyperbolic orbit depend continuously on compact parts in the  $C^1$  topology with respect to the vector field, then  $\mathcal{K}^r(N)$  is an open subset of  $\mathcal{G}^r$ .

It remains to prove that  $\mathcal{K}^r(N)$  is dense in  $\mathcal{G}^r$ , for then the set

$$\mathcal{K}^r := \bigcap_{N \in \mathbb{N}} \mathcal{K}^r(N),$$

is the residual subset we are looking for.

We see first that in order to prove the density of  $\mathcal{K}^r(N)$  it is enough to make small local perturbations. Let  $\gamma, \eta$  be two hyperbolic periodic orbits  $\gamma, \eta$  of period  $\leq N$ . Observe that if the two invariant manifolds  $W^u(\gamma), W^s(\eta)$  intersect, then they intersect along complete

orbits. If they intersect transversally, then they are transversal along the whole orbit of the intersection point.

A fundamental domain for  $W^u(\gamma)$  is a compact subset  $K \subset W^u(\gamma)$  such that every orbit in  $W^u(\gamma)$  intersects  $K$ . Such fundamental domain can be constructed for example inside the strong unstable manifold of  $\gamma$  using Hartman's theorem (i.e. the linearization of the Poincaré return map in a neighbourhood of  $\gamma$ ). Moreover there are fundamental domains which are arbitrarily small and arbitrarily near to  $\gamma$ . Hence it is enough to make  $W^u(\gamma) \cap B(\gamma, N)$  transversal to  $W^s(\eta) \cap B(\gamma, N)$  in a fundamental domain for  $W^u(\gamma)$ .

We will use the following perturbation lemma whose proof will be given after completing the proof of theorem 2.5:

**2.6. Lemma.** *For every point  $\theta \in W^u(\gamma)$  such that the projection  $\pi|_{W^u(\gamma)}$  is a diffeomorphism in a neighbourhood of  $\theta$ , and sufficiently small neighbourhoods  $\theta \in V \subset \bar{V} \subset U$  in  $SM$ , there are riemannian metrics  $\bar{g}$  such that*

- (1)  $\bar{g}$  is arbitrarily near  $g$  in the  $C^r$ -topology;
- (2)  $g$  and  $\bar{g}$  coincide outside  $\pi(U)$ ;
- (3)  $\gamma$  and  $\eta$  are periodic orbits for  $\bar{g}$ ;
- (4) the connected component of  $W^u(\gamma) \cap B(\gamma, N) \cap V$  containing  $\theta$  and the submanifold  $W^s(\eta)$  are transversal.

Let  $\theta$  be in a fundamental domain  $K$  for  $W^u(\gamma)$ . By the inverse function theorem the projection  $\pi|_{W^u(\gamma)}$  is a local diffeomorphism at  $\theta$  if and only if the tangent space of  $W^u(\gamma)$  at  $\theta$  is transversal to the vertical subspace i.e.  $T_\theta W^u(\gamma) \cap \ker d_\theta \pi = \{0\}$ .

Observe that the manifolds  $W^u(\gamma)$  and  $W^s(\gamma)$  are lagrangian. A well known property of the geodesic flow (cf. [41]) asserts that if  $W$  is a lagrangian subspace, then the set of times  $t$  for which  $d_\theta \phi_t(W) \cap \ker d_{\phi_t \theta} \pi \neq \{0\}$  is discrete and hence at most countable.

By flowing a bit the point  $\theta$  we obtain another point  $\phi_t(\theta)$  satisfying the conditions of the lemma. We can also choose  $t$  such that  $\pi_t(\theta)$  does not intersect any closed geodesic of period  $\leq N$ . One chooses the neighbourhood  $U$  in lemma 2.6 such that the support of the perturbation  $\pi(U)$  does not intersect any closed geodesic of period  $\leq N$ . Choose a neighbourhood  $V$  such that  $\phi_t(\theta) \in V \subset \bar{V} \subset U$ . Applying lemma 2.6 we obtain a new riemannian metric  $\bar{g}$  such that  $\bar{g}|_{\pi(U)^c} = g|_{\pi(U)^c}$  and the connected component of  $W^u(\gamma) \cap B(\gamma, N) \cap \bar{V}$  containing  $\phi_t(\theta)$  is transversal to  $W^s(\gamma)$ . If the perturbation is small enough, flowing backwards a bit we obtain a neighbourhood  $\bar{V}_1$  of  $\theta$ , where  $W^u(\gamma) \cap B(\gamma, N)$  and  $W^s(\eta)$  are transversal.

Now cover the compact fundamental domain  $K$  by a finite number of these neighbourhoods  $\bar{V}_1$  and call them, let us say,  $W_1, \dots, W_r$ . Observe that in lemma 2.6 the perturbations are arbitrarily small but the neighbourhood  $V$  of transversality is fixed. Since transversality of compact parts of stable (unstable) manifolds is an open condition on  $g$ , one can make the perturbation on  $W_{i+1}$  small enough so that the invariant manifolds are still transversal on  $W_1, \dots, W_i$ .

In order to make now  $W^u(\eta)$  transverse to  $W^s(\gamma)$  one can use the invariance of the geodesic flow under the flip  $F(x, v) = (x, -v)$ , so that  $W^s(\gamma) = W^u(F(\gamma))$  or repeat the same arguments for the geodesic flow with the time inverted.

This completes the proof of the density of  $\mathcal{K}^r(N)$ .  $\square$

**Proof of lemma 2.6:**

Perhaps, the easiest way to prove lemma 2.6 is to use a perturbation result for general hamiltonian systems. The Legendre transform  $\mathcal{L}(x, v) = \langle v, \cdot \rangle_g$  conjugates the geodesic flow with the hamiltonian flow of

$$H(x, p) := \frac{1}{2} \sum_{ij} g^{ij}(x) p_i p_j$$

on the cotangent bundle  $T^*M$  with the canonical (and *fixed*) symplectic form  $\omega = \sum_i dp_i \wedge dx_i$ . Here  $g^{ij}(x)$  is the inverse of the matrix of the riemannian metric.

Observe that the stable and unstable manifolds are lagrangian submanifolds of  $T^*M$ . Since they are in the energy level  $H^{-1}\{\frac{1}{2}\}$  by lemma A.1, they are invariant.

Now use a local perturbation for the hamiltonian flow (e.g. [47, claim a, th. 3] or A.3 in appendix A) such that the new hamiltonian flow has  $W^u(\gamma)$  transversal to the old  $W^s(\eta)$  in the neighbourhood  $V$ . The stable manifold  $W^s(\eta)$  only depends on the future times and on the future it only accumulates on the periodic orbit  $\eta$  so up to the perturbation it does not change.

If the perturbation is small enough, then the new piece of unstable manifold  $\widetilde{W}^u(\gamma)$  in the support of the perturbation  $U$ , still projects injectively into  $M$ . Let  $p : \pi(U) \rightarrow T^*_{\pi(U)}M$  be such that

$$\text{connected component of } \widetilde{W}^u(\gamma) \cap U = \text{Graph}(p) = \{ (x, p(x)) \mid x \in \pi(U) \}.$$

Define a new riemannian metric by

$$\bar{g}_{ij}(x) = \begin{cases} 2H(x, p(x)) g_{ij}(x) & \text{if } x \in \pi(U), \\ g_{ij}(x) & \text{if } x \notin \pi(U). \end{cases}$$

Then  $\bar{g}$  is  $C^r$  near  $g$ , coincides with  $g$  on the complement of  $\pi(U)$  and its hamiltonian satisfies

$$\begin{aligned} \bar{H}(x, p(x)) &= \frac{1}{2} \sum_{ij} \bar{g}^{ij}(x) p_i(x) p_j(x) = \frac{1}{2} \sum_{ij} \frac{g^{ij}(x)}{2H(x, p(x))} p_i(x) p_j(x) \\ (2) \quad &= \frac{H(x, p(x))}{2H(x, p(x))} = \frac{1}{2}, \quad \text{for } x \in \pi(U). \end{aligned}$$

Then  $\widetilde{W}^u(\gamma)$  is a lagrangian submanifold of  $T^*M$  which is in the energy level  $\bar{H} \equiv \frac{1}{2}$  of the hamiltonian for  $\bar{g}$  and which coincides with the unstable manifold of  $\gamma$  in a neighbourhood of  $\gamma$ . Hence  $\widetilde{W}^u(\gamma)$  is invariant under the geodesic flow of  $\bar{g}$  and is the unstable manifold of  $\gamma$  for  $\bar{g}$ .  $\square$

### 3. Twist maps and topological entropy.

We say that a homeomorphism of the annulus  $f : [0, 1] \times S^1 \leftrightarrow$  is a *twist map* if for all  $\theta \in S^1$  the function  $[0, 1] \ni r \mapsto \pi_2 \circ f(r, \theta) \in S^1$  is strictly monotonous.

For a proof of the Birkhoff's normal form below see Birkhoff [7], Siegel and Moser [50] or Le Calvez [33, Th. 1.1]. For a higher dimensional version for symplectic maps see Klingenberg [29, p. 101].

#### 3.1. Birkhoff's normal form.

Let  $f$  be a  $C^\infty$  diffeomorphism defined on a neighborhood of 0 in  $\mathbb{R}^2$  such that  $f(0) = 0$ , it preserves the area form  $dx \wedge dy$ , and the eigenvalues of  $d_0 f$  satisfy  $|\lambda| = 1$  and  $\lambda^n \neq 1$  for all  $n \in \{1, \dots, q\}$  for some  $q \geq 4$ .

Then there exists a  $C^\infty$  diffeomorphism  $h$ , defined on a neighborhood of 0 such that  $h(0) = 0$ ,  $h$  preserves the form  $dx \wedge dy$  and in complex coordinates  $z = x + iy \approx (x, y)$  one has:

$$h \circ f \circ h^{-1}(z) = \lambda z e^{2\pi i P(z\bar{z})} + o(|z|^{q-1}),$$

where  $P(X) = a_1 X + \dots + a_m X^m$  is a real polynomial of degree  $m$  with  $2m + 1 < q$ .

The coefficients  $a_i$ ,  $1 \leq i \leq m \leq \frac{q}{2} - 1$  are uniquely determined by  $f$ .

In polar coordinates the function  $g = h \circ f \circ h^{-1}$  is written as

$$(r, \theta) \mapsto (r + \mu(r, \theta), \theta + \alpha + a_1 r^2 + \dots + a_m r^{2m} + \nu(r, \theta)),$$

where  $\lambda = e^{2\pi i \alpha}$ . If  $a_1 \neq 0$  and  $|r| \leq \varepsilon$  is small enough, then  $\frac{\partial}{\partial r}(\pi_2 \circ g)$  has the same non-zero sign as  $a_1$  and hence  $g$  is a twist map in  $[0, \varepsilon] \times S^1$ .

We shall use following result:

#### 3.2. Proposition (Le Calvez [34, Remarques p. 34]).

Let  $f$  be a diffeomorphism of the annulus  $\mathbb{R} \times S^1$  such that it is a twist map, it is area preserving, the form  $f^*(Rd\theta) - Rd\theta$  is exact and

- (i) If  $x$  is a periodic point for  $f$  and  $q$  is its least period, the eigenvalues of  $d_x f^q$  are not roots of unity.
- (ii) The stable and unstable manifolds of hyperbolic periodic orbits of  $f$  intersect transversally (i.e. whenever they meet, they meet transversally).

Then  $f$  has periodic orbits with homoclinic points.

We are now ready to show:

#### 3.3. Proposition.

Let  $g_0$  be a metric of class  $C^r$ ,  $r \geq 4$ , on a surface  $M$  with a nonhyperbolic closed geodesic. Then there exists a  $C^\infty$  metric  $g$  arbitrarily close to  $g_0$  in the  $C^r$  topology with a non-trivial hyperbolic basic set. In particular,  $h_{top}(g) > 0$ .

**Proof:** Let  $Q \subset J_s^3(1)$  be

$$Q = \{ \sigma f_{\alpha, a_1} \sigma^{-1} \mid \sigma \in J_s^3(1), a_1 \neq 0, \alpha \notin \{ \frac{n}{k} \mid n, k \in \mathbb{Z}, k \neq 0 \} \},$$

where  $f_{\alpha, a_1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is given by  $f_{\alpha, a_1}(r, \theta) = (r, \theta + \alpha + a_1 r^2) + o(r^3)$  in polar coordinates. By Birkhoff's normal form (with  $q = 4$ ), the set  $Q$  is open, dense and invariant.

By the Kupka-Smale theorem 2.5 we can  $C^r$ -approximate  $g_0$ ,  $r \geq 4$  by a  $C^\infty$  metric  $g$  with a nonhyperbolic closed orbit  $\gamma$  such that the 3-jet of its Poincaré map is in  $Q$  and  $g$  satisfies the conditions (i) and (ii) in proposition 3.2. The symplectic form on  $TM$  induced by the riemannian metric, induces a symplectic form on a local transverse section  $\Sigma$  to  $\gamma$ , which is preserved by the Poincaré map  $\mathcal{P}_g(\Sigma, \gamma)$ . By Darboux's theorem, using a change of coordinates we can assume that  $\Sigma$  is a neighbourhood of 0 in  $\mathbb{R}^2$  and that the symplectic form on  $\Sigma$  is the area form of  $\mathbb{R}^2$ .

By the definition of  $Q$ , the Poincaré map  $f = \mathcal{P}_g(\Sigma, \gamma)$  is conjugate to a twist map  $f_0 = h f h^{-1}$  when written in polar coordinates. In order to apply proposition 3.2 we show below a change of coordinates which transforms  $f_0$  into an exact twist map of the annulus  $\mathbb{R}^+ \times S^1$ . Then the existence of an homoclinic orbit implies the existence of a non-trivial hyperbolic basic set.

Consider the following maps

$$\begin{array}{ccccc} (x, y) & \longrightarrow & (r, \theta) & \longrightarrow & (\frac{1}{2}r^2, \theta) = (R, \theta) \\ \mathbb{D} & \xrightarrow{P} & \mathbb{R}^+ \times S^1 & \longrightarrow & \mathbb{R}^+ \times S^1 \\ f_0 \downarrow & & \downarrow & & \downarrow T \\ \mathbb{D} & \longrightarrow & \mathbb{R}^+ \times S^1 & \longrightarrow & \mathbb{R}^+ \times S^1 \end{array}$$

where  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ ,  $P^{-1}(r, \theta) = (r \cos \theta, r \sin \theta)$ . Write  $G(x, y) = (\frac{1}{2}r^2, \theta) = (R, \theta)$ , the upper composition. Then  $G^*(R d\theta) = \frac{1}{2}(x dy - y dx) =: \lambda$ . Observe that  $d\lambda = dx \wedge dy$  is the area form in  $\mathbb{D}$ . Since  $\mathbb{D}$  is contractible, then  $f_0^*(\lambda) - \lambda$  is exact. Then  $T^*(R d\theta) - R d\theta$  is exact. Since  $R(r) = \frac{1}{2}r^2$  is strictly increasing on  $r > 0$ , then  $T$  is a twist map iff  $f_0$  is a twist map.

□

#### 4. Franks' lemma for geodesic flows.

Let  $\gamma = \{\phi_t^g(v) \mid t \in [0, 1]\}$  be a piece of an orbit of length 1 of the geodesic flow  $\phi_t^g$  of the metric  $g \in \mathcal{G}^r$ . Let  $\Sigma_0$  and  $\Sigma_t$  be sections at  $v$  and  $\phi_t(v)$  respectively. We have a Poincaré map  $\mathcal{P}_g(\Sigma_0, \Sigma_t, \gamma)$  going from  $\Sigma_0$  to  $\Sigma_t$ . One can choose  $\Sigma_t$  such that the *linearized Poincaré map*

$$P_g(\gamma)(t) \stackrel{\text{def}}{=} d_v \mathcal{P}_g(\Sigma_0, \Sigma_t, \gamma)$$

is a linear symplectic map from  $\mathcal{N}_0 := N(v) \oplus N(v)$  to  $\mathcal{N}_t := N(\phi_t v) \oplus N(\phi_t v)$  and

$$P_g(\gamma)(t)(J(0), \dot{J}(0)) = (J(t), \dot{J}(t)),$$

where  $J$  is a normal Jacobi field along the geodesic  $\pi \circ \gamma$  and  $\dot{J}$  denotes the covariant derivative along the geodesic. Let us identify the set of all linear symplectic maps from  $\mathcal{N}_0$

to  $\mathcal{N}_t$  with the symplectic group

$$Sp(1) := \{ X \in \mathbb{R}^{2 \times 2} \mid X^* \mathbb{J} X = \mathbb{J} \},$$

where  $\mathbb{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ . The group  $Sp(1)$  is naturally endowed with a norm.

Suppose that the geodesic arc  $\pi \circ \gamma(t)$ ,  $t \in [0, 1]$ , does not have any self intersection and let  $W$  be a tubular neighborhood of it. We denote by  $\mathcal{G}^r(\gamma, g, W)$  the set of metrics  $\bar{g} \in \mathcal{G}^r$  for which  $\gamma$  is a piece of orbit of length 1 and such that the support of  $\bar{g} - g$  lies in  $W$ .

When we apply the following theorem to a piece of a closed geodesic we cannot avoid to have self intersections of the whole geodesic. Given any finite set of non-self intersecting geodesic segments  $\mathfrak{F} = \{\eta_1, \dots, \eta_m\}$  defined on  $[0, 1]$  with the following properties:

1. The endpoints of  $\eta_i$  are not contained in  $W$ ;
2. The segment  $\pi \circ \gamma|_{[0,1]}$  intersects each  $\eta_i$  transversally;

denote by  $\mathcal{G}^r(\gamma, g, W, \mathfrak{F})$  the set of metrics  $\bar{g} \in \mathcal{G}^r(\gamma, g, W)$  such that  $\bar{g} = g$  in a small neighbourhood of  $W \cap \cup_{i=1}^m \eta_i([0, 1])$ .

Consider the map  $S : \mathcal{G}^r(\gamma, g, W) \rightarrow Sp(1)$  given by  $S(\bar{g}) = P_{\bar{g}}(\gamma)(1)$ . The following result is the analogue of Franks' lemma [19, lem. 1.1] for geodesic flows.

**4.1. Theorem.** *Let  $g_0 \in \mathcal{G}^r$ ,  $r \geq 4$ . Given  $\mathcal{U} \subset \mathcal{G}^2$  a neighborhood of  $g_0$ , there exists  $\delta = \delta(g_0, \mathcal{U}) > 0$  such that given  $g \in \mathcal{U}$ ,  $\gamma$ ,  $W$  and  $\mathfrak{F}$  as above, the image of  $\mathcal{U} \cap \mathcal{G}^r(\gamma, g, W, \mathfrak{F})$  under the map  $S$  contains a ball of radius  $\delta$  centered at  $S(g_0)$ .*

The time 1 in the preceding statement was chosen to simplify the exposition and the same result holds for any time  $\tau$  chosen in a closed interval  $[a, b] \subset ]0, +\infty[$ ; now with  $\delta = \delta(g_0, \mathcal{U}, a, b) > 0$ . In order to fix the setting, take  $[a, b] = [\frac{1}{2}, 1]$  and assume that the injectivity radius of  $M$  is larger than 1. This implies that there are no periodic orbits with period smaller than  $\frac{1}{2}$  and that any periodic orbit can be cut into non self-intersecting geodesic segments of length  $\tau$  with  $\tau \in [\frac{1}{2}, 1]$ . We shall apply theorem 4.1 to such segments of a periodic orbit choosing the supporting neighborhoods carefully as we now describe.

Given  $g \in \mathcal{G}^r$  and  $\gamma$  a prime periodic orbit of  $g$  let  $\tau \in [\frac{1}{2}, 1]$  be such that  $m\tau = \text{period}(\gamma)$  with  $m \in \mathbb{N}$ . For  $0 \leq k < m$ , let  $\gamma_k(t) := \gamma(t + k\tau)$  with  $t \in [0, \tau]$ . Given a tubular neighborhood  $W$  of  $\pi \circ \gamma$  and  $0 \leq k < m$  let  $S_k : \mathcal{G}^r(\gamma, g, W) \rightarrow Sp(1)$  be the map  $S_k(\bar{g}) = P_{\bar{g}}(\gamma_k)(\tau)$ .

Let  $W_0$  be a small tubular neighborhood of  $\gamma_0$  contained in  $W$ . Let  $\mathcal{F}_0 = \{\eta_1^0, \dots, \eta_{m_0}^0\}$  be the set of geodesic segments  $\eta$  given by those subsegments of  $\gamma$  of length  $\tau$  whose endpoints are outside  $W_0$  and which intersect  $\gamma_0$  transversally at  $\eta(\tau/2)$  (see Figure 1). We now apply Theorem 4.1 to  $\gamma_0$ ,  $W_0$  and  $\mathcal{F}_0$ . The proof of this theorem also selects a neighborhood  $U_0$  of  $W_0 \cap \cup_{i=1}^{m_0} \eta_i^0([0, \tau])$ . We now consider  $\gamma_1$  and we choose a tubular neighborhood  $W_1$  of  $\gamma_1$  small enough so that if  $\gamma_1$  intersects  $\gamma_0$  transversally, then  $W_1$  intersected with  $W_0$  is contained in  $U_0$  (see Figure 1). By continuing in this fashion we select recursively tubular neighborhoods  $W_0, \dots, W_{m-1}$ , all contained in  $W$ , to which we successively apply Theorem 4.1. This choice of neighborhoods ensures that there is no interference between one perturbation and the next. In the end we obtain the following:

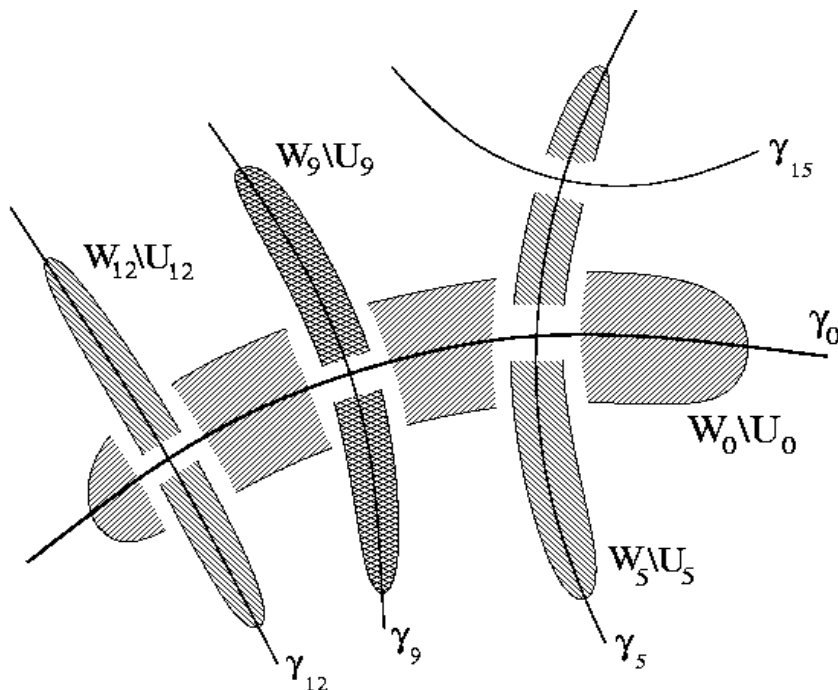


FIGURE 1. Avoiding self-intersections.

#### 4.2. Corollary.

Let  $g_0 \in \mathcal{G}^r$ ,  $r \geq 4$ . Given a neighbourhood  $\mathcal{U}$  of  $g_0$  in  $\mathcal{G}^2$ , there exists  $\delta = \delta(g_0, \mathcal{U}) > 0$  such that if  $g \in \mathcal{U}$ ,  $\gamma$  is a prime closed orbit of  $\phi^g$  and  $W$  is a tubular neighborhood of  $c = \pi \circ \gamma$ , then the image of  $\mathcal{U} \cap \mathcal{G}^r(\gamma, g_0, W) \rightarrow \prod_{k=0}^{m-1} Sp(1)$  under the map  $(S_0, \dots, S_{m-1})$  contains the product of balls of radius  $\delta$  centered at  $S_k(g_0)$  for  $0 \leq k < m$ .

The arguments below can be used to show that  $\bar{g} - g$  can be supported not only outside a finite number of intersecting segments but outside any given set<sup>2</sup> of measure zero in  $\gamma$ . This is done by adjusting the choice of the function  $h$  in (10).

The nature of these results (i.e. the independence on the size of the neighbourhood  $W$ ) fixes the  $C^1$  topology on the perturbation of the geodesic flow, thus the  $C^2$  topology on the metric. The size  $\delta(g_0, \mathcal{U}) > 0$  in theorem 4.1 and corollary 4.2 depends on the  $C^4$ -norm of  $g_0$ .

#### Proof of theorem 4.1.

Let us begin by describing informally the strategy that we shall follow to prove theorem 4.1. At the beginning we fix most of the constants and bump functions that are needed. Next we introduce Fermi coordinates along the geodesic  $c = \pi \circ \gamma$  and we consider a family of perturbations following Klingenberg and Takens in [30]. We show that the map  $S$  is a submersion when restricted to a suitable submanifold of the set of perturbations. To obtain a size  $\delta$  that depends only on  $g_0$  and  $\mathcal{U}$  and that works for all  $g \in \mathcal{U}$ ,  $\gamma$  and  $W$  we find

<sup>2</sup>But to use this argument to support  $\bar{g} - g$  outside a given infinite set of geodesic segments of length  $\geq \frac{1}{2}$  one needs to bound from below their angle of intersection with  $c$ .



a uniform lower bound for the norm of the derivative of  $S$  using the constants and the bump functions that we fixed before. This uniform estimate can only be obtained in the  $C^2$  topology.

The technicalities of the proof can be summarized as follows. To obtain a  $C^2$  perturbation of the metric preserving the geodesic segment  $c = \pi \circ \gamma$  one needs a perturbation of the form (12), with  $\alpha(t, x) = \varphi(x) \beta_A(t)$ ; where  $\varphi(x)$  is a bump function supported in an  $\varepsilon$ -neighbourhood in the transversal direction to  $c$  and  $\beta_A(t)$  is given by formula (31). In  $\beta_A(t)$ , the second factor is used to make the derivative of  $S$  surjective,<sup>3</sup> and the first factor  $h(t)$  is an approximation of a characteristic function used to support the perturbation outside a neighbourhood of the intersecting segments in  $\mathfrak{F} = \{\eta_1, \dots, \eta_m\}$ . Then inequality (8) shows that if the neighbourhood  $W$  of  $c$  is taken small enough, the  $C^2$  norm of the perturbation is essentially bounded by only the  $C^0$  norm of  $\beta_A(t)$ . In order to bound the  $C^2$  norm of  $\beta_A$  from (31) in equation (8), we use the hypothesis  $g_0 \in \mathcal{G}^4$  to have a bound for the second derivative of the curvature  $K_0(t, 0)$  of  $g_0$  along the geodesic  $c$ .

By shrinking  $\mathcal{U}$  if necessary, we can assume that

$$(3) \quad \|g\|_{C^2} \leq \|g_0\|_{C^2} + 1 \quad \text{for all } g \in \mathcal{U}.$$

Let  $k_1 = k_1(\mathcal{U}) > 1$  be such that if  $g \in \mathcal{U}$  and  $\phi_t$  is the geodesic flow of  $g$ , then

$$(4) \quad \|d_v \phi_t\| \leq k_1 \quad \text{and} \quad \|d_v \phi_t^{-1}\| \leq k_1 \quad \text{for all } t \in [0, 1]$$

and all  $v \in S_g^1 M$ . Let  $0 < \lambda \ll \frac{1}{2}$  and let  $k_2 = k_2(\mathcal{U}, \lambda) > 0$  be such that

$$(5) \quad \max_{|t-1/2| \leq \lambda} \|d_v \phi_t - d_v \phi_{1/2}\| \leq k_2 \quad \text{and} \quad \max_{|t-1/2| \leq \lambda} \|d_v \phi_t^{-1} - d_v \phi_{1/2}^{-1}\| \leq k_2$$

for all  $g \in \mathcal{U}$  and all  $v \in S_g^1 M$ . If  $\lambda = \lambda(g_0, \mathcal{U})$  is small enough, then

$$(6) \quad 0 < k_2 < \frac{1}{16 k_1^3} < 1 < k_1.$$

Let  $\delta_\lambda$  and  $\Delta_\lambda : [0, 1] \rightarrow [0, +\infty[$  be  $C^\infty$  functions such that  $\delta_\lambda$  has support on  $[\frac{1}{2} - \lambda, \frac{1}{2}[$ ,  $\Delta_\lambda$  has support on  $]\frac{1}{2}, \frac{1}{2} + \lambda]$ ,  $\int \delta_\lambda(t) dt = \int \Delta_\lambda(t) dt = 1$  and the support of  $\Delta_\lambda$  is an interval.

Let  $k_3 = k_3(g_0, \mathcal{U}, \lambda) = k_3(g_0, \mathcal{U})$  be

$$(7) \quad k_3 := k_1^2 [ \|\delta_\lambda\|_{C^0} + \|\delta'_\lambda\|_{C^0} + \|\Delta_\lambda\|_{C^0} (1 + \|g_0\|_{C^2}) + \|\Delta''_\lambda\|_{C^0} ]$$

where  $\delta'_\lambda$  and  $\Delta''_\lambda$  are the first and second derivatives of the functions  $\delta_\lambda$  and  $\Delta_\lambda$  with respect to  $t$ .

Given  $\varepsilon$  with  $0 < \varepsilon < 1$ , let  $\varphi_\varepsilon : \mathbb{R} \rightarrow [0, 1]$  be a  $C^\infty$  function such that  $\varphi_\varepsilon(x) = 1$  if  $x \in [-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]$  and  $\varphi_\varepsilon(x) = 0$  if  $x \notin [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ . In Lemma 4.5 it is proven that  $\varphi_\varepsilon(x)$  can be chosen such that

$$(8) \quad \|\varphi_\varepsilon(x) \beta(t) x^2\|_{C^2} \leq k_4 \|\beta\|_{C^0} + k_4 \varepsilon \|\beta\|_{C^1} + \varepsilon^2 \|\beta\|_{C^2}.$$

<sup>3</sup>The functions  $\delta_\lambda(t)$  and  $\Delta_\lambda(t)$  are approximations to a Dirac delta at  $t = \frac{1}{2}$ .

for some fixed  $k_4 > 0$  (independent of  $\varepsilon$ ) and any  $\beta : [0, 1] \rightarrow \mathbb{R}$  of class  $C^2$ .

Choose  $0 < \varrho \ll 1/(4k_1^2 k_3)$ . From (6), we have that

$$(9) \quad \frac{1}{k_1^2} - k_3 \varrho - 4k_1 k_2 > \frac{1}{2k_1^2}.$$

Let  $h : [0, 1] \rightarrow [0, 1]$  be a  $C^\infty$  function supported outside a neighbourhood of the intersecting points and the endpoints of the support of  $\Delta_\lambda$ ,

$$\text{supp}(h) \subset [0, 1] \setminus \left[ \gamma^{-1}(\cup_{i=1}^m \eta_i) \cup \partial \text{supp}(\Delta_\lambda) \right]$$

and such that

$$(10) \quad \int_0^1 |h(t) - 1| dt \leq \varrho.$$

We now introduce Fermi coordinates along the geodesic arc  $c = \pi \circ \gamma$ . All the facts that we will use about Fermi coordinates can be found in [22, 29]. Take an orthonormal frame  $\{\dot{c}(0), E\}$  in  $T_{c(0)}M$ . Let  $E(t)$  denote the parallel translation of  $E$  along  $c$ . Consider the differentiable map  $\Phi : [0, 1] \times \mathbb{R} \rightarrow M$  given by

$$\Phi(t, x) = \exp_{c(t)}(xE(t)).$$

This map has maximal rank at  $(t, 0)$ ,  $t \in [0, 1]$ . Since  $c(t)$  has no self intersections on  $t \in [0, 1]$  there exists a neighbourhood  $V$  of  $[0, 1] \times \{0\}$  in which  $\Phi|_V$  is a diffeomorphism.

Choose

$$(11) \quad \varepsilon_1 = \varepsilon_1(g_0, \mathcal{U}, \gamma, \mathfrak{F}) > 0$$

such that the segments  $\eta_i$  do not intersect the points with coordinates  $(t, x)$  with  $|x| < \varepsilon_1$  and  $t \in \text{supp}(h)$  and such that  $[0, 1] \times [-\varepsilon_1, \varepsilon_1] \subset V$  and  $\Phi([0, 1] \times [-\varepsilon_1, \varepsilon_1]) \subset W$ .

Let  $[g_0(t, x)]_{ij}$  denote the components of the metric  $g_0$  in the chart  $(\Phi, V)$ . Let  $\alpha(t, x)$  denote a  $C^\infty$  function on  $[0, 1] \times \mathbb{R}$  with support contained in  $V \setminus \Phi^{-1}[\cup_{i=0}^m \eta_i([0, 1])]$ . We can define a new riemannian metric  $g$  by setting

$$(12) \quad \begin{aligned} g_{00}(t, x) &= [g_0(t, x)]_{00} + \alpha(t, x) x^2; \\ g_{01}(t, x) &= [g_0(t, x)]_{01}; \\ g_{11}(t, x) &= [g_0(t, x)]_{11}; \end{aligned}$$

where we index the coordinates by  $x_0 = t$  and  $x_1 = x$ .

For any such metric  $g$  we have that (cf. [22, 29]):

$$\begin{aligned} g^{ij}(t, 0) &= g_{ij}(t, 0) = \delta_{ij}, & 0 \leq i, j \leq 1; \\ \partial_k g^{ij}(t, 0) &= \partial_k g_{ij}(t, 0) = 0, & 0 \leq i, j, k \leq 1; \end{aligned}$$

where  $[g^{ij}]$  is the inverse matrix of  $[g_{ij}]$ .

We need the differential equations for the geodesic flow  $\phi_t$  in hamiltonian form. It is well known that the geodesic flow is conjugated to the hamiltonian flow of the function

$$H(x, y) = \frac{1}{2} \sum_{ij} g^{ij}(x) y_i y_j.$$

Hamilton's equations are

$$\begin{aligned} \frac{d}{dt} x_i &= H_{y_i} = \sum_j g^{ij}(x) y_j, \\ \frac{d}{dt} y_k &= -H_{x_k} = -\frac{1}{2} \sum_{i,j} \frac{\partial}{\partial x_k} g^{ij}(x) y_i y_j. \end{aligned}$$

Let  $\mathcal{F}$  be the set of the riemannian metrics given by (12) endowed with the  $C^2$  topology. One easily checks that  $\mathcal{F} \subset \mathcal{G}^r(\gamma, g_0, W, \mathfrak{F})$ . Let

$$\mathcal{V} := \mathcal{F} \cap \mathcal{U}.$$

Using the identity  $\frac{d}{dt} (d\phi_t) = (dX \circ \phi_t) \cdot d\phi_t$ , with  $X = \frac{d}{dt} \phi_t|_{t=0}$ , we obtain the differential equations for the linearized hamiltonian flow, on the geodesic  $c(t)$  (given by:  $t, x = 0, y_0 = 1, y_1 = 0$ ), which we call the Jacobi equation:

$$(13) \quad \frac{d}{dt} \Big|_{(t,x=0)} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & -K \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix},$$

where

$$(14) \quad K(t, 0) = \frac{1}{2} \frac{\partial^2}{\partial x^2} g^{00}(t, 0) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} g_{00}(t, 0).$$

Let

$$K_0(t, 0) := \frac{1}{2} \frac{\partial^2}{\partial x^2} g_0^{00}(t, 0).$$

It is easy to check that

$$(15) \quad K(t, 0) = K_0(t, 0) - \alpha(t, 0).$$

By comparison with the usual Jacobi equation<sup>4</sup> we get that  $K(t, 0)$  is the curvature at the point  $c(t)$  for the metric  $g$ .

Observe from (12) that the conditions<sup>5</sup>

$$(16) \quad \begin{aligned} a_0(t) &= \langle h, \dot{c} \rangle_g = \sum_i g^{0i}(t, 0) a_i(t) \equiv 0, \\ b_0(t) &= \dot{a}_0(t) = \langle b, \dot{c} \rangle_g \equiv 0, \end{aligned}$$

<sup>4</sup>The geometric notion of curvature is not really used. The reader might just use equation (14) as the definition of curvature in this section.

<sup>5</sup>Here the products  $\langle h, \dot{c} \rangle_g$  and  $\langle b, \dot{c} \rangle_g$  are not needed to follow the argument. In fact, here  $\dot{c}(t)$  is the hamiltonian orbit corresponding to the geodesic  $c(t)$  in the cotangent bundle and  $\langle \cdot, \cdot \rangle_g$  is the riemannian metric in the cotangent bundle induced by  $g$ , whose coefficients are those of the inverse matrix  $[g^{ij}]$ . These products are included in (16) to suggest the reader that the following subspace  $\mathcal{N}_t$  is just the reduction of the space of Jacobi fields to those Jacobi fields which are orthogonal to the geodesic.

are invariant among the metrics  $g \in \mathcal{F}$  and satisfy (13). In particular the subspaces

$$\mathcal{N}_t = \{ (a, b) \in T_{c(t)}TM \mid a_0 = b_0 = 0 \} \approx \mathbb{R} \times \mathbb{R}$$

are invariant under (13) for all  $g \in \mathcal{F}$ . From now on reduce the Jacobi equation (13) to the subspaces  $\mathcal{N}_t$ .

We need uniform estimates for all  $g \in \mathcal{V}$ . Fix  $g \in \mathcal{V}$  and write

$$(17) \quad \mathbb{A}_t = \mathbb{A}_t^g = \begin{bmatrix} 0 & 1 \\ -K(t, 0) & 0 \end{bmatrix}_{2 \times 2}$$

where  $K$  is from (15). Let  $X_t = X_t^g = d\phi_t|_{\mathcal{N}_0} : \mathcal{N}_0 \rightarrow \mathcal{N}_t$  be the solution of the Jacobi equation (13) for  $g$ :

$$(18) \quad \dot{X}_t = \mathbb{A}_t X_t.$$

The *time 1* map  $X_1$  is a symplectic linear isomorphism:  $X_1^* \mathbb{J} X_1 = \mathbb{J}$ , where  $\mathbb{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ . Differentiating this equation we get the tangent space of the symplectic isomorphisms at  $X_1$ :  $\mathcal{T}_{X_1} = \{ Y \in \mathbb{R}^{2 \times 2} \mid X_1^* \mathbb{J} Y \text{ is symmetric} \}$ . Observe that, since  $X_1$  is symplectic:

$$(19) \quad \mathcal{T}_{X_1} = X_1 \cdot \mathcal{T}_I$$

and that  $\mathcal{T}_I$  is the space of  $2 \times 2$  matrices of the form<sup>6</sup>  $Z = \begin{bmatrix} b & c \\ a & -b \end{bmatrix}$ .

Let us consider the map given by

$$\mathcal{F} \ni g \xrightarrow{H} X_1^g \in Sp(1).$$

Equivalently,  $H$  is the restriction of  $S : \mathcal{G}^r(\gamma, g_0, W) \rightarrow Sp(1)$  to  $\mathcal{F}$ . We shall show that  $H$  is a submersion at any  $g \in \mathcal{V}$ . We start by finding a uniform lower bound for the norm of  $d_g H$  restricted to a suitable subspace.

#### 4.3. Lemma.

Consider a small parameter  $s$  near zero and write  $g_s = g + \alpha^s x_1^2 dx_0 \otimes dx_0 \in \mathcal{F}$  where

$$\alpha^s(t, x) := \varphi_\varepsilon(x) \beta^s(t),$$

where  $\beta^s(t)$  satisfies  $\beta^{s=0}(t) \equiv 0$  and

$$(20) \quad \left. \frac{\partial \beta^s(t)}{\partial s} \right|_{s=0} = h(t) \left\{ \delta(t) a + \delta'(t) b - (\Delta_\lambda(t) K^s(t, 0) + \frac{1}{2} \Delta_\lambda''(t)) c \right\},$$

where  $a, b, c \in \mathbb{R}$ ,  $0 \leq h(t) \leq 1$  satisfies (10),  $K(t, 0)$  is the curvature of  $g$  at  $(t, 0)$  and  $0 < \varepsilon < \varepsilon_1$ . In particular  $\alpha^s$  has support contained in  $V$ .

Then

$$\|d_g H\left(\left.\frac{d}{ds} g_s\right|_{s=0}\right)\| \geq \frac{1}{2k_1^3} \left\| \begin{bmatrix} b & c \\ a & -b \end{bmatrix} \right\|$$

We use  $h(t)$  to support the perturbation of the riemannian metric outside the intersecting segments and also to bound the  $C^2$  norm of the term  $\frac{\Delta_\lambda''(t)}{\Delta_\lambda(t)} (e^{-h\Delta_\lambda c} - 1)$  in equation (31).

---

<sup>6</sup>If  $\dim M > 2$  the elements of  $\mathcal{T}_I$  have the form  $\begin{bmatrix} b & c \\ a & d \end{bmatrix}$ , with  $a$  and  $c$  symmetric and  $d = -b^*$ . The arguments shown here are not sufficient to cover this case.

**Proof:** From (13), we see that  $X_t^{g_s}$  satisfies

$$\dot{X}_t^{g_s} = (\mathbb{A}_t + D_t^s) X_t^{g_s},$$

where  $\mathbb{A}_t$  is from (17) and  $D_t = \begin{bmatrix} 0 & 0 \\ \alpha^s(t,0) & 0 \end{bmatrix}$ . Thus the derivative of the map  $H$  satisfies  $d_g H(\frac{d}{ds} g_s|_{s=0}) = Z_1$ , where

$$\dot{Z}_t = \mathbb{A}_t Z_t + E_t X_t,$$

where  $E_t = \frac{d}{ds}|_{s=0} D_t^s = h(t) \begin{bmatrix} 0 & 0 \\ \frac{\partial \beta}{\partial s}|_{s=0}(t) & 0 \end{bmatrix}$ . Writing  $Z_t = X_t W_t$  and using that  $\dot{X}_t = \mathbb{A}_t X_t$ , we get that  $\dot{W}_t = X_t^{-1} E_t X_t$ . Hence

$$(21) \quad Z_1 = X_1 \int_0^1 X_t^{-1} E_t X_t dt.$$

Write  $A := \begin{bmatrix} b & c \\ a & -b \end{bmatrix}$ . We have to prove that

$$\|Z_1\| \geq \frac{1}{2k_1^3} \|A\| \quad \text{for all } g \in \mathcal{V}.$$

We compute the integral in (21). Write  $B = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$  and  $C = \begin{bmatrix} 0 & 0 \\ -c/2 & 0 \end{bmatrix}$ . Then, using (18),

$$\begin{aligned} \int_0^1 X_t^{-1} \delta'_\lambda(t) B X_t dt &= - \int_0^1 \delta_\lambda(t) \left[ (X_t^{-1})' B X_t + X_t^{-1} B X_t' \right] dt \\ &= \int_0^1 \delta_\lambda(t) X_t^{-1} [\mathbb{A}_t B - B \mathbb{A}_t] X_t dt \\ &= \int_0^1 \delta_\lambda(t) X_t^{-1} \begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix} X_t dt. \end{aligned}$$

$$\begin{aligned} \int_0^1 X_t^{-1} \Delta''_\lambda(t) C X_t dt &= \int_0^1 \Delta'_\lambda(t) X_t^{-1} \begin{bmatrix} -\frac{c}{2} & 0 \\ 0 & \frac{c}{2} \end{bmatrix} X_t dt \\ &= \int_0^1 \Delta_\lambda(t) X_t^{-1} \left[ \mathbb{A}_t \begin{bmatrix} -c/2 & 0 \\ 0 & c/2 \end{bmatrix} - \begin{bmatrix} -c/2 & 0 \\ 0 & c/2 \end{bmatrix} \mathbb{A}_t \right] X_t dt \\ &= \int \Delta_\lambda(t) X_t^{-1} \begin{bmatrix} \frac{1}{2}(K(t,0)c + cK(t,0)) & c \\ 0 & 0 \end{bmatrix} X_t dt. \end{aligned}$$

Hence,

$$\int_0^1 X_t^{-1} \frac{E_t}{h(t)} X_t dt = \int_0^1 \delta_\lambda(t) X_t^{-1} \begin{bmatrix} b & 0 \\ a & -b \end{bmatrix} X_t dt + \int_0^1 \Delta_\lambda(t) X_t^{-1} \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} X_t dt.$$

Write  $P(t) = X_t^{-1} \frac{E_t}{h(t)} X_t$ ,  $Q_1(t) = X_t^{-1} \begin{bmatrix} b & 0 \\ a & -b \end{bmatrix} X_t$ ,  $Q_2(t) = X_t^{-1} \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} X_t$  and  $\mathcal{Q}(t) = X_t^{-1} \begin{bmatrix} b & c \\ a & -b \end{bmatrix} X_t$ . Then

$$(22) \quad \int_0^1 P(t) dt = \int_0^1 \delta_\lambda(t) Q_1(t) dt + \int_0^1 \Delta_\lambda(t) Q_2(t) dt.$$

Using (4) we have that

$$\begin{aligned} \|\delta_\lambda(t) Q_1(t)\| &\leq \|\delta_\lambda\|_{C^0} \|X_t^{-1}\| \sqrt{2} \max\{|a|, |b|\} \|X_t\| \\ &\leq \|\delta_\lambda\|_{C^0} \cdot k_1 \cdot \|A\| \cdot k_1. \end{aligned}$$

Similarly

$$\|\Delta_\lambda(t) Q_1(t)\| \leq \|\Delta_\lambda\|_{C^0} k_1^2 |c| \leq \|\Delta_\lambda\|_{C^0} k_1^2 \|A\|.$$

Hence, using (7), we have that

$$\begin{aligned} \|P\|_0 &\leq k_1^2 (\|\delta_\lambda\|_{C^0} + \|\Delta_\lambda\|_{C^0}) \|A\| \\ (23) \quad &\leq k_3(\lambda) \|A\|. \end{aligned}$$

From (21), we have that

$$(24) \quad Z_1 = X_1 \int_0^1 h(t) P(t) dt.$$

Observe that

$$\begin{aligned} \left\| \int_0^1 \delta_\lambda(t) Q_1(t) dt - Q_1\left(\frac{1}{2}\right) \right\| &\leq \int_0^1 \delta_\lambda(t) \|Q_1(t) - Q_1\left(\frac{1}{2}\right)\| dt \leq \mathcal{O}_\lambda(Q_1, \frac{1}{2}), \\ \left\| \int_0^1 \Delta_\lambda(t) Q_2(t) dt - Q_2\left(\frac{1}{2}\right) \right\| &\leq \int_0^1 \Delta_\lambda(t) \|Q_2(t) - Q_2\left(\frac{1}{2}\right)\| dt \leq \mathcal{O}_\lambda(Q_2, \frac{1}{2}), \end{aligned}$$

where  $\mathcal{O}_\lambda(Q_i, \frac{1}{2}) := \max_{|t-1/2| \leq \lambda} \|Q_i(t) - Q_i(\frac{1}{2})\|$ . Thus, using (22),

$$\begin{aligned} \left\| \int_0^1 h(t) P(t) dt - Q\left(\frac{1}{2}\right) \right\| &= \left\| \int h P dt - \int_0^1 P dt + \int P dt - Q\left(\frac{1}{2}\right) \right\| \\ &\leq \left\| \int (h-1) P \right\| + \left\| \int \delta_\lambda(t) Q_1(t) - Q_1\left(\frac{1}{2}\right) \right\| + \left\| \int \Delta_\lambda(t) Q_2(t) - Q_2\left(\frac{1}{2}\right) \right\|, \\ &\leq \|P\|_0 \int |h-1| + \mathcal{O}_\lambda(Q_1, \frac{1}{2}) + \mathcal{O}_\lambda(Q_2, \frac{1}{2}) \\ &\leq \|P\|_0 \int |h-1| + 2 \mathcal{O}_\lambda(Q, \frac{1}{2}), \quad \text{because } Q = Q_1 + Q_2. \end{aligned}$$

If  $f, g : [0, 1] \rightarrow \mathbb{R}^{2 \times 2}$ , by adding and subtracting  $f(t)g(\frac{1}{2})$ , we obtain the formula

$$(25) \quad \mathcal{O}_\lambda(fg, \frac{1}{2}) \leq \|f\|_0 \mathcal{O}_\lambda(g, \frac{1}{2}) + \mathcal{O}_\lambda(f, \frac{1}{2}) \|g(\frac{1}{2})\|.$$

Also, if  $e \in \mathbb{R}^{2 \times 2}$  is constant, then

$$(26) \quad \mathcal{O}_\lambda(ef, \frac{1}{2}) \leq \|e\| \mathcal{O}_\lambda(f, \frac{1}{2}).$$

Write  $A = \begin{bmatrix} b & c \\ a & -b \end{bmatrix}$ . Using formulas (25), (26); from (4) and (5) we have that

$$\begin{aligned} \mathcal{O}_\lambda(\mathcal{Q}, \tfrac{1}{2}) &= \mathcal{O}_\lambda(X_t^{-1} A X_t, \tfrac{1}{2}) \\ &\leq \|X_t^{-1}\|_0 \mathcal{O}_\lambda(A X_t, \tfrac{1}{2}) + \mathcal{O}_\lambda(X_t^{-1}, \tfrac{1}{2}) \|A\| \|X_{1/2}\|_0 \\ &\leq \|X_t^{-1}\|_0 \|A\| \mathcal{O}_\lambda(X_t, \tfrac{1}{2}) + \mathcal{O}_\lambda(X_t^{-1}, \tfrac{1}{2}) \|A\| \|X_{1/2}\|_0 \\ &\leq 2 k_1 k_2 \|A\|. \end{aligned}$$

Also, from (23) and (10),

$$\|P\|_0 \int |h-1| \leq \|A\| k_3(\lambda) \int |h-1| \leq k_3 \|A\| \varrho.$$

Moreover

$$\|A\| = \|X_{1/2} \mathcal{Q}(\tfrac{1}{2}) X_{1/2}^{-1}\| \leq k_1^2 \|\mathcal{Q}(\tfrac{1}{2})\|.$$

Hence, using (9),

$$\begin{aligned} \left\| \int h P dt \right\| &\geq \|\mathcal{Q}(\tfrac{1}{2})\| - \left\| \int h P - \mathcal{Q}(\tfrac{1}{2}) \right\| \\ &\geq \left( \frac{1}{k_1^2} - k_3 \varrho - 4 k_1 k_2 \right) \|A\| \\ &\geq \frac{1}{2k_1^2} \|A\|. \end{aligned}$$

This implies that the transformation  $\mathcal{T}_I \ni A \mapsto \int_0^1 h(t) P(t) dt \in \mathcal{T}_I$  is onto. From (19) and (24), the map  $\mathcal{T}_I \ni A \mapsto Z_1 \in \mathcal{T}_{X_1}$  is surjective. Moreover, using (4) and (24),

$$k_1 \|Z_1\| \geq \|X_1^{-1} Z_1\| = \left\| \int_0^1 h P dt \right\| \geq \frac{1}{2k_1^2} \|A\|.$$

Thus

$$\|Z_1\| \geq \frac{1}{2k_1^3} \|A\| \quad \text{for all } g \in \mathcal{V}.$$

□

We shall combine lemma 4.3 with the next lemma to prove the theorem.

#### 4.4. Lemma.

Let  $\mathcal{N}$  be a smooth connected riemannian 3-manifold and let  $F : \mathbb{R}^3 \rightarrow \mathcal{N}$  be a smooth map such that

$$(27) \quad |d_x F(v)| \geq a > 0 \quad \text{for all } (x, v) \in T\mathbb{R}^3 \text{ with } |v| = 1 \text{ and } |x| \leq r.$$

Then for all  $0 < b < ar$ ,

$$\{w \in \mathcal{N} \mid d(w, F(0)) < b\} \subseteq F\{x \in \mathbb{R}^3 \mid |x| < \frac{b}{a}\}.$$

**Proof:** Let  $w \in \mathcal{N}$  with  $d(w, F(0)) < b$ . Let  $\beta : [0, 1] \rightarrow \mathcal{N}$  be a differentiable curve with  $\beta(0) = F(0)$ ,  $\beta(1) = w$  and  $|\dot{\beta}| < b$ . Let  $\tau = \sup(A)$ , where  $A \subset [0, 1]$  is the set of  $t \in [0, 1]$  such that there exist a unique  $C^1$  curve  $\alpha : [0, t] \rightarrow \mathbb{R}^3$  such that  $\alpha(0) = 0$ ,  $|\alpha(s)| < r$  and

$F(\alpha(s)) = \beta(s)$  for all  $s \in [0, t]$ . By the inverse function theorem  $\tau > 0$ ,  $A$  is open in  $[0, 1]$  and there exist a unique  $\alpha : [0, \tau] \rightarrow \mathbb{R}^3$  such that  $F \circ \alpha = \beta$ . By (27),

$$(28) \quad |\dot{\beta}(s)| = \|d_{\alpha(s)}F\| \cdot |\dot{\alpha}(s)| \geq a |\dot{\alpha}(s)|, \quad \text{for all } s \in [0, \tau].$$

Thus,  $|\dot{\alpha}| \leq \frac{1}{a} \max_{0 \leq t \leq 1} |\dot{\beta}(t)|$ . This implies that  $\alpha$  is Lipschitz and hence it can be extended continuously to  $[0, \tau]$ . Observe that  $|\alpha(\tau)| < r$ , for if  $|\alpha(\tau)| \geq r$ , then

$$b \geq b\tau \geq \int_0^\tau |\dot{\beta}(s)| ds \geq a \int_0^\tau |\dot{\alpha}(s)| ds \geq ar,$$

contradicting the hypothesis  $b < ar$ . This implies that the set  $A$  is also closed in  $[0, 1]$ . Thus  $A = [0, 1]$  and  $\tau = 1$ . From (28), writing  $x = \alpha(1) \in F^{-1}\{w\}$ ,

$$|x| \leq \text{length}(\alpha) = \int_0^1 |\dot{\alpha}(t)| dt \leq \frac{1}{a} \int_0^1 |\dot{\beta}(t)| dt < \frac{b}{a}.$$

□

We now see that the condition (27) of Lemma 4.4 holds in our setting. Let  $k_5 = k_5(g_0, \mathcal{U}, \gamma, \mathfrak{F})$  and  $k_6 = k_6(g_0, \mathcal{U}, \gamma, \mathfrak{F})$  be

$$\begin{aligned} k_5 &:= \|\delta_\lambda\|_0 + \|\delta'_\lambda\|_0 + [\|\Delta_\lambda\|_0 \|g_0\|_{C^2} + \frac{1}{2}\|\Delta''_\lambda\|_0] e^{\|\Delta_\lambda\|_0}, \\ k_6 &:= \max_{|c| \leq 1} \left\{ 2 \|h\|_{C^2} \left[ \|\delta_\lambda\|_{C^2} + \|\delta'_\lambda\|_{C^2} \right] + 2 \|g_0\|_{C^4} \|(e^{-h\Delta_\lambda c} - 1)\|_{C^2} \right. \\ &\quad \left. + \left\| \frac{\Delta''_\lambda}{2\Delta_\lambda} (e^{-h\Delta_\lambda c} - 1) \right\|_{C^2} \right\}, \end{aligned}$$

observe that since  $\Delta_\lambda > 0$  on  $\text{supp}(h)$ , the last term in  $k_6$  is finite.

Let  $0 < \rho_1 < 1$  be such that the closed ball

$$(29) \quad \overline{B_{\mathcal{G}^2}(g_0, \rho_1)} \subseteq \mathcal{U}.$$

Choose  $0 < \varepsilon = \varepsilon(g_0, \mathcal{U}, \gamma, \mathfrak{F}) < \varepsilon_1$ , small enough so that

$$(\varepsilon k_4 + \varepsilon^2) k_6 \leq \frac{1}{2} \rho_1.$$

Choose  $0 < \delta < 1$  such that

$$(30) \quad k_4 k_5 (2k_1^3 \delta) + (\varepsilon k_4 + \varepsilon^2) k_6 \leq \rho_1 < 1 \quad \text{and} \quad 2k_1^3 \delta \leq 1.$$

For  $A = \begin{bmatrix} b & c \\ a & -b \end{bmatrix}$ , let

$$(31) \quad \beta_A(t) := h(t) \left\{ \delta_\lambda(t) a + \delta'_\lambda(t) b \right\} + \left( K_0(t, 0) + \frac{\Delta''_\lambda(t)}{2\Delta_\lambda(t)} \right) (e^{-h(t)\Delta_\lambda(t)c} - 1);$$

and let  $g_A \in \mathcal{G}^{r-2}(\gamma, g_0, W, \mathfrak{F})$  be the riemannian metric

$$g_A := g_0 + \varphi_\varepsilon(x) \beta_A(t) x^2 dt \otimes dt.$$

Observe that  $\beta_A = 0$  when  $h(t) = 0$ , so that  $g = g_0$  in a neighbourhood of the intersections of the segments  $\eta_i$  with  $c = \pi \circ \gamma$ . Then the choice of  $\varepsilon < \varepsilon_1$  from (11), ensures that  $g = g_0$  in a neighbourhood of the intersecting segments.



Observe that

$$\begin{aligned} \frac{\partial \beta_A}{\partial a} &= h(t) \delta_\lambda(t), & \frac{\partial \beta_A}{\partial b} &= h(t) \delta'_\lambda(t), \\ \frac{\partial \beta_A}{\partial c} &= -h(t) \left\{ \Delta_\lambda(t) (K_0(t, 0) + \beta_A(t)) + \frac{1}{2} \Delta''_\lambda(t) \right\}, \end{aligned}$$

In particular, its directional derivatives of the map  $\mathcal{T}_I \ni A \mapsto \beta_A$  are given by formula (20). (Note that  $K_0(t, 0) + \beta_A(t)$  is the curvature of  $g_A$  at  $(t, 0)$ .) Indeed,

$$\begin{aligned} \frac{\partial \beta_A}{\partial c} + h(t) \Delta_\lambda(t) \beta_A(t) &= h(t)^2 \Delta_\lambda(t) \left\{ \delta_\lambda(t) a + \delta'_\lambda(t) b \right\} - h(t) \left\{ K_0(t, 0) \Delta_\lambda(t) + \frac{1}{2} \Delta''_\lambda(t) \right\} \\ &= -h(t) \left\{ K_0(t, 0) \Delta_\lambda(t) + \frac{1}{2} \Delta''_\lambda(t) \right\}, \end{aligned}$$

because  $\Delta_\lambda(t) \delta_\lambda(t) \equiv 0$  and  $\Delta_\lambda(t) \delta'_\lambda(t) \equiv 0$ .

Define  $F : \mathcal{T}_I \rightarrow Sp(1)$  by

$$F(A) = S(g_A) = d_{\dot{c}(0)} \phi_1^{g_A} \Big|_{\mathcal{N}_1}.$$

Applying lemma 4.3, we get that if  $g_B \in \mathcal{V}$ , then the derivative  $d_B F$  satisfies

$$(32) \quad \|(d_B F) \cdot A\| \geq \frac{1}{2k_1^3} \|A\|, \quad \text{if } g_B \in \mathcal{V}.$$

Let  $G : \mathcal{T}_I \rightarrow \mathcal{G}^{r-2}(\gamma, g_0, W, \mathfrak{F})$  be the map  $G(A) = g_A$ . By lemma 4.5, we have that

$$(33) \quad \begin{aligned} \|G(A) - g_0\|_{C^2} &= \|\varphi_\varepsilon(x) \beta_A(t) x^2\|_{C^2} \\ &\leq k_4 \|\beta_A\|_{C^0} + \varepsilon k_4 \|\beta_A\|_{C^1} + \varepsilon^2 \|\beta_A\|_{C^2} \end{aligned}$$

Observe that for  $|c| \leq 1$  we have that

$$\left| e^{-h\Delta_\lambda c} - 1 \right| \leq |c| \max_{|c| \leq 1} \left| \frac{\partial}{\partial c} (e^{-h\Delta_\lambda c} - 1) \right| \leq |c| \Delta_\lambda e^{\|\Delta_\lambda\|_0}, \quad \text{if } |c| \leq 1.$$

Then, if  $|c| \leq 1$ ,

$$\begin{aligned} \left| K_0(t, 0) + \frac{\Delta''_\lambda(t)}{2\Delta_\lambda(t)} \right| \left| e^{-h\Delta_\lambda c} - 1 \right| &\leq [ |K_0| \Delta_\lambda + \frac{1}{2} |\Delta''_\lambda| ] e^{\|\Delta_\lambda\|_0} |c| \\ &\leq |c| \left[ \|g_0\|_{C^2} \|\Delta_\lambda\|_0 + \frac{1}{2} \|\Delta''_\lambda\|_0 \right] e^{\|\Delta_\lambda\|_0} \end{aligned}$$

Hence

$$\|\beta_A\|_{C^0} \leq k_5 \|A\|, \quad \text{if } \|A\| \leq 1.$$

Since  $\|f \cdot g\|_{C^2} \leq 2 \|f\|_{C^2} \|g\|_{C^2}$  then  $\|\beta_A\|_{C^1} \leq \|\beta_A\|_{C^2} \leq k_6$ . Then from (33) we get that

$$\|G(A) - g_0\|_{C^2} \leq k_4 k_5 \|A\| + (\varepsilon k_4 + \varepsilon^2) k_6, \quad \text{if } \|A\| \leq 1 \text{ and } G(A) \in \mathcal{U}.$$

By definition of  $\rho_1$  in (29), we can write  $\mathcal{W} := B_{\mathcal{G}^2}(g_0, \rho_1) \cap G(\mathcal{T}_I) \subset \mathcal{V} \subset \mathcal{U}$ . Then (30) implies that

$$G(B_{\mathcal{T}_I}(0, 2k_1^3 \delta)) \subseteq \mathcal{W} \subset \mathcal{V}.$$

Hence the hypothesis  $g_B \in \mathcal{V}$  of (32) is satisfied.

$$\begin{array}{ccc} \mathcal{T}_I \supset B(0, 2k_1^3\delta) & \xrightarrow{G} & \mathcal{W} \subset \mathcal{V} \subset \mathcal{G}^2 \\ & \searrow F & \vdots H \\ & & Sp(1) \end{array}$$

Applying lemma 4.4 to  $F$  in (32), with  $r = 2k_1^3\delta$  and  $a = \frac{1}{2k_1^3}$ , we get that

$$B_{Sp(1)}(S(g_0), \delta) \subseteq F(B_{\mathcal{T}_I}(0, 2k_1^3\delta)) \subseteq F(G^{-1}(\mathcal{W})) \subset S(\mathcal{U} \cap \mathcal{G}^r(\gamma, g_0, W, \mathfrak{F})).$$

□

### Bump functions

**4.5. Lemma.** *There exist  $k_4 > 0$  and a family of  $C^\infty$  functions  $\varphi_\varepsilon : [-\varepsilon, \varepsilon]^{n-1} \rightarrow [0, 1]$  such that  $\varphi_\varepsilon(x) \equiv 1$  if  $x \in [-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]^{n-1}$ ,  $\varphi_\varepsilon(x) \equiv 0$  if  $x \notin [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^{n-1}$  and for any map  $B : [0, 1] \rightarrow \mathbb{R}^{(n-1) \times (n-1)}$ ,*

$$\|\varphi_\varepsilon(x) x^* B(t) x\|_{C^2} \leq k_4 \|B\|_{C^0} + \varepsilon k_4 \|B\|_{C^1} + \varepsilon^2 \|B\|_{C^2},$$

with  $k_4$  independent of  $0 < \varepsilon < 1$ .

**Proof:** Let  $\psi : [-1, 1] \rightarrow [0, 1]$  be a  $C^\infty$  function such that  $\psi(x) \equiv 1$  for  $|x| \leq \frac{1}{4}$  and  $\psi(x) \equiv 0$  for  $|x| \geq \frac{1}{2}$ . Given  $\varepsilon > 0$  let  $\varphi = \varphi_\varepsilon : [-\varepsilon, \varepsilon]^{n-1} \rightarrow [0, 1]$  be defined by  $\varphi(x) = \prod_{i=1}^{n-1} \psi(\frac{x_i}{\varepsilon})$ . Let  $B \in \mathbb{R}^{(n-1) \times (n-1)}$  and let  $\beta(x) = \varphi(x) x^* Bx$ . Then

$$(34) \quad \|\beta\|_0 \leq \varepsilon^2 \|B\|$$

$$d_x \beta = (d_x \varphi) x^* Bx + \varphi(x) x^* (B + B^*)$$

$$\frac{\partial \varphi}{\partial x_i} = \frac{1}{\varepsilon} \psi' \left( \frac{x_i}{\varepsilon} \right) \prod_{k \neq i} \psi \left( \frac{x_k}{\varepsilon} \right)$$

$$(35) \quad \|d_x \varphi\| \leq \frac{1}{\varepsilon} \|d\psi\|_0$$

$$(36) \quad \|d_x \beta\| \leq 3\varepsilon \|B\| \|\psi\|_{C^1}$$

$$d_x^2 \beta = (d_x^2 \varphi) x^* Bx + 2(d_x \varphi) x^* (B + B^*) + \varphi(x) (B + B^*)$$

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \frac{1}{\varepsilon^2} \psi'' \left( \frac{x_i}{\varepsilon} \right) \prod_{k \neq i} \psi \left( \frac{x_k}{\varepsilon} \right) \delta_{ij} + \frac{1}{\varepsilon^2} \psi' \left( \frac{x_i}{\varepsilon} \right) \psi' \left( \frac{x_j}{\varepsilon} \right) \prod_{k \neq i, j} \psi \left( \frac{x_k}{\varepsilon} \right) (1 - \delta_{ij}).$$

$$\|d_x^2 \varphi\| \leq \frac{1}{\varepsilon^2} \max \{ \|d^2 \psi\|_0, \|d\psi\|_0^2 \} \leq \frac{1}{\varepsilon^2} \|\psi\|_{C^2}^2.$$

$$\|d_x^2 \beta\| \leq \|\psi\|_{C^2}^2 \|B\| (1 + 4 + 2)$$

$$(37) \quad \leq 7 \|\psi\|_{C^2}^2 \|B\|.$$

Let  $k_4 := 4 + 3 \|\psi\|_{C^1} + 7 \|\psi\|_{C^2}^2$ . Then from (34), (36) and (37), we have that

$$(38) \quad \|\beta\|_{C^2} \leq k_4 \|B\|.$$

Now let  $\alpha(t, x) := \varphi(x) x^* B(t) x$ . Observe that

$$\begin{aligned} \|\alpha\|_{C^2} &\leq \|\alpha\|_{C^2(x)} + \|\alpha\|_{C^2(t)} + 2 \left\| \frac{\partial^2 \alpha}{\partial x \partial t} \right\|_0 \\ &\leq \|\beta\|_{C^2} + \varepsilon^2 \|B\|_{C^2} + 2 \left\| \frac{\partial^2 \alpha}{\partial x \partial t} \right\|_0. \end{aligned}$$

But, using (35),

$$\begin{aligned} \frac{\partial^2 \alpha}{\partial x \partial t} &= d_x \varphi \cdot x^* B'(t) x + \varphi(x) [x^* B'(t) + B'(t) x] \\ \frac{\partial^2 \alpha}{\partial x \partial t} &\leq \varepsilon \|\psi\|_{C^1} \|B'\|_0 + 2\varepsilon \|B'\|_0 \\ &\leq \frac{1}{2} k_4 \varepsilon \|B\|_{C^1}. \end{aligned}$$

Hence, using (38),

$$\|\alpha\|_{C^2} \leq k_4 \|B\|_{C^0} + k_4 \varepsilon \|B\|_{C^1} + \varepsilon^2 \|B\|_{C^2}.$$

□

## 5. Dominated splittings for geodesic flows.

We say that a linear map  $T : \mathbb{R}^N \rightarrow \mathbb{R}^N$  is *hyperbolic* if it has no eigenvalue of modulus 1. The *stable* and *unstable* subspaces of  $T$  are

$$E^s(T) := \{v \in \mathbb{R}^N \mid \lim_{n \rightarrow +\infty} T^n(v) = 0\}, \quad E^u(T) := \{v \in \mathbb{R}^N \mid \lim_{n \rightarrow +\infty} T^{-n}(v) = 0\}.$$

### 5.1. Periodic sequences of symplectic maps.

Let  $GL(\mathbb{R}^N)$  be the group of linear isomorphisms of  $\mathbb{R}^N$ . We say that a sequence  $\xi : \mathbb{Z} \rightarrow GL(\mathbb{R}^N)$  is *periodic* if there exists  $n_0 \geq 1$  such that  $\xi_{j+n_0} = \xi_j$  for all  $j \in \mathbb{Z}$ . We say that a periodic sequence  $\xi$  is *hyperbolic* if the linear map  $\prod_{i=0}^{n_0-1} \xi_i$  is hyperbolic. In this case the stable and unstable subspaces of  $\prod_{i=0}^{n_0-1} \xi_{i+j}$  are denoted by  $E_j^s(\xi)$  and  $E_j^u(\xi)$  respectively.

Given two families of sequences in  $GL(\mathbb{R}^N)$ ,  $\xi = \{\xi^{(\alpha)} \mid \alpha \in \mathcal{A}\}$  and  $\eta = \{\eta^{(\alpha)} \mid \alpha \in \mathcal{A}\}$ , define

$$d(\xi, \eta) = \sup \{ \|\xi_n^{(\alpha)} - \eta_n^{(\alpha)}\| \mid \alpha \in \mathcal{A}, n \in \mathbb{Z} \}.$$

We say that two families are *periodically equivalent* if they have the same indexing set  $\mathcal{A}$  and for all  $\alpha \in \mathcal{A}$  the minimum periods of  $\xi^{(\alpha)}$  and  $\eta^{(\alpha)}$  coincide. We say that a family  $\xi$  is *hyperbolic* if for all  $\alpha \in \mathcal{A}$ , the periodic sequence  $\xi^{(\alpha)}$  is hyperbolic. Finally, we say that a hyperbolic periodic family  $\xi$  is *stably hyperbolic* if there exists  $\varepsilon > 0$  such that any periodically equivalent family  $\eta$  satisfying  $d(\eta, \xi) < \varepsilon$  is also hyperbolic.

**5.1. Theorem** (Mañé, [35, lemma II.3]).

If  $\{\xi^{(\alpha)} \mid \alpha \in \mathcal{A}\}$  is a stably hyperbolic family of periodic sequences of isomorphisms of  $\mathbb{R}^N$ , then there exist constants  $m \in \mathbb{Z}^+$  and  $0 < \lambda < 1$  such that for all  $\alpha \in \mathcal{A}$ ,  $j \in \mathbb{Z}$ :

$$\left\| \prod_{i=0}^{m-1} \xi_{j+i}^{(\alpha)} \Big| E_j^s(\xi^{(\alpha)}) \right\| \cdot \left\| \left[ \prod_{i=0}^{m-1} \xi_{j+i}^{(\alpha)} \right]^{-1} \Big| E_{j+m}^u(\xi^{(\alpha)}) \right\| \leq \lambda.$$

Denote by  $Sp(1) = SL(2, \mathbb{R})$  the group of symplectic linear maps in  $\mathbb{R}^2$ . Lemma 5.4 below shows that if a periodic sequence  $\xi$  of symplectic maps in  $\mathbb{R}^2$  is stably hyperbolic among the periodic sequences in  $Sp(1)$  and  $\sup_{\alpha} \|\xi^{(\alpha)}\| < \infty$ , then it is also stably hyperbolic among the sequences in  $GL(\mathbb{R}^2)$ . Thus we get:

**5.2. Corollary.**

If  $\{\xi^{(\alpha)} \mid \alpha \in \mathcal{A}\}$  is a family of periodic sequences in  $Sp(1)$  which is stably hyperbolic in  $Sp(1)$ , and  $\sup_{\alpha} \|\xi^{(\alpha)}\| < \infty$ . Then there exist constants  $m \in \mathbb{Z}^+$  and  $0 < \lambda < 1$  such that for all  $\alpha \in \mathcal{A}$ ,  $j \in \mathbb{Z}$ :

$$\left\| \prod_{i=0}^{m-1} \xi_{j+i}^{(\alpha)} \Big| E_j^s(\xi^{(\alpha)}) \right\| \cdot \left\| \left[ \prod_{i=0}^{m-1} \xi_{j+i}^{(\alpha)} \right]^{-1} \Big| E_{j+m}^u(\xi^{(\alpha)}) \right\| \leq \lambda.$$

**5.3. Remark.** Write  $T_j^N := \prod_{i=0}^{N-1} \xi_{j+i}^{(\alpha)}$ . Using that  $\|AB\| \leq \|A\| \|B\|$  for  $A, B \in GL(\mathbb{R}^2)$  we get that for all  $N \geq 1$  and all  $\alpha \in \mathcal{A}$ ,  $j \in \mathbb{Z}$ ,

$$\left\| T_j^{mN} \Big| E_j^s(\xi^{(\alpha)}) \right\| \cdot \left\| [T_j^{mN}]^{-1} \Big| E_{j+Nm}^u(\xi^{(\alpha)}) \right\| < \lambda^N.$$

**5.4. Lemma.** If  $F_k \in GL(\mathbb{R}^2)$ ,  $T_k \in Sp(1)$ ,  $\|F_k - T_k\| < \varepsilon$  for  $k = 1, \dots, N$ ;  $\mathbb{T} = T_N \circ T_{N-1} \circ \dots \circ T_1$  is hyperbolic and  $\mathbb{F} = F_N \circ F_{N-1} \circ \dots \circ F_1$  is not hyperbolic. Suppose that

$$2\varepsilon \left( 1 + 2 \max_{1 \leq j \leq N} \|T_j\| \right) < \frac{1}{2}.$$

Then there exist  $A_k \in Sp(1)$  such that

$$\|A_k - T_k\| < 16\varepsilon \left( 2 + \max_{1 \leq j \leq N} \|T_j\| \right)^2$$

and  $\mathbb{A} := A_N \circ A_{N-1} \circ \dots \circ A_1$  is not hyperbolic.

**Proof:**

Suppose first that  $\mathbb{F}$  has complex eigenvalues  $\lambda$  and  $\bar{\lambda}$ . Since  $\mathbb{F}$  is not hyperbolic then  $|\lambda| = |\bar{\lambda}| = 1$ , and hence  $\det \mathbb{F} = +1$ .

Let  $e_1 = (1, 0)$ ,  $e_2 = (0, 1)$  and

$$\lambda_k := \det F_k = \omega(F_k e_1, F_k e_2).$$

Since  $\omega(a, b) \leq |a| |b|$ , we have that

$$\begin{aligned} |\lambda_k - 1| &= |\omega(F_k e_1, F_k e_2) - \omega(T_k e_1, T_k e_2)| \\ &\leq |\omega(F_k e_1 - T_k e_1, F_k e_2) - \omega(T_k e_1, T_k e_2 - F_k e_2)| \\ &\leq \varepsilon \|F_k\| + \varepsilon \|T_k\| \\ &\leq 2\varepsilon [2\|T_k\| + 1] < \frac{1}{2}. \end{aligned}$$

Since  $\left|1 - \frac{1}{\sqrt{x}}\right| \leq 2|x - 1|$  for  $\frac{1}{2} \leq x \leq \frac{3}{2}$ , then

$$\left|1 - \frac{1}{\sqrt{\lambda_k}}\right| \leq 4\varepsilon [2\|T_k\| + 1].$$

Since  $\prod_{k=1}^N \lambda_k = \det \mathbb{F} = 1$ , then

$$\prod_{k=1}^N \frac{1}{\sqrt{\lambda_k}} = 1.$$

Observe that  $Sp(1) = \{A \in GL(\mathbb{R}^2) \mid \det A = +1\}$ . Write

$$A_k := \frac{1}{\sqrt{\lambda_k}} F_k.$$

Then  $A_k \in Sp(1)$ . Also

$$\mathbb{A} = A_N \circ \dots \circ A_1 = \left(\prod_{k=1}^N \frac{1}{\sqrt{\lambda_k}}\right) \mathbb{F} = \mathbb{F}$$

is not hyperbolic. Finally,

$$\begin{aligned} \|A_k - T_k\| &\leq \|A_k - F_k\| + \|F_k - T_k\| \\ &\leq \left|1 - \frac{1}{\sqrt{\lambda_k}}\right| \|F_k\| + \varepsilon \\ &\leq 4\varepsilon [2\|T_k\| + 1]^2 + \varepsilon \\ &\leq 4\varepsilon [2\|T_k\| + 2]^2. \end{aligned}$$

Now suppose that  $\mathbb{F}$  has an eigenvalue 1. The case of an eigenvalue  $-1$  follows from this case using  $-T_1$  and  $-F_1$  instead of  $T_1$  and  $F_1$ .

Take  $a_1 \neq 0$  such that  $\mathbb{F}(a_1) = a_1$ . Define inductively

$$a_{k+1} := F_k(a_k), \quad u_k := \frac{a_k}{|a_k|}.$$

We shall construct a symplectic map  $A_k \in Sp(1)$  such that  $\|A_k - T_k\| < (3 + \|T_k\|)\varepsilon$  and  $A_k(u_k) = F_k(u_k)$ . This will imply that  $A_k(a_k) = a_{k+1}$ ,  $\mathbb{A}(a_1) = a_1$  and thus that  $\mathbb{A}$  is not hyperbolic.

Let  $J(x, y) := (-y, x)$  and

$$\lambda_k := \frac{\omega(T_k J u_k, T_k u_k)}{\omega(F_k J u_k, F_k u_k)} = \frac{1}{\omega(F_k J u_k, F_k u_k)}.$$

Define  $A_k \in GL(\mathbb{R}^2)$  by  $A_k(u_k) = F_k(u_k)$  and  $A_k(Ju_k) = \lambda_k F_k(Ju_k)$ . Then

$$\omega(A_k Ju_k, A_k u_k) = \lambda_k \omega(F_k Ju_k, F_k u_k) = 1 = \omega(Ju_k, u_k),$$

so that  $A_k \in Sp(1)$ .

Since  $\omega(a, b) \leq |a| |b|$ , we have that

$$\begin{aligned} \left| \frac{1}{\lambda_k} - 1 \right| &= \left| \omega(F_k Ju_k, F_k u_k) - \omega(T_k Ju_k, T_k u_k) \right| \\ &= \left| \omega(F_k Ju_k, F_k u_k - T_k u_k) + \omega(F_k Ju_k - T_k Ju_k, T_k u_k) \right| \\ &\leq \varepsilon (\|F_k\| + \|T_k\|) \\ &\leq \varepsilon (2 \|T_k\| + 1). \end{aligned}$$

Since  $|x - 1| \leq 4 \left| 1 - \frac{1}{x} \right|$  for  $\frac{1}{2} < x < \frac{3}{2}$ , then

$$\begin{aligned} |A_k(Ju_k) - T_k(Ju_k)| &= |\lambda_k F_k(Ju_k) - T_k(Ju_k)| \\ &\leq |\lambda_k - 1| |F_k(Ju_k)| + |F_k(Ju_k) - T_k(Ju_k)| \\ &\leq |\lambda_k - 1| \|F_k\| + \|F_k - T_k\| \\ &\leq 4\varepsilon [2 \|T_k\| + 1] (\|T_k\| + 1) + \varepsilon \\ &\leq 4\varepsilon (2 + 2 \|T_k\|)^2. \end{aligned}$$

Also,

$$|A_k(u_k) - T_k(u_k)| = |F_k(u_k) - T_k(u_k)| \leq \varepsilon.$$

Since the basis  $\{u_k, Ju_k\}$  is orthonormal, we have that

$$\|A_k - T_k\| \leq 16\varepsilon (1 + \|T_k\|)^2 + \varepsilon.$$

□

## 5.2. The hyperbolic splitting.

Let  $M$  be a closed 2-dimensional smooth manifold and let  $\mathcal{R}^1(M)$  be the set of  $C^r$  riemannian metrics,  $r \geq 4$  on  $M$  all of whose closed geodesics are hyperbolic, endowed with the  $C^2$  topology and let  $\mathcal{F}^1(M) = \text{int}(\mathcal{R}^1(M))$  be the interior of  $\mathcal{R}^1(M)$  in the  $C^2$  topology.

Given  $g \in \mathcal{G}^r(M)$  let  $\text{Per}(g)$  be the union of the hyperbolic (prime) periodic orbits of  $g$ . We say that a closed  $\phi^g$ -invariant subset  $\Lambda \subset SM$  is hyperbolic if there exists a (continuous) splitting  $T_\Lambda(SM) = E^s \oplus E^c \oplus E^u$  such that

- $E^c = \langle X^g \rangle$  is generated by the vector field of  $\phi^g$ .
- There exist constants  $K > 0$  and  $0 < \lambda < 1$  such that

$$\begin{aligned} |d_\theta \phi_t^g(\xi)| &\leq K \lambda^t |\xi|, & \text{for all } t \geq 0, \theta \in \Lambda, \xi \in E^s(\theta); \\ |d_\theta \phi_{-t}^g(\xi)| &\leq K \lambda^t |\xi|, & \text{for all } t \geq 0, \theta \in \Lambda, \xi \in E^u(\theta). \end{aligned}$$

We shall show now

**Theorem D.**

*If  $g \in \mathcal{F}^1(M)$ , then the closure  $\overline{\text{Per}(g)}$  is a hyperbolic set.*

We state a local version which implies theorem D. Let  $U \subseteq SM$  be an open subset and let  $\mathcal{R}^1(U)$  be the set of riemannian metrics  $g \in \mathcal{G}^r(M)$  such that all the periodic orbits of  $\phi^g$  contained in  $U$  are hyperbolic. Let  $\text{Per}(g, U)$  be the union of the periodic orbits of  $\phi^g$  entirely contained in  $U$ . Let  $\mathcal{F}^1(U) = \text{int}_{C^2}(\mathcal{R}^1(U))$ .

**5.5. Proposition.** *If  $g \in \mathcal{F}^1(U)$ , then the closure  $\overline{\text{Per}(g, U)}$  is hyperbolic.*

**Proof:** Observe that on a  $C^2$  neighbourhood  $\mathcal{U}$  of  $g$  each periodic orbit in  $\text{Per}(g, U)$  can be continued and its continuation (see section 2) is hyperbolic, because otherwise one could produce a non-hyperbolic orbit.

Cut the closed geodesics in  $\text{Per}(g, U)$  into segments of length in  $[\frac{1}{4}\ell, \frac{1}{2}\ell]$  where  $\ell$  is the injectivity radius of  $g$ . Given a closed geodesic  $\gamma$  in  $\text{Per}(g, U)$  construct normal local transversal sections  $\Sigma_i$  to  $\phi^g$  passing through the cutting points  $\gamma(t_i)$  of  $\gamma$ . Given a nearby metric  $\bar{g}$ , cut the continuation  $\gamma^{\bar{g}}$  of  $\gamma$  along the  $\Sigma_i$ 's:  $\gamma^{\bar{g}}(t_i^{\bar{g}}) \in \Sigma_i$ . Then  $\gamma^{\bar{g}}$  is cut in the same number of segments as  $\gamma$  is, so that the families

$$\mathcal{F}(\bar{g}) = \{ d_{\gamma^{\bar{g}}(t_i^{\bar{g}})} \phi_{t_{i+1}^{\bar{g}} - t_i^{\bar{g}}}^{\bar{g}} |_{\mathcal{N}^{\bar{g}}(\gamma^{\bar{g}}(t_i^{\bar{g}}))} \mid \gamma \in \text{Per}(g, U), 0 \leq i \leq n(\gamma) \}$$

in  $Sp(\mathcal{N}^{\bar{g}}(\theta))$  are periodically equivalent, where

$$\mathcal{N}^{\bar{g}}(\theta) = \{ \xi \in T_\theta S(M, \bar{g}) \mid \langle d\pi\xi, \theta \rangle_{\bar{g}} = 0 \}$$

and  $n(\gamma)$  is the number of segments in which we cut  $\gamma$ .

**5.6. Lemma.** *If  $g \in \mathcal{F}(U)$  then the family  $\mathcal{F}(g)$  is stably hyperbolic.*

**Proof:** Since  $g \in \mathcal{F}^1(U)$  then there exists a  $C^2$ -neighbourhood  $\mathcal{U}$  of  $g$  in  $\mathcal{G}^r(M)$  such that for all  $\bar{g} \in \mathcal{U}$ , the family  $\mathcal{F}(\bar{g})$  is hyperbolic. Let  $\delta = \delta(g, \mathcal{U}) > 0$  be given by corollary 4.2. For  $\gamma \in \text{Per}(g, U)$ , write

$$\xi_i^{(\gamma)} := d_{\dot{\gamma}(t_i)} \phi_{t_{i+1}^g - t_i^g}^g |_{\mathcal{N}_i}, \quad t_i := t_i^g, \quad \mathcal{N}_i := \mathcal{N}^g(\dot{\gamma}^g(t_i)).$$

Suppose that the family

$$\mathcal{F}(g) = \{ \xi_i^{(\gamma)} \mid \gamma \in \text{Per}(g, U), 1 \leq i \leq n_i(\gamma) \}$$

is not stably hyperbolic. Then there exist a periodic orbit  $\gamma \in \text{Per}(g, U)$  for  $g$  and a sequence of symplectic linear maps  $\eta_i : \mathcal{N}_i \rightarrow \mathcal{N}_{i+1}$  such that  $\|\eta_i - \xi_i^{(\gamma)}\| < \delta$  and  $\prod_{i=1}^{n(\gamma)} \eta_i$  is not hyperbolic. Observe that the perturbations of Franks' lemma 4.1 do not change the subspaces  $\mathcal{N}(\theta)$  along the selected segment of  $c(t)$ . By corollary 4.2 there is another riemannian metric  $\bar{g} \in \mathcal{U}$  such that  $\gamma$  is also a closed geodesic for  $\bar{g}$ ,  $t_i^{\bar{g}} = t_i$ ,  $\mathcal{N}^{\bar{g}}(t_i^{\bar{g}}) = \mathcal{N}_i$

and  $d_{\dot{\gamma}(t_i)}\phi_{t_{i+1}-t_i}^{\bar{g}}|_{\mathcal{N}_i} = \eta_i$ . Let  $\tau(\gamma) := \sum_{i=1}^{n(\gamma)} t_i$  be the period of  $\gamma$ . Since the linearized Poincaré map for  $(\gamma, \bar{g})$  is

$$d_{\dot{\gamma}(0)}\phi_{\tau(\gamma)}^{\bar{g}}|_{\mathcal{N}_1} = \prod_{i=1}^{n(\gamma)} \eta_i,$$

then the closed geodesic  $\gamma$  is not hyperbolic for the metric  $\bar{g} \in \mathcal{U}$ . This contradicts the choice of  $\mathcal{U}$ .  $\square$

Applying corollary 5.2 – and remark 5.3 if necessary (the time spacing between cut points may vary) –, we get that there exist  $\lambda < 1$  and  $T > 0$  such that

$$(39) \quad \|d_\theta\phi_T|_{E^s(\theta)}\| \cdot \|d_{\phi_T\theta}\phi_{-T}|_{E^u(\phi_T\theta)}\| < \lambda \quad \text{for all } \theta \in \text{Per}(g, U);$$

where  $\phi = \phi^g$ .

Write  $\Lambda(g) = \overline{\text{Per}(g, U)}$ . For  $\theta \in \Lambda(g)$  let

$$S(\theta) := \text{span} \left\{ \xi \in \mathcal{N}^g(\theta) \mid \begin{array}{l} \exists \langle \theta_n \rangle \subseteq \text{Per}(g, U), \lim_n \theta_n = \theta; \\ \exists \xi_n \in E^s(\theta_n), \lim_n \xi_n = \xi. \end{array} \right\}$$

$$U(\theta) := \text{span} \left\{ \xi \in \mathcal{N}^g(\theta) \mid \begin{array}{l} \exists \langle \theta_n \rangle \subseteq \text{Per}(g, U), \lim_n \theta_n = \theta; \\ \exists \xi_n \in E^u(\theta_n), \lim_n \xi_n = \xi. \end{array} \right\}$$

Then the domination condition (39) implies that

$$(40) \quad \|d_\theta\phi_T|_{S(\theta)}\| \cdot \|d_{\phi_T\theta}\phi_{-T}|_{U(\phi_T\theta)}\| < \lambda, \quad \text{for all } \theta \in \Lambda(g).$$

We show now that the domination condition (40) implies that  $S \oplus U$  is a continuous splitting of  $\mathcal{N}|_{\Lambda(g)} = S \oplus U$ . First observe that  $S(\theta) \cap U(\theta) = \{0\}$  for all  $\theta \in \Lambda(g)$ ; because if  $\xi_0 \in S(\theta) \cap U(\theta)$ , writing  $\xi_T := d_\theta\phi_T(\xi_0)$ , we would have that

$$|\xi_T| \leq \|d_\theta\phi_T|_{S(\theta)}\| \cdot |\xi_0| \leq \|d_\theta\phi_T|_{S(\theta)}\| \cdot \|d_{\phi_T\theta}\phi_{-T}|_{U(\phi_T\theta)}\| \cdot |\xi_T| < \lambda |\xi_T|.$$

But the definitions of  $S$  and  $U$  imply that  $\dim S(\theta) \geq \dim E^s(\theta_n)$  and that  $\dim U(\theta) \geq \dim E^u(\theta_n)$  if  $\lim_n \theta_n = \theta$  and  $\theta_n \in \text{Per}(g, U)$ . Therefore  $\mathcal{N}(\theta) = S(\theta) \oplus U(\theta)$  and  $\lim_n S(\theta_n) = S(\theta)$ ,  $\lim U(\theta_n) = U(\theta)$  in the appropriate Grassmann manifold.

The continuity of the bundles  $S$  and  $U$  and their definition imply that  $S(\theta) = E^s(\theta)$  and  $U(\theta) = E^u(\theta)$  when  $\theta \in \text{Per}(g, U)$ . Observe that if  $\theta \in \text{Per}(g, U)$  then  $E^s(\theta)$  and  $E^u(\theta)$  are<sup>7</sup> lagrangian subspaces of  $\mathcal{N}(\theta)$  because for example

$$\omega_g(u, v) = \lim_{t \rightarrow +\infty} \omega_g(d_\theta\phi_t(u), d_\theta\phi_t(v)) = 0,$$

where  $\omega_g$  is the symplectic form induced by  $g$ . The continuity of the bundles  $S$  and  $U$  and the continuity of  $\omega_g$  imply that the subspaces  $S(\theta)$  and  $U(\theta)$  are lagrangian for all  $\theta \in \Lambda(g)$ . Then the next proposition due to Ruggiero [49, proposition 2.1] (cf. also [14]) shows that  $\overline{\text{Per}(g, U)}$  is hyperbolic.

**5.7. Proposition.** *Let  $S(\theta) \oplus U(\theta)$  be a continuous, invariant lagrangian splitting defined on a compact invariant set  $X \subset SM$ . The splitting is dominated if and only if it is hyperbolic.*

$\square$

<sup>7</sup>This is trivial in our case of  $\dim \mathcal{N}(\theta) = 2$  and  $\dim E^s(\theta) = \dim E^u(\theta) = 1$ .



A hyperbolic set  $\Lambda$  is said *locally maximal* if there exists an open neighbourhood  $U$  of  $\Lambda$ , such that  $\Lambda$  is the maximal invariant subset of  $U$ , i.e.

$$\Lambda = \bigcap_{t \in \mathbb{R}} d\phi_t^g(U).$$

A *basic set* is a locally maximal hyperbolic set with a dense orbit. It is *non-trivial* if it is not a single closed orbit.

Given a continuous flow  $\phi_t$  on a topological space  $X$  a point  $x \in X$  is said *wandering* if there is an open neighbourhood  $U$  of  $x$  and  $T > 0$  such that  $\phi_t(U) \cap U = \emptyset$  for all  $t > T$ . Denote by  $\Omega(\phi_t|_X)$  the set of non-wandering points for  $(X, \phi_t)$ . Recall

**5.8. Smale's spectral decomposition theorem for flows.** [51], [23, ]

*If  $\Lambda$  is a locally maximal hyperbolic set for a flow  $\phi_t$ , then there exists a finite collection of basic sets  $\Lambda_1, \dots, \Lambda_N$  such that the non-wandering set of the restriction  $\phi_t|_\Lambda$  satisfies*

$$\Omega(\phi_t|_\Lambda) = \bigcup_{i=1}^N \Lambda_i.$$

**5.9. Corollary.** *If the number of geometrically distinct periodic geodesics is infinite and  $g \in \mathcal{F}^1(M)$ , then  $\overline{\text{Per}(g)}$  contains a non-trivial hyperbolic basic set.*

**Proof:** Let  $\Lambda = \overline{\text{Per}(g)}$ . Since  $g \in \mathcal{F}^1(M)$  then, by theorem D,  $\Lambda$  is a hyperbolic set. By proposition 6.4.6 in [23], there exists an open neighbourhood  $U$  of  $\Lambda$  such that the set

$$\Lambda_U := \bigcap_{t \in \mathbb{R}} \phi_t^g(\overline{U})$$

is hyperbolic. Since  $\Lambda = \overline{\text{Per}(g)}$ , then its non-wandering set is  $\Omega(\phi_t|_\Lambda) = \Lambda$ . By definition of  $\Lambda_U$ ,  $\Lambda \subseteq \Lambda_U$  and hence  $\Lambda = \Omega(\phi_t|_\Lambda) \subseteq \Omega(\phi_t|_{\Lambda_U})$ . By corollary 6.4.20 in [23], the periodic orbits are dense in the non-wandering set  $\Omega(\phi_t|_{\Lambda_U})$  of the locally maximal hyperbolic set  $\Lambda_U$ . Thus  $\Lambda \subseteq \Omega(\phi_t|_{\Lambda_U}) \subseteq \overline{\text{Per}(g)} = \Lambda$ . By theorem 5.8, the set  $\Lambda = \Omega(\phi_t|_{\Lambda_U})$  decomposes into a finite collection of basic sets. Since the number of periodic orbits in  $\Lambda$  is infinite, at least one of the basic sets  $\Lambda_i$  is not a single periodic orbit, i.e. it is non-trivial.  $\square$

N. Hingston proves in [26] that if  $M$  is a simply-connected manifold rational homotopy equivalent to a compact rank-one symmetric space with a metric all of whose closed geodesics are hyperbolic then

$$\liminf n(\ell) \frac{\log(\ell)}{\ell} > 0,$$

where  $n(\ell)$  is the number of geometrically distinct closed geodesics of length  $\leq \ell$ .

Rademacher proves

**5.10. Theorem** (Rademacher [45, cor. 2]).

*For a  $C^4$ -generic metric on a compact riemannian manifold with finite fundamental group there are infinitely many geometrically distinct closed geodesics.*

Thus we have,

**1.1. Theorem.**

*A  $C^4$  metric on a surface can be  $C^2$ -approximated by one having an elliptic closed geodesic or by one with a non-trivial hyperbolic basic set.*

Theorem 1.1 together with proposition 3.3 complete the proof of theorem A.

APPENDIX A. PERTURBATION OF LAGRANGIAN MANIFOLDS.

In this appendix we prove a perturbation lemma for invariant lagrangian submanifolds of an autonomous hamiltonian suitable for our application in the Kupka-Smale theorem for geodesic flows.

Let  $V$  be a  $2n$ -dimensional vector space. A symplectic form  $\omega$  on  $V$  is an antisymmetric bilinear map which is non-degenerate, i.e. for all  $v \in V \setminus \{0\}$  there exists  $w \in V$  such that  $\omega(v, w) \neq 0$ . We say that a subspace  $E \subset V$  is *isotropic* if  $\omega|_E \equiv 0$  and that it is *lagrangian* if  $E$  is isotropic and  $\dim E = n = \frac{1}{2}\dim V$ . This is the maximal dimension that an isotropic subspace can have.

A symplectic manifold  $(\mathcal{M}, \omega)$  is a  $2n$ -dimensional smooth manifold together with a symplectic form  $\omega$ , i.e. a 2-form which is non-degenerate at each tangent space. A *lagrangian* submanifold  $\mathcal{N} \subset \mathcal{M}$  is a submanifold such that each tangent space  $T_x\mathcal{N}$  is a lagrangian subspace of  $T_x\mathcal{M}$ . In particular,  $\dim \mathcal{N} = n$ .

**A.1. Lemma.** *Let  $(\mathcal{M}, \omega)$  be a symplectic manifold and  $H : \mathcal{M} \rightarrow \mathbb{R}$  be a smooth function. If  $\mathcal{N}$  is a lagrangian submanifold of  $(\mathcal{M}, \omega)$  such that  $\mathcal{N} \subset H^{-1}\{k\}$  for some  $k \in \mathbb{R}$  then the hamiltonian vector field  $X$  of  $H$  is tangent to  $\mathcal{N}$ . In particular,  $\mathcal{N}$  is a union of orbit segments of the hamiltonian flow.*

**Proof:** The hamiltonian vector field  $X$  is defined by  $i_X\omega = -dH$ . In particular, on level set  $\Sigma = H^{-1}\{k\}$  we have that  $i_X\omega|_\Sigma = dH|_\Sigma \equiv 0$ . Then  $i_X\omega|_{\mathcal{N}} \equiv 0$ . Then for all  $x \in \mathcal{N}$  the subspace  $E_x := T_x\mathcal{N} \oplus \langle X(x) \rangle$  is isotropic. If  $X(x) \notin T_x\mathcal{N}$  then  $\dim E_x = n + 1$  that is impossible. Then  $X(x) \in T_x\mathcal{N}$ .  $\square$

We shall use a special coordinate system associated to a lagrangian submanifold that we shall call Darboux coordinates for the lagrangian manifold.

**A.2. Lemma.** *Let  $\mathcal{N}$  be a lagrangian submanifold contained in an energy level  $H^{-1}\{k\}$  of a hamiltonian  $H : \mathcal{M} \rightarrow \mathbb{R}$  on a symplectic manifold  $(\mathcal{M}, \omega)$ . Let  $\theta \in \mathcal{N}$  and suppose that  $\theta$  is not a singularity of the hamiltonian vectorfield of  $H$ . Then there exist a neighbourhood  $U$  in  $\mathcal{M}$  and a coordinate system  $(x, p) : U \rightarrow \mathbb{R}^n \times \mathbb{R}^n$  such that*

- (a)  $\omega = \sum_i dp_i \wedge dx_i$ .
- (b)  $\mathcal{N} \cap U = [p \equiv 0]$ .
- (c) *The hamiltonian vector field of  $H$  on  $\mathcal{N}$  is given by  $X_H|_{\mathcal{N}} = \frac{\partial}{\partial x_0}$ .*

**Proof:** By Weinstein's theorem [54], [2, th. 5.3.18], [37, th. 3.32] there is a neighbourhood  $W_1$  of  $\mathcal{N}$  which is symplectomorphic to a neighbourhood of the zero section of  $T^*\mathcal{N}$  with its canonical symplectic form, sending  $\mathcal{N}$  to the zero section  $\mathcal{N} \times 0$ .

By lemma A.1,  $\mathcal{N}$  is invariant under the hamiltonian flow  $\phi_t$  of  $H$ . Let  $V$  be a flow box for the restriction  $\phi_t|_{\mathcal{N}}$  containing  $\theta \in V$  and choose a local chart  $x : V \rightarrow \mathbb{R}^n$  for  $\mathcal{N}$  such that  $x(\theta) = 0 \in \mathbb{R}^n$  and  $X|_V = \frac{\partial}{\partial x_0}$ , where  $X|_V$  is the restriction of the hamiltonian vector field to  $\mathcal{N} \cap V$ .

The canonical symplectic coordinates associated to the chart  $(V, x)$  are given by  $(x, p) : V \times \mathbb{R}^n \rightarrow T_V^*\mathcal{N}$ ,  $p_i = dx_i$ . The pull-back of the canonical symplectic form for  $T^*\mathcal{N}$  in these coordinates is  $\sum_i dp_i \wedge dx_i$ . The zero section  $V \times 0 \subset \mathcal{N} \times 0 \subset T^*\mathcal{N}$  is given by  $[p \equiv 0]$ . Now compose this chart  $(x, p)$  with the symplectomorphism to obtain the required chart.  $\square$

This is our perturbation lemma for invariant lagrangian submanifolds.

**A.3. Lemma.** *Let  $\mathcal{N}$  and  $\mathcal{K}$  be two lagrangian submanifolds inside an energy level  $H^{-1}\{k\}$  of a hamiltonian  $H : \mathcal{M} \rightarrow \mathbb{R}$  of a symplectic manifold  $(\mathcal{M}, \omega)$ . Let  $\theta \in \mathcal{N}$  be a non-singular point for the hamiltonian vector field. Let  $(t, x; p)$ ,  $t \equiv x_0$ , be Darboux coordinates for  $\mathcal{N}$ ,  $0 \leq t \leq 1$ ,  $|x| < \varepsilon$  as in lemma A.2. Choose  $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$ . Then there exist a sequence  $\mathcal{N}_n$  of lagrangian submanifolds of  $(\mathcal{M}, \omega)$  such that*

- (a)  $\mathcal{N}_n \rightarrow \mathcal{N}$  in the  $C^\infty$  topology.
- (b)  $\mathcal{N}_n \cap A = \mathcal{K} \cap A$ , where  $A := \{ (t, x; p) \mid \min_i |x_i| \geq \varepsilon_1 \text{ or } 0 \leq t \leq \frac{1}{4} \}$ .
- (c)  $H(\mathcal{N}_n \cap B) = \{k\}$ , where  $B = A \cup \{ (t, x; p) \mid \frac{1}{2} \leq t \leq 1 \}$ .
- (d)  $\mathcal{N}_n \cap D$  is transversal to  $\mathcal{K}$ , where  $D = \{ (t, x; p) \mid t = 1, \text{ and } \max_i |x_i| < \varepsilon_2 \}$ .

**Proof:** Let  $\varphi : [-\varepsilon, \varepsilon]^{n-1} \rightarrow [0, 1]$  be a  $C^\infty$  function such that  $\varphi(x) = 0$  if  $\max_i |x_i| > \varepsilon_1$  and  $\varphi(x) = 1$  if  $x \in [-\varepsilon_2, \varepsilon_2]^{n-1}$ . Given  $s = (s_1, \dots, s_{n-1}) \in \mathbb{R}^{n-1}$  with  $|s|$  small, let  $h_s : [-\varepsilon, \varepsilon]^{n-1} \rightarrow \mathbb{R}$  be the function  $h_s(x) := 1 + \varphi(x) \sum_{i=1}^{n-1} s_i x_i$ . Then

$$\begin{aligned} d_x h_s &= (s_1, \dots, s_{n-1}), & \text{if } x \in [-\varepsilon_2, \varepsilon_2]^{n-1}; \\ d_x h_s &= 0, & \text{if } \max_i |x_i| > \varepsilon_1. \end{aligned}$$

Let  $p^s : \{1\} \times [-\varepsilon, \varepsilon]^{n-1} \rightarrow \mathbb{R}^n$  be defined by  $p^s(1, x) := (p_0^s(x), d_x h_s)$ , where  $p_0^s(x) : [-\varepsilon, \varepsilon]^{n-1} \rightarrow \mathbb{R}$  is defined by the equation

$$(41) \quad H((1, x); p^s(x)) = k.$$

Since the curves  $t \mapsto (t, x, p = 0)$  are solutions of the hamiltonian equations, then

$$H_{p_0}((1, x), 0) \equiv 1 \neq 0.$$

By the implicit function theorem, for  $s$  small we can solve equation (41) for  $(s, x) \mapsto p_0^s(x)$  and this is a  $C^\infty$  function on  $s$  and  $x$ .

The graph of  $p^s$ :

$$\text{Graph}(p^s) := \{ ((1, x); p^s(x)) \mid x \in [-\varepsilon, \varepsilon]^{n-1} \} \subset H^{-1}\{k\}$$

is an isotropic submanifold of  $H^{-1}\{\frac{1}{2}\}$ . Indeed, its tangent vectors are generated by  $\xi_i = ((0, e_i); \frac{\partial p^s}{\partial x_i})$  and

$$\begin{aligned} dp \wedge dx(\xi_i, \xi_j) &= \sum_{k=0}^{n-1} \frac{\partial p_k^s}{\partial x_i} dx^k(e_j) - \frac{\partial p_k^s}{\partial x_j} dx^k(e_i) \\ &= \frac{\partial p_j^s}{\partial x_i} - \frac{\partial p_i^s}{\partial x_j} = \frac{\partial^2 h}{\partial x_i \partial x_j} - \frac{\partial^2 h}{\partial x_j \partial x_i} = 0. \end{aligned}$$

When  $s$  is near zero, the submanifold  $\text{Graph}(p^s)$  is  $C^\infty$  near

$$\text{Graph}(p^0) := (\{1\} \times [-\varepsilon, \varepsilon]^{n-1}) \times \{e_0\} \subset \mathcal{N}.$$

The tangent subspace to  $\text{Graph}(p^0)$  is generated by the vectors  $\xi_i^0 = ((0, e_i); 0)$ . Condition (c) in lemma A.2 implies that, the hamiltonian vector field on  $\mathcal{N}$  is  $X = ((1, 0); 0)$ . Then  $X$  is transversal to  $\text{Graph}(p^0)$ . Then for  $s$  small, the hamiltonian vector field  $X$  is also transversal to  $\text{Graph}(p^s)$ .

Let

$$N_s = [\frac{1}{2} \leq t \leq 1] \cap [|x| < \varepsilon] \cap \phi_{[-2\varepsilon, 0]}(\text{Graph}(p^s)),$$

We are adding the flow direction to the isotropic submanifold  $\text{Graph}(p^s)$  of the energy level  $[H \equiv k]$ . Then  $N_s$  is also isotropic. Since  $\dim N_s = n$ , then  $N_s$  is a lagrangian submanifold. Since the projection  $\pi|_{\mathcal{N}}$  is a diffeomorphism and when  $s \rightarrow 0$ ,  $N_s$  converges to  $\mathcal{N}$  in the  $C^\infty$  topology, then  $\pi|_{N_s}$  is also a diffeomorphism for  $s$  small. Then  $N_s$  is the graph of a 1-form  $\eta(t, x) \in T^*B$  defined on  $[\frac{1}{2}, 1] \times [-\varepsilon, \varepsilon]^{n-1}$ . Since  $N_s$  is a lagrangian submanifold, the 1-form  $\eta_s$  is closed. Since its domain is contractible, then  $\eta_s$  is exact:  $\eta_s = d_{(t,x)}u_s$ . Adding a constant if necessary we can assume that  $u_s = h_s$  on  $\{1\} \times [-\varepsilon, \varepsilon]$ . Extend  $u_s$  to a  $C^\infty$  function on  $B$  such that

$$u_s(t, x) = t, \quad \text{if } \max_i |x_i| > \varepsilon_1 \quad \text{or} \quad t < \frac{1}{4}.$$

$$\lim_{s \rightarrow 0} u_s(t, x) = t, \quad \text{in the } C^\infty \text{ topology.}$$

This can be done using the Whitney extension theorem [18].

By construction  $H(du_s) \equiv k \equiv H(\mathcal{K})$  on  $t \in [\frac{1}{2}, 1]$ . Since  $\text{Graph}(du_s)$  and  $\mathcal{K}$  are lagrangian submanifolds, then they are invariant under the hamiltonian vector field. Hence  $\text{Graph}(du_s)$  and  $\mathcal{K}$  are transversal over  $(t, x) \in [\frac{1}{2}, 1] \times [-\varepsilon_2, \varepsilon_2]$  if and only if their intersections with  $[t = 1]$ ,  $(x, \partial_x u_s(1, x))$  and  $\mathcal{K} \cap [t = 1]$  are transversal over  $x \in [-\varepsilon_2, \varepsilon_2]$ . By construction of  $u_s$  we have that

$$\partial_x u_s(1, x) = s \in \mathbb{R}^{n-1} \quad \text{for } x \in [-\varepsilon_2, \varepsilon_2].$$

Observe that the submanifolds  $\text{Graph}(du_s)$  on  $(t, x) \in [\frac{1}{2}, 1] \times [-\varepsilon_2, \varepsilon_2]$ , parametrized by  $s$  are a foliation of  $(t, x; p) \in [\frac{1}{2}, 1] \times [-\varepsilon_2, \varepsilon_2] \times [-\delta, \delta]$ . The projection of  $\mathcal{K} \cap [t = 1]$  into the transverse direction to the foliation is given by  $[-\varepsilon_2, \varepsilon_2] \ni x \mapsto d_x v$  where the function  $v : [-\varepsilon, \varepsilon] \rightarrow \mathbb{R}$  is defined by  $\mathcal{K} \cap [t = 1] = \text{Graph}(dv)$ . Therefore  $\text{Graph}(du_s)$  is transversal

to  $\mathcal{K}$  if and only if  $s$  is a regular point for  $x \mapsto d_x v$ . By Sard's theorem the set of regular points of  $dv$  has total measure, in particular there is a sequence  $s_n \rightarrow 0$  of regular points. The sets  $\mathcal{N}_n := \text{Graph}(du_{s_n})$  are the required lagrangian manifolds.  $\square$

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