NON-GENERICITY OF MINIMISING PERIODIC ORBITS

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Abstract. We answer in the negative Problem IV of [Mn95], for configuration spaces of dimension ≥ 3. A positive answer is given for the two-dimensional case in [Mt02].

1. Introduction

Let M be a smooth, closed, connected manifold and L be a Lagrangian on the tangent bundle TM, that is, a $C^r, r \geq 2$ function on TM which is convex and superlinear when restricted to any fiber. The Euler-Lagrange equation then defines a complete flow Φ_t on TM.

Given a closed one-form ω, $L - \omega$ is again a Lagrangian and its Euler-Lagrange flow is the same as that of L. We are interested in probability measures on the tangent bundle TM, that are invariant under the Euler-Lagrange flow, and minimise the action of $L-\omega$, that is, the integral $\int_{TM}(L-\omega)d\mu$. Actually this action only depends on the cohomology class c of ω (see [Mr91] and the next section). The measures achieving the minimum are called c-minimising, or simply minimising if $c = 0$.

We say a property is true for a generic Lagrangian if, given a Lagrangian L, there exists a residual (countable intersection of open and dense subsets) subset $\mathcal{O}$ of $C^\infty(M)$ such that the property holds for $L + f, \forall f \in \mathcal{O}$. Mañé ([Mn96]) proved for a generic Lagrangian, there exists a unique minimising measure and proposed in [Mn95] (Problem IV)(see also [Mn96], Problem III) the following

Problem 1 (Mañé)**.** *Is it true that for a generic Lagrangian, there exists a dense open subset of $\mathcal{U}$ of $H^1(M,\mathbb{R})$ such that for any c in $\mathcal{U}$ there is a unique c-minimising measure and it is supported on a periodic orbit ?*

The answer is yes when M is a closed, orientable surface (see [Mt02]). It turns out to be no in higher dimensions, as shown our next Theorem.

If the conjecture was true, we could find a sequence f_n of C^∞ functions on M, going to zero in the C^∞ topology, such that for every n, there exists an open dense subset U_n of $H^1(M,\mathbb{R})$, such that for any c in U_n, the conjecture holds. The intersection U over $\mathbb{N}$ of the U_n is dense in $H^1(M,\mathbb{R})$. So for every c in U, the set of functions f such that $L + c + f$ has a minimising periodic orbit accumulates at zero.

Given a Lagrangian L and a cohomology class c, denote $\mathcal{O}_{L,c}$ the set of $f \in C^\infty(M)$ such that for ω in c, $L+\omega+f$ has a minimising periodic orbit.

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Theorem 2. *Let M be a manifold of dimension ≥ 3. There exists a Lagrangian L on M and an open neighborhood U of 0 in $H^1(M,\mathbb{R})$ such that for any c in U, the set $\mathcal{O}_{L,c}$ does not accumulate at zero in the C^4-topology.*

See [Mn97] and [CDI97], for a stronger conjecture where we perturb only by a function ; and [Mt02a] for a disproof thereof when M is the two-torus.

The idea here is, first, to construct a Lagrangian on M, the minimising set of which is contained in a contractible part of M. Theorem 1 of [Mt02] then ensures that for a small enough cohomology class $c = [\omega]$, the minimising measure of $L - \omega$ is the same as that of L. Besides, we make up the Lagrangian so the Euler-Lagrange flow restricted to the support of its minimising measure is an irrational flow on an imbedded two-torus, with the slope quadratic. From there the idea is to use the Diophantine approximation properties of the slope as in [Mt02a], to prove that a C^4 perturbation by a function on M cannot create a minimising periodic orbit.

2. Prerequisites

Given a C^1 curve γ defined on some compact interval I into M, the L-action of γ is the integral $\int_I L(\gamma,\dot\gamma)ds$. The curve γ is said to be minimising if it minimises the L-action over all C^1 curves defined over the same interval, with the same endpoints. A C^1 curve $\gamma\colon \mathbb{R} \longrightarrow M$ is said to be minimising if its restriction to any compact interval is. An orbit $\gamma\colon \mathbb{R} \longrightarrow TM$ is said to be minimising if its projection to M is. We denote by $\mathcal{G}(L)$ the union in TM of all minimising orbits. Note that the support of a minimising measure is always contained in $\mathcal{G}(L)$. The Aubry set, denoted $\mathcal{A}_0(L)$, is the projection to M of a special set of of minimising orbits, containing all supports of minimising measures (see [Fa00] for more information).

Mather's α-function is defined in [Mr91] as

$$\alpha(\omega) = -\min\{\int_{TM}(L-\omega)d\mu\colon\ \mu \in \mathcal{M}\}$$

where $\mathcal{M}$ is the set of closed measures on TM, that is (see [Ba99]) the compactly supported probability measures μ on TM such that $\int df\, d\mu = 0$ for every C^1 function f on M. In other words, those are the measures with a well-defined homology class. The measures achieving the minimum are invariant by the Euler-Lagrange flow Φ_t of L (see [Ba99]).

The quantity α defines a convex and superlinear function on $H^1(M,\mathbb{R})$. It may not be stricly convex, however. It turns out ([Mt02]) that whenever there exists a closed, non-exact one-form ω supported away from $\mathcal{A}_0(L)$, the α-function has a flat. That is to say, its epigraph contains a piece of affine subspace, and the underlying vector space of this affine subspace contains the cohomology class of ω.

3. The Lagrangian

Let M be a 3-dimensional manifold and let B be an embedding into M of the unit ball of $\mathbb{R}^3$. Consider an embedding into B of $\mathbb{T}^2\times]-1,1[$, the two-torus times an open interval, equipped with coordinates (x,y,z). Take a Riemannian metric on M, such that its restriction to $\mathbb{T}^2\times]-1,1[$ is $dx^2+dy^2+dz^2$. Let p,q be real numbers such that $p^2+q^2=1$ and p/q is irrational and quadratic. We define a differential 1-form α in $\mathbb{T}^2\times]-1,1[$ by $(x,y,z,u,v,\zeta)\mapsto -(pu+qv)$ where (x,y,z,u,v,ζ) is a tangent vector to M at the point of coordinates (x,y,z). Extend α to a 1-form on M. Let ϕ be a C^∞ function on M, the restriction of which to $\mathbb{T}^2\times]-1,1[$ is $(x,y,z)\mapsto z^2$ and such that $f(P)\geq 1$ for all P in $M\setminus\mathbb{T}^2\times]-1,1[$. Our Lagrangian is then defined as the sum of the quadratic form that comes with the Riemannian metric, the 1-form α, and the function ϕ. In particular, in $\mathbb{T}^2\times]-1,1[$ it takes the form

$$L(x,y,z,u,v,\zeta)=\frac{1}{2}(u^2+v^2+1)-pu-qv+z^2.$$

Furthermore we choose α so that L is a non-negative function on TM, vanishing only on the set hereafter defined.

Proposition 3. *The minimising set of the Lagrangian L is*

$$\{(x,y,0,p,q,0)\colon\ (x,y)\in\mathbb{T}^2\}.$$

Proof. The vector field $(x,y,0,p,q,0)$ defines an irrational foliation of $\mathbb{T}^2\times 0$, hence it admits a unique, ergodic invariant measure which we denote μ. First note that this measure is the L-minimising. Indeed its L-action is zero, since $L(x,y,0,p,q,0)=0$ for any (x,y), while the action of any measure is non-negative since L itself is non-negative.

Then observe that μ is the only minimising measure. Indeed if a measure is not supported inside $\mathbb{T}^2\times\{0\}$, it must have positive action. But then a minimising measure, which must be invariant by the Euler-Lagrange flow of L, must be invariant by the vector field $(x,y,0,p,q,0)$, which is uniquely ergodic.

Thus any minimising orbit must be asymptotic, positively and negatively, to supp(μ) ([Fa00]). Assume that a minimising orbit $\gamma\colon\mathbb{R}\longrightarrow TM$ is not contained in supp(μ). Then there exists $\delta>0$ and a,b in $\mathbb{R}$ such that for every $s\leq a$, $t\geq b$, we have $\int_s^t L(\gamma(r))dr\geq\delta$. On the other hand, γ being asymptotic, positively and negatively, to supp(μ), there exists $S\leq a, T\geq b$ in $\mathbb{R}$ such that for any $t\geq T$, $s\leq S$, the point $\gamma(t)$ (resp. $\gamma(s)$) may be joined to a point P_t (resp. P_s) in supp(μ) by a path of L-action less than $\delta/3$. The orbits of the vector field $(x,y,0,p,q,0)$ are dense in $\mathbb{T}^2\times 0$, and have zero L-action, so there exists a path in $\mathbb{T}^2\times 0$ of L-action less than $\delta/3$, joining P_t and P_s. Hence we can build a path between $\gamma(t)$ and $\gamma(s)$ of L-action strictly less than δ, contradicting the fact that γ is minimising. □

Corollary 4. *There exists a neighborhood U of 0 in $H^1(M,\mathbb{R})$ such that for any ω in U, the only ω-minimising measure is μ.*

Proof. Since the projection to M of $\mathcal{G}(L)$, hence the Aubry set $\mathcal{A}_0(L)$, is contained in B which is contractible, there exists 1-forms $\omega_1,\ldots\omega_n$, supported away from $\mathcal{A}_0(L)$, the cohomology classes of which generate $H^1(M,\mathbb{R})$. By [Mt02], Theorem 1, this implies that the α-function of L has a face of codimension zero containing the null cohomology class in its interior. Such a face is a neighborhood of the origin. Call U its interior. Then by [Mt02], Proposition 6, for every 1-form ω with $[\omega]$ in U, the Aubry sets for L and $L-\omega$ coincide. In particular, every $L-\omega$-minimising measure is also L-minimising, hence μ is the only $L-\omega$-minimising measure. □

4. Coverings

Assume that for some ω in U there exists a sequence f_n of C^∞ functions on M converging to zero in the C^2-topology, and closed curves $\gamma_n\colon [0,t_n] \longrightarrow M$ such that the probability measure evenly distributed along γ_n is $L+f_n$-minimising. First note that by semi-continuity of $\mathcal{G}(L)$ with respect to L ([Mt02a], Proposition 3) for n large enough, for any $c \in U$, $\mathcal{G}(L+c+f_n)$ is contained in $\mathbb{T}^2\times]-1,1[$. Then we may write $\gamma_n(t) = (x_n(t), y_n(t), z_n(t))$ in $(\mathbb{R}/\mathbb{Z})^2\times]-1,1[$.

The closed curve γ_n represents an integer homology class in $H_1(T^2\times]-1,1[,\mathbb{R})$ which is generated by the curves $\{x=z=0\}, \{y=z=0\}$. Let (p_n, q_n) be the corresponding coordinates of $[\gamma_n]$.

Lift this curve to the universal cover $\mathbb{R}^2\times]-1,1[$, keeping the same notations. Then the coordinates x_n and y_n belong to $\mathbb{R}$ and we have $x_n(t+t_n) = x_n(t)+p_n$, $y_n(t+t_n) = y_n(t)+q_n$. By semi-continuity of $\mathcal{G}$, for n large enough, the tangent vector to $\gamma_n(t)$, being close to $(p,q,0)$, is not orthogonal to $\partial/\partial x$, so the function $t \mapsto x_n(t)$ is injective. For the same reason, the derivative $\dot{x}_n(t)$ does not vanish for large n's. Define, for any real number s, $\gamma_{n,s}(t) = (x_n(t), y_n(t)+s, z_n(t))$. So $\gamma_{n,s_1}(t_1) = \gamma_{n,s_2}(t_2)$ implies

$$\begin{cases} x_n(t_1) &= x_n(t_2) \\ y_n(t_1)+s_1 &= y_n(t_2)+s_2 \\ z_n(t_1) &= z_n(t_2). \end{cases}$$

By injectivity the first equation implies $t_1 = t_2$, whence $s_1 = s_2$ from the second equation. Hence the $\gamma_{n,s}$ foliate a surface S_n homeomorphic to $\mathbb{R}^2$, endowed with the (possibly not free) action of $\mathbb{Z}^2$ which takes $(x_n(t), y_n(t)+s, z_n(t))$ to $(x_n(t)+a, y_n(t)+s+b, z_n(t))$ for (a,b) in $\mathbb{Z}^2$. The tangent space to S_n at $(x_n(t), y_n(t)+s, z_n(t))$ is generated by $(\dot{x}_n(t), \dot{y}_n(t), \dot{z}_n(t))$ and $(0,1,0)$, thus it contains the vector

$$(p,q,\dot{z}_n(t)) = \frac{p}{\dot{x}_n(t)}(\dot{x}_n(t), \dot{y}_n(t), \dot{z}_n(t)) + (q - \frac{p}{\dot{x}_n(t)})(0,1,0).$$

The above formula defines a vector field Y on the surface S_n. Note that while the aforementioned $\mathbb{Z}^2$-action on S_n may not be free, the action of the subgroup $p_n\mathbb{Z}\times\{0\}+\{0\}\times q_n\mathbb{Z}$ is free. Indeed, assume for some t_1, s_1 and t_2, s_2 and integer k, k' we have

$$(x_n(t_1)+kp_n, y_n(t_1)+s_1, z_n(t_1)) = (x_n(t_2)+k'p_n, y_n(t_2)+s_2, z_n(t_2)).$$

Then we have

$$\begin{cases} x_n(t_1) &= x_n(t_2)+(k'-k)p_n \\ y_n(t_1)+s_1 &= y_n(t_2)+s_2. \end{cases}$$

Now since γ_n is t_n-periodic with homology (p_n, q_n) the first equation reads $x_n(t_1) = x_n(t_2+(k'-k)t_n)$ and by injectivity of $t \mapsto x_n$ this implies $t_1 = t_2+(k'-k)t_n$. Then $y_n(t_1)+s_1 = y_n(t_2)+(k-k')q_n+s_1$ whence $s_2 = s_1+(k-k')q_n$. In particular if $k=k'$ we have $t_1=t_2$ and $s_1=s_2$ so the action is free. Its quotient is a two-torus $\mathbb{T}^2_{p_n,q_n}$ which covers a (possibly not embedded) $\mathbb{T}^2$ in $\mathbb{T}^2\times]-1,1[$ with covering group $\mathbb{Z}/p_n\mathbb{Z}\times\mathbb{Z}/q_n\mathbb{Z}$.

The vector field Y descends to a vector field on $\mathbb{T}^2_{p_n,q_n}$ and defines an irrational foliation there, since the ratio of p and q is irrational. Hence Y

admits a unique, ergodic invariant measure μ'_n. This measure is closed since it is invariant by a flow (see [Mr91]).

5. Proof of the Theorem

From now on we work in $\mathbb{T}^2_{p_n,q_n}\times]-1,1[$, still denoting f_n the composition $f_n\circ\pi$, where π is the projection of the cover $T^2_{p_n,q_n}\times]-1,1[\longrightarrow\mathbb{T}^2\times]-1,1[$. So f_n is now a $\mathbb{Z}/p_n\mathbb{Z}\times\mathbb{Z}/q_n\mathbb{Z}$-periodic function on $\mathbb{T}^2_{p_n,q_n}\times]-1,1[$. Neither do we change notations for γ_n.

Since the curve γ_n is $L+\omega+f_n$-minimising, its lift to $\mathbb{T}^2_{p_n,q_n}\times]-1,1[$ is again minimising ([Fa98, CP02]) and we have

$$\int(L+\omega+f_n)d\gamma_n\leq\int(L+\omega+f_n)d\mu'_n, \tag{1}$$

where we denote γ_n the probability measure evenly distributed on the curve γ_n. Note that $\int\omega d\gamma_n=\int\omega d\mu'_n=0$ since both γ_n and μ'_n are supported in a contractible region of M. Besides, we have $L(x_n(t),y_n(t),z_n(t),p,q,\dot z_n(t))=z_n^2(t)$ so Equation 1 becomes

$$\int(\frac{1}{2}(u^2+v^2+1)-pu-qv)d\gamma_n\leq\int(z^2+f_n)d\mu'_n-\int(z^2+f_n)d\gamma_n \tag{2}$$

The ratio p/q being quadratic, the left-hand term in the above equation is greater than or equal to C/q_n^4 for some positive C (see [Mt02a], 2.3).

Define on the circle $\mathbb{T}^2_{p_n,q_n}\cap\{x=0\}\cong\mathbb{R}/p_n\mathbb{Z}$ the function $\phi_n(y)$ as the mean value of f_n+z^2 on the leaf of the foliation going through y. Then ϕ_n is C^k if f_n is C^k. Besides, since the derivatives with respect to y of z^2 are everywhere zero, $\phi_n^{(k)}(y)$ is the mean value of $\partial^{(k)}f_n/\partial y^{(k)}$ on the leaf of the foliation going through y, that is

$$\forall k\in\mathbb{N}\setminus\{0\},\ \phi_n^{(k)}(y)=\frac{1}{T_n}\int_0^{T_n}\frac{\partial^k f_n}{\partial y^k}(x,y)dx.$$

Note that the C^4-norm of f_n is greater than or equal to that of ϕ_n. Indeed so if for some y we have $\phi_n^{(4)}(y)\geq K$ for some K, then there exists x such that $\partial^k f_n/\partial y^k(x,y)\geq K$. Besides the mean value of ϕ_n over $\{x=0\}$ equals the mean value of f_n over $\mathbb{T}^2$. Note that ϕ_n is 1-periodic and C^∞, so for any k, $\phi_n^{(k)}$ vanishes at least once in $[0,1]$.

Assume for definiteness that γ_n crosses $\{x=0\}$ at $y=0$. Since γ_n is minimising in particular it minimises among its translates so we may assume, up to adding a constant, that $\phi_n\geq 0=\phi_n(0)$. Since γ_n crosses $\{x=0;y\in[0,1]\}$ q_n times, there exists at least one interval in $\{x=0;y\in[0,1]\}$ of length $\leq 1/q_n$ which is crossed exactly once by all leaves of the foliation. Changing the origin if we have to, to another point of γ_n, we may assume this interval is $[0,a_n]$. So every value of ϕ_n and its derivatives is taken at least once in $[0,a_n]$. Thus for every k in $\mathbb{N}$, there exists x_k in $[0,a_n]$ such that $\phi_n^{(k)}(x_k)=0$.

Proposition 7 of [Mt02a] (see below) then shows shows that ϕ_n, hence f_n, does not go to zero in the C^4-topology.

Proposition 5. *Let ϕ_n be a sequence of real-valued,non-negative, C^∞, 1-periodic functions with $\phi_n(0) = 0$. Assume there exists a sequence of integers $q_n \longrightarrow \infty$ such that*

- *the mean value of ϕ_n is $\geq 1/q_n^4$*
- *every value of ϕ_n and its derivatives is taken at least once in an interval $[0, a_n]$ with $a_n \leq 1/q_n$.*

Then for all k in $\mathbb{N}$, there exists y_k in $[0, a_n]$ such that $\phi_n^{(k)}(y_k) \geq C q_n^{k-4}$.

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