ON THE BURGERS EQUATION WITH A STOCHASTIC STEPPING – STONE NOISY TERM

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On the Burgers Equation with a stochastic stepping-stone noisy term

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Abstract

We consider the one-dimensional Burgers equation perturbed by a white noise term with Dirichlet boundary conditions and a non-Lipschitz coefficient. We obtain existence of a weak solution proving tightness for a sequence of polygonal approximations for the equation and solving a martingale problem for the weak limit.

Key words: *Burgers equation, white noise, strong solutions, polygonal approximations, martingale problem.*

1 Introduction

The one-dimensional Burgers equation

$$\frac{\partial}{\partial t}u(t,x) = \nu\Delta u(t,x) - \frac{\lambda}{2}\nabla u^2(t,x),$$

where $\Delta = \frac{\partial^2}{\partial x^2}$ and $\nabla = \frac{\partial}{\partial x}$, has been proposed as a model for turbulent fluid motion (see [3, 4, 8]).

Burgers equations perturbed by space-time white noises with Lipschitz coefficients have been studied recently by several authors (see e.g. [1, 6, 2] and the references therein).

Our aim in this paper is to study a one-dimensional Burgers equation perturbed by a stochastic noise term with a non-Lipschitz coefficient, namely,

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \Delta u(t,x) + \lambda\nabla u^2(t,x) + \gamma\sqrt{u(t,x)(1-u(t,x))}\,\frac{\partial^2}{\partial t\partial x}W(t,x), \\ u(t,0) &= u(t,1) = 0, \\ u(0,x) &= f(x),\, x\in[0,1], \end{aligned} \tag{1.1}$$

where $f(x):[0,1]\to[0,1]$ is a continuous function and $\frac{\partial^2}{\partial t\partial x}W(t,x)$ is the space-time white noise (see [16] for the definition and properties of the white noise). The stochastic term in

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this equation corresponds to continuous-time stepping-stone models in population genetics ([5, 12]), where $u(t,x)$ models the gene frequency in colonies.

Equation (1.1) is interpreted in the weak sense, which means that for each $\varphi \in C^2([0,1])$,

$$\begin{aligned}\int_{[0,1]} u(t,x)\varphi(x)\,dx &= \int_{[0,1]} u(0,x)\varphi(x)\,dx + \int_{[0,1]} u(t,x)\varphi''(x)\,dx \\ &- \lambda \int_0^t \int_{[0,1]} u^2(s,x)\varphi'(x)\,dx\,ds \\ &+ \gamma \int_0^t \int_{[0,1]} \sqrt{u(s,x)(1-u(s,x))}\varphi(x)\,W(ds,dx).\end{aligned} \tag{1.2}$$

Since the coefficients of (1.1) are non-Lipschitz, the standard results on existence and uniqueness of solutions cannot be applied.

In this paper we prove existence of a non-negative weak solution of (1.1) (Theorem 4.2). Our method of proof is briefly described as follows. Following Funaki [7], in Section 1 we define a discrete version of (1.1), which is a finite-dimensional system (2.2) of stochastic differential equations. In Section 2 we prove existence of a weak solution for this system, and use the method of Le Gall [11] to obtain pathwise uniqueness of weak solutions. This yields existence of a unique strong solution $x_k^N(t)$ of (2.2). Next, in Section 3, we define a system of polygonal approximations $u_N(t,x)$ of $x_k^N(t)$, and use the multidimensional Kolmogorov-Totoki criterion to obtain tightness of $\{u_N(t,x),\ N=1,2,\ldots\}$. In the last section we use a martingale problem to conclude the proof of existence of a weak solution of equation (1.1).

2 Existence of a solution of the discretized version

Let us fix an integer $N \geq 1$ and consider the discretized version of (1.1) on the set $\{\frac{k}{N}, 1 \leq k \leq N\}$:

$$\begin{aligned}\frac{\partial}{\partial t} X_N\left(t,\frac{k}{N}\right) &= \Delta_N X_N\left(t,\frac{k}{N}\right) + \nabla_N\left(X_N^2\left(t,\frac{k}{N}\right)\right) \\ &\quad + \sqrt{N X_N\left(t,\frac{k}{N}\right)\left(1 - X_N\left(t,\frac{k}{N}\right)\right)}\,dB_k(t), \\ X_N\left(0,\frac{k}{N}\right) &= f\left(\frac{k}{N}\right), \quad 1 \leq k \leq N,\ t \geq 0.\end{aligned} \tag{2.1}$$

Here $\mathbb{N}$ is the set of the non-negative integer numbers, $\{B_k(t)\}_{1\leq k\leq N}$ is an infinite system of independent one-dimensional Brownian motions and ∇_N and Δ_N are, respectively, the discrete approximations of the first and second derivative with respect to the variable x:

$$\Delta_N X_N\left(t,\frac{k}{N}\right) = \frac{X_N\left(t,\frac{k+1}{N}\right) - 2X_N\left(t,\frac{k}{N}\right) + X_N\left(t,\frac{k-1}{N}\right)}{\frac{1}{N^2}},$$

$$\nabla_N h\left(s,\frac{k}{N}\right)=\frac{h\left(s,\frac{k+1}{N}\right)-h\left(s,\frac{k}{N}\right)}{\frac{1}{N}},\quad 1\le k\le N.$$

Let us write $x_k^N(t)=X_N\left(t,\frac{k}{N}\right)$. Substituting the above expressions in equation (2.1) we obtain the finite-dimensional system of stochastic differential equations

$$\begin{aligned} dx_i^N(t) &= N^2\left[x_{i+1}^N(t)-2x_i^N(t)+x_{i-1}^N(t)\right]+Nx_{i+1}^N(t)-Nx_i^N(t) \\ &+ \sqrt{Nx_i^N(t)\left(1-x_i^N(t)\right)}\,dB_i(t),\ i=1,...,N, \end{aligned}$$

which can be written in the more compact form

$$\begin{aligned} dx_i^N(t) &= \left(\sum_{j=1}^N a_{ij}^N x_j^N(t)+b_{ij}^N x_j^N(t)^2\right)dt+\sqrt{Nx_i^N(t)\left(1-x_i^N(t)\right)}\,dB_i(t) \\ x_i^N(0) &= f(i/N),\ 1\le i,j\le N, \end{aligned} \tag{2.2}$$

where

$$a_{ij}^N=\begin{cases} N^2 & \text{if} \quad j=i+1,i-1, \\ -2N^2 & \text{if} \quad j=i, \\ 0 & \quad\text{otherwise} \end{cases}$$

and

$$b_{ij}^N=\begin{cases} N & \text{if} \quad j=i+1, \\ -N & \text{if} \quad j=i, \\ 0 & \quad\text{otherwise.} \end{cases}$$

Note that for this system we cannot apply the standard results on existence and uniqueness of solution because Lipschitz assumptions on the drift and diffusion coefficients fail. We prove the following result.

Theorem 2.1. *For any initial random condition* $X^N(0)=(x_1^N,...,x_N^N)\in[0,1]^N$*, the system*

$$\begin{aligned} dx_i^N(t) &= \left(\sum_j a_{ij}^N x_j^N(t)+\sum_j b_{ij}^N x_j^N(t)^2\right)dt+\sqrt{Nx_i^N(t)(1-x_i^N(t))}\,dB_i(t) \\ x_i^N(0) &= x_i,\ i=1,...,N, \end{aligned} \tag{2.3}$$

admits a unique strong solution $X^N(t)=(x_1^N(t),\dots,x_N^N(t))\in C([0,\infty),[0,1]^N)$.

Proof. Let us consider the re-scaled system

$$\begin{aligned} dx_i^N(t) &= \left(\sum_j a_{ij}^N x_j^N(t)+\sum_j b_{ij}^N x_j^N(t)^2\right)dt+\sqrt{g(x_i^N(t))}\,dB_i(t) \\ x_i^N(0) &= x_i,\ i=1,...,N, \end{aligned} \tag{2.4}$$

where $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = Nx(1-x)$ for $0 \le x \le 1$, and $g(x) = 0$ otherwise. Since the coefficients of (2.4) are continuous, by the Skorohod's existence theorem ([14, 10]) we conclude that there exists on some probability space a weak solution $X^N(t)$ of (2.4). We will prove that for each weak solution $X^N(t) = (x_1^N(t), \dots, x_N^N(t))$ of this system, $x_i^N(t) \in [0,1]$ for all $i = 1, \dots, N$ and $t \ge 0$, thus showing that $X^N(t)$ is a solution of (2.3).

First we show that $x_i^N(t) \ge 0$ for each $i = 1 \dots, N$. Since the coefficients of the system are non-Lipschitz, the solution may explode in finite time. Let $\tau_1 \le \infty$ denote the explosion time of the solution. If some of the solution coordinates are negative, then there exists a random time $0 < \tau_2 \le \infty$ such that for $0 < t \le \tau_2$ all such coordinates are between -1 and 0. This is so because (2.4) is finite-dimensional, and its solution is continuous.

We shall use the following lemma ([11]).

Lemma 2.2. *Let $Z \equiv \{Z(t),\ t \ge 0\}$ be a real-valued semimartingale. Suppose that there exists a function $\rho : [0,\infty) \to [0,\infty)$ such that $\int_0^\varepsilon \frac{du}{\rho(u)} = +\infty$ for all $\varepsilon > 0$, and $\int_0^t \frac{1_{\{Z_s>0\}}}{\rho(Z_s)}\, d\langle Z\rangle_s < \infty$ for all $t > 0$ a.s. Then the local time at zero of Z, $L_t^0(Z)$, is identically zero for all t a.s.*

Applying Lemma 2.2 to $x_i^N(t)$ with $\rho(u) = u$ and using the Tanaka formula ([13]), we obtain for $x_i^N(t)_- := \max[0, -x_i^N(t)]$,

$$\begin{aligned}
\sum_{i=1}^N x_i^N(t)_- &= -\int_0^t \sum_{i=1}^N 1_{x_i^N(s)<0} \sum_{j=1}^N (a_{ij}^N x_j(s) + b_{ij}^N x_j(s)^2)\, ds \\
&\le \int_0^t \sum_{i,j=1}^N 1_{x_i^N(s)<0} a_{ij}^N x_j(s)_-\, ds + N \int_0^t \sum_{i=1}^N 1_{x_i^N(s)<0} x_i(s)^2\, ds \\
&\le \int_0^t \sum_{i,j=1}^N a_{ij}^N x_j(s)_-\, ds + N \int_0^t \sum_{i=1}^N x_i^N(s)_-\, ds \\
&= N \int_0^t \sum_{i=1}^N x_i^N(s)_-\, ds,
\end{aligned}$$

where we used that $\sum_i a_{ij}^N = 0$ to obtain the last equality. Then by Gronwall's lemma we obtain that $\sum_{i=1}^N x_i^N(t)_- = 0$, and hence that the solution is non-negative for each $t \ge 0$. By a similar argument applied to $(1 - x_i^N(t))_-$, it follows that $x_i^N(t) \le 1$ for each $1 \le i \le N$.

Using Lemma 2.2 we shall prove pathwise uniqueness of weak solutions of (2.3). Let $X^{1,N} = (x_1^{1,N}, \dots, x_N^{1,N})$ and $X^{2,N} = (x_1^{2,N}, \dots, x_N^{2,N})$ be two solutions of (2.3) with the same initial conditions and the same Brownian motions. We define

$$d_i\left(X^{l,N}(t)\right) = a_{ij}^N x_j^{l,N}(t) + b_{ij}^N x_j^{l,N}(t)^2,\ t \ge 0,\ l = 1, 2.$$

Then

$$
\begin{aligned}
x_i^{1,N}(t) - x_i^{2,N}(t) \quad &= \quad \int_0^t \left[d_i\left(X^{1,N}(s)\right) - d_i\left(X^{2,N}(s)\right)\right] ds \\
&\quad + \int_0^t \left[\sqrt{N x_i^{1,N}(s)(1 - x_i^{1,N}(s))} - \sqrt{N x_i^{2,N}(s)(1 - x_i^{2,N})(s)}\right] dB_i(s), \\
&\quad i = 1, \ldots, N,
\end{aligned}
$$

Since

$$
\langle X \rangle_t = \int_0^t \left[\sqrt{N x_i^{1,N}(s)(1 - x_i^{1,N}(s))} - \sqrt{N x_i^{2,N}(s)(1 - x_i^{2,N})(s)}\right]^2 ds
$$

and

$$
\begin{aligned}
&\int_0^t \frac{\left[\sqrt{N x_i^{1,N}(s)(1 - x_i^{1,N}(s))} - \sqrt{N x_i^{2,N}(s)(1 - x_i^{2,N})(s)}\right]^2}{x_i^{1,N}(s) - x_i^{2,N}(s)} 1_{x_i^{1,N}(s) - x_i^{2,N}(s) > 0}\, ds \\
&\leq \int_0^t 2N 1_{x_i^{1,N}(s) - x_i^{2,N}(s) > 0}\, ds \; < \; 2Nt
\end{aligned}
$$

(where we used that $(\sqrt{x(1-x)} - \sqrt{y(1-y)}\,)/(x-y) < 2$ for $x, y \in [0,1]$, $x > y$, which follows from L'Hospital rule), we can apply Lemma 2.2 to $X(t) = x_i^{1,N}(t) - x_i^{2,N}(t)$ with $\rho(x) = x$. Therefore, $L_t^0\left(x_i^{1,N}(s) - x_i^{2,N}(s)\right) = 0$ for all $i \in \{1, \ldots, N\}$.

By Tanaka's formula,

$$
\begin{aligned}
&\left|x_i^{1,N}(t) - x_i^{2,N}(t)\right| \\
&\quad = \int_0^t \operatorname{sgn}\left(x_i^{1,N}(s) - x_i^{2,N}(s)\right)\left(d_i\left(X^{1,N}(s)\right) - d_i\left(X^{2,N}(s)\right)\right) ds \\
&\qquad + \int_0^t \operatorname{sgn}\left(x_i^{1,N}(s) - x_i^{2,N}(s)\right)\left[\sqrt{N x_i^{1,N}(s)\left(1 - x_i^{1,N}(s)\right)} - \sqrt{N x_i^{2,N}(s)\left(1 - x_i^{2,N}(s)\right)}\right] \\
&\qquad \cdot dB_i(s), \quad i = 1, \ldots, N.
\end{aligned}
$$

Since a_{ij}^N and b_{ij}^N are bounded, it follows that

$$
\begin{aligned}
\mathbb{E}\sum_{i=1}^N \left|x_i^{1,N}(t) - x_i^{2,N}(t)\right| \quad &\leq \quad \int_0^t \mathbb{E}\sum_{i=1}^N \left|d_i\left(X^{1,N}(s)\right) - d_i\left(X^{2,N}(s)\right)\right| ds \\
&\leq \quad \int_0^t K(N)\,\mathbb{E}\sum_{i=1}^N \left|x_i^{1,N}(s) - x_i^{2,N}(s)\right| ds,
\end{aligned}
$$

where $K(N)$ is a constant depending on N. From Gronwall's inequality we conclude that

$$\mathbb{E}\sum_{i=1}^{d}\left|x_i^{1,N}(t)-x_i^{2,N}(t)\right|=0$$

for all $t\geq 0$, thus proving pathwise uniqueness. By a classical theorem of Yamada and Watanabe [17], this is sufficient for existence of a unique strong solution of (2.3). □

3 Tightness of the approximating processes

As a consequence of Theorem 2.1, there exists a strong solution of the system approximating (1.1)

$$\begin{aligned} dx_i^N(t) &= \sum_{1\leq j\leq N} a_{ij}^N x_j^N(t)+\sum_{1\leq j\leq N} b_{ij}^N x_j^N(t)^2+\sqrt{Nx_i^N(t)\left(1-x_i^N(t)\right)}\,dB_i(t) \quad (3.1)\\ x_i^N(0) &= f\left(\frac{i}{N}\right),\ i=1,2,...,N, \end{aligned}$$

where $N\in\mathbb{N}$ is fixed. We denote by $u_N(t,x)$ the polygonal approximation of $x_i^N(t)$,

$$\begin{aligned} u_N(t,x) &= X_N\left(t,\frac{[Nx]+1}{N}\right)(Nx-[Nx])+X_N\left(t,\frac{[Nx]}{N}\right)([Nx]+1-Nx), \quad (3.2)\\ & \quad t\geq 0,\ x\in[0,1], \end{aligned}$$

where by definition $[y]=\frac{k}{N}$ for $\frac{k}{N}\leq y<\frac{k+1}{N}$. Therefore we have $X_N(t,\frac{k}{N})=x_k^N(t)=u_N\left(t,\frac{k}{N}\right)$, $1\leq k\leq N$, and $0\leq u_N(t,x)\leq 1$ for all $t\geq 0$, $x\in[0,1]$.

Let $p_N\left(t,\frac{i}{N},\frac{j}{N}\right)$, $t\geq 0$, $0\leq i,j\leq N+1$, be the fundamental solution of Δ_N, i.e.,

$$\begin{aligned} \frac{\partial}{\partial t}p_N\left(t,\frac{i}{N},\frac{j}{N}\right) &= \Delta_N p_N\left(t,\frac{i}{N},\frac{j}{N}\right),\ t>0,\ 1\leq i,j\leq N\\ p_N\left(0,\frac{i}{N},\frac{j}{N}\right) &= N\delta_{ij}, \end{aligned}$$

with the boundary conditions

$$p_N\left(t,0,\frac{j}{N}\right) = p_N\left(t,\frac{N+1}{N},\frac{j}{N}\right)=0,\ t>0,\ 1\leq j\leq N.$$

Then (3.1) is equivalent to the system (see [9])

$$x_i^N(t) = \sum_{j=1}^{N}\frac{1}{N}p_N\left(t,\frac{i}{N},\frac{j}{N}\right)x_j^N(0)+\int_0^t\left[\sum_{j=1}^{N}\frac{1}{N}p_N\left(t-s,\frac{i}{N},\frac{j}{N}\right)b(i,j)x_j^N(s)^2\right]ds$$

$$+\int_0^t \sum_{j=1}^{N} \left[p_N \left(t-s, \frac{i}{N}, \frac{j}{N} \right) \sqrt{N x_j^N(s) \left(1 - x_j^N(s)\right)} \right] dB_j(s),\ 1 \le i \le N,$$

where in the last integral we used that $\{\frac{1}{N} B_j(s),\ 1 \le j \le N\}$ is an independent system of Brownian motions which we also denote by $\{B_j(s)\}$.

Let us define the re-scaled polygonal interpolation G_N of p_N in $[0,1]$ by

$$G_N(t, x, \frac{j}{N}) \;=\; p_N \left(t, \frac{[Nx]+1}{N}, \frac{j}{N} \right) (Nx - [Nx]) + p_N \left(t, \frac{[Nx]}{N}, \frac{j}{N} \right) ([Nx] + 1 - Nx) \,.$$

Using (3.2) and (3.3) we obtain the following representation for the approximate solution. For $x \in [\frac{i}{N}, \frac{i+1}{N})$,

$$\begin{aligned} u_N(t,x) \;=\;& \sum_{j=1}^{N} \frac{1}{N} G_N \left(t, x, \frac{j}{N} \right) u_N \left(0, \frac{j}{N} \right) \\ &+ \int_0^t \sum_{j=1}^{N} \left[\frac{1}{N} p_N \left(t-s, \frac{i+1}{N}, \frac{j}{N} \right) b(i+1, j) x_j^N(s)^2 (Nx - [Nx]) \right. \\ &+ \left. \frac{1}{N} p_N \left(t-s, \frac{i}{N}, \frac{j}{N} \right) b(i,j) x_j^N(s)^2 ([Nx] + 1 - Nx) \right] ds \\ &+ \int_0^t \sum_{j=1}^{N} G_N \left(t-s, x, \frac{j}{N} \right) \sqrt{N u_N \left(s, \frac{j}{N} \right) \left(1 - u_N \left(s, \frac{j}{N} \right) \right)}\, dB_j(s)\, ds \\ :=\;& u_N^{(1)}(t,x) + u_N^{(2)}(t,x) + u_N^{(3)}(t,x). \end{aligned}$$

Then $u_N(t,x)$ satisfies the boundary conditions in (1.1).

Theorem 3.1. *For each $T > 0$, the sequence $\{u_N(t,x),\ N \ge 1\}$ is tight in the space $C([0,T], A)$, where $A = C([0,1],[0,1])$.*

Proof. Using the fact that $u_N(t,x) \in [0,1]$, we obtain, as in the proof of Lemma 2.2 and Proposition 2.1 in [7], that for each $T < \infty$ and $p \in \mathbb{N}$ there exists $C = C(T,p) > 0$ such that

$$E \left| u_N^{(3)}(t_1, x) - u_N^{(3)}(t_2, y) \right|^{2p} \;\le\; C(|t_1 - t_2|^{p/2} + |x-y|^{p/2}) \tag{3.3}$$

for every $t_1, t_2 \in [0,T]$, $x, y \in [0,1]$ and $N \in \mathbb{N}$, and that

$$\lim_{N\to\infty} \sup_{(t,y)\in[0,T]\times[0,1]} |u_N^{(1)}(t,y) - u(t,y)| = 0, \tag{3.4}$$

where $u(t,y)$ is the fundamental solution of Δ. Since

$$u_N^{(2)} \left(t, \frac{k}{N} \right) = \int_0^t \left[p_N \left(t-s, \frac{k}{N}, \frac{k+1}{N} \right) x_{k+1}^N(s)^2 - p_N \left(t-s, \frac{k}{N}, \frac{k}{N} \right) x_k^N(s)^2 \right] ds$$

and p_N is a fundamental solution of Δ_N, it follows that

$$\frac{\partial}{\partial t} u_N^{(2)}\left(t, \frac{k}{N}\right) = \Delta_N u_N^{(2)}\left(t, \frac{k}{N}\right) + u_N\left(t, \frac{k+1}{N}\right)^2 - u_N\left(t, \frac{k}{N}\right)^2 .$$

Also, $u_N^{(2)}(t,0) = u_N^{(2)}(t,1) = 0$ by the boundary condition (3.3). Integration by parts and an application of Gronwall's inequality give, as in [9] (Thm. 4.2),

$$\max_{1\le k\le N} \left| u_N^{(2)}\left(t, \frac{k}{N}\right)\right| \le e^{2t} \int_0^t \max_{1\le k\le N} \left| u_N^{(1)}\left(s, \frac{k}{N}\right) + u_N^{(3)}\left(s, \frac{k}{N}\right)\right| ds.$$

Hence, from the polygonal form of $u_N^{(2)}$ and (3.3), (3.4) we obtain that for every $T < \infty$ and $p \in \mathbb{N}$ there exists $C_1 = C_1(T,p) > 0$ such that

$$E\left| u_N^{(2)}(t_1, x) - u_N^{(2)}(t_2, y)\right|^{2p} \quad \le \quad C_1(|t_1 - t_2|^{p/2} + |x-y|^{p/2}) \tag{3.5}$$

for every $t_1, t_2 \in [0,T]$, $x, y \in [0,1]$ and $N \in \mathbb{N}$. It follows from the multidimensional Kolmogorov's-Totoki criterion [15] that $u_N(t,x) \in C([0,T], A)$ and that the family $\{u_N(t,x), N \in \mathbb{N}\}$ is tight in $C([0,T], A)$ for each positive T. □

4 The martingale problem for the SPDE

Since the sequence $\{u_N(t,x),\ N \ge 1\}$ is tight by Theorem 3.1, there exists a subsequence, which we denote again by $\{u_{N_k}(t,x)\}$, that converges weakly in $C([0,T], A)$ to a limit $v(t,x)$. By the Skorokhod's representation theorem we can construct processes $\{v_N(t,x)\}$, $u(t,x)$ on some probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, such that $\{u_N\} \stackrel{\mathcal{D}}{=} \{v_N\}$, $u \stackrel{\mathcal{D}}{=} v$, and $\{v_N(t,x)\}$ converges to $u(t,x)$ uniformly on compact subsets of $[0,T] \times \mathbb{R}$ for any $T > 0$ as $N \to \infty$. Obviously $u(t,x)$ satisfies the boundary conditions in (1.1). We shall show that $u(t,x)$ is a weak solution of (1.1) by solving the corresponding martingale problem.

Theorem 4.1. *For any $\varphi \in C_c^2$,*

$$\begin{aligned}\mathcal{M}_\varphi(t) \quad := \quad & \int_{[0,1]} u(t,x)\varphi(x)\,dx - \int_{[0,1]} u(0,x)\varphi(x)\,dx - \int_{[0,1]} u(t,x)\varphi''(x)\,dx \\ & - \int_0^t \int_{[0,1]} u^2(s,x)\varphi'(x)\,dx\end{aligned}$$

is an $\{\mathcal{F}_t\}$-martingale with $\langle \mathcal{M}_\varphi \rangle_t = \int_0^t \int_{[0,1]} u(s,x)((1-u(s,x))\varphi^2(x)\,dx\,ds$.

Proof. Using that

$$\sum_{n\in\mathbb{Z}} a_n\,(b_{n+1} - b_n) + \sum_{n\in\mathbb{Z}} b_{n+1}\,(a_{n+1} - a_n) = 0$$

and multiplying both sides of (2.1) by $\varphi(\frac{k}{N})\frac{1}{N}$, we obtain for fixed $N \geq 1$,

$$
\begin{aligned}
\mathcal{M}_\varphi^N(t) \quad := \quad & \sum_{k=1}^N u_N\left(t, \frac{k}{N}\right)\varphi\left(\frac{k}{N}\right)\frac{1}{N} - \sum_{k=1}^N u_N\left(0, \frac{k}{N}\right)\varphi\left(\frac{k}{N}\right)\frac{1}{N} \\
& - \int_0^t \sum_{k=1}^N \Delta_N u_N\left(s, \frac{k}{N}\right)\varphi\left(\frac{k}{N}\right)\frac{1}{N} - \int_0^t \sum_{k=1}^N \nabla_N\left\{u_N^2\left(s, \frac{k}{N}\right)\right\}\varphi\left(\frac{k}{N}\right)\frac{1}{N} \\
= \quad & \sum_k v_N\left(t, \frac{k}{N}\right)\varphi\left(\frac{k}{N}\right)\frac{1}{N} - \sum_k v_N\left(0, \frac{k}{N}\right)\varphi\left(\frac{k}{N}\right)\frac{1}{N} \\
& - \int_0^t \sum_k v_N\left(s, \frac{k}{N}\right)\Delta_N\varphi\left(\frac{k}{N}\right)\frac{1}{N} - \int_0^t \sum_k v_N^2\left(s, \frac{k}{N}\right)\nabla_N\varphi\left(\frac{k}{N}\right)\frac{1}{N} \\
= \quad & \sum_k \varphi\left(\frac{k}{N}\right)\frac{1}{N}\int_0^t \sqrt{N v_N\left(s, \frac{k}{N}\right)\left(1 - v_N\left(s, \frac{k}{N}\right)\right)}\, dB_k(s). \qquad (4.1)
\end{aligned}
$$

Hence $M_\varphi^N(t)$ is a martingale because by (4.1) $M_\varphi^N(t)$ is the sum of a finite number of martingales. Moreover, $\{\mathcal{M}_\varphi^N(t)\}$ is uniformly integrable because $\sup_{N\in\mathbb{N}} \mathbb{E}(\mathcal{M}_\varphi^N(t))^2 < \infty$ uniformly in $t \in [0, T]$. Indeed, since φ^2 is integrable,

$$
\begin{aligned}
\mathbb{E}(\mathcal{M}_\varphi^N(t))^2 \quad = \quad & \sum_k \varphi^2\left(\frac{k}{N}\right)\frac{1}{N}\int_0^t \left[v_N\left(s, \frac{k}{N}\right)\left(1 - v_N\left(s, \frac{k}{N}\right)\right)\right] ds \\
\leq \quad & T\sum_k \frac{1}{N}\varphi^2\left(\frac{k}{N}\right) \\
< \quad & C(\varphi, T),
\end{aligned}
$$

where $C(\varphi, T)$ is a finite constant depending only on φ and T, but not on N. Therefore $\mathcal{M}_\varphi^N(t) \to \mathcal{M}_\varphi(t)$ as $N \to \infty$, where

$$
\begin{aligned}
\mathcal{M}_\varphi(t) \quad = \quad & \int_{[0,1]} v(t,x)\varphi(x)\, dx - \int_{[0,1]} v(t,0)\varphi(x)\, dx - \int_0^t \int_0^1 v(s,x)\varphi''(x)\, dx\, ds \\
& - \int_0^t \int_0^1 v^2(s,x)\varphi'(x)\, dx\, ds \qquad (4.2)
\end{aligned}
$$

is a martingale. From (4.1) we obtain the quadratic variation of $\mathcal{M}_\varphi(t)$, which is given by

$$
\begin{aligned}
\langle \mathcal{M}_N(\varphi)\rangle_t \quad = \quad & \left\langle \sum_k \varphi\left(\frac{k}{N}\right)\frac{1}{N}\int_0^t \sqrt{N v_N\left(s, \frac{k}{N}\right)\left(1 - v_N\left(s, \frac{k}{N}\right)\right)}\, dB_k(s)\right\rangle_t \\
= \quad & \int_0^t \sum_k N v_N\left(s, \frac{k}{N}\right)\left(1 - v_N\left(s, \frac{k}{N}\right)\right)\varphi^2\left(\frac{k}{N}\right)\frac{1}{N^2}\, ds.
\end{aligned}
$$

Hence $\lim_{N\to\infty} \langle \mathcal{M}_N(\varphi)\rangle_t = \int_0^t \int_{[0,1]} v(s,x)\,(1-v(s,x))\,\varphi^2(x)\,dx\,ds = \langle \mathcal{M}(\varphi)\rangle_t$, and the theorem is proved. □

Now we proceed to the proof of the main result.

Theorem 4.2. *$u(t,x)$ is a weak solution of the stochastic partial differential equation (1.1).*

Proof. To the quadratic variation $\langle \mathcal{M}(\varphi)\rangle_t$ there corresponds a martingale measure $\mathcal{M}(dt,dx)$ with quadratic measure $\nu(dx\,dt) = u(t,x)(1-u(t,x))\,dx\,dt$ (see [16]). Let $\widetilde{W}$ be a white noise independent of $\mathcal{M}$ (defined possibly on a extended probability space). Let us define

$$\begin{aligned} W_t(\varphi) &= \int_{[0,1]} \int_0^t \frac{1}{u(s,x)\,(1-u(s,x))} 1_{\{u(s,x)\notin\{0,1\}\}}\varphi(x)\mathcal{M}(ds,dx) \\ &\quad + \int_{[0,1]} \int_0^t 1_{\{u(s,x)\in\{0,1\}\}}\varphi(x)\widetilde{W}(ds,dx). \end{aligned}$$

Then W_t corresponds to a space -time white noise $W(ds,dx)$ such that

$$\mathcal{M}_t(\varphi) = \int_{[0,1]} \int_0^t \sqrt{u(s,x)(1-u(s,x))}\varphi(x)\,W(ds,dx).$$

From (4.2) we conclude that $u(t,x)$ satisfies (1.2), and hence that $u(t,x)$ is a weak solution of (1.1). The theorem is proved. □

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