FRACTIONAL BILINEAR STOCHASTIC EQUATIONS WITH THE DRIFT IN THE FIRST FRACTIONAL CHAOS

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Abstract. In the paper we compute the explicit form of the fractional chaos decomposition of the solution of a fractional stochastic bilinear equation with the drift in the fractional chaos of order one and initial condition in a finite fractional chaos.

The large deviations principle is also obtained for the one-dimensional distributions of the solution of the equation perturbed by a small noise.

1. Introduction

In recent years many phenomena in telecommunication network, mathematical finance, filtering theory, biology, etc., are modeled by fractional Brownian motion (fBm for short). Multiple integrals has been introduced and studied by Dasgupta and Kallianpur [3], Duncan, Hu and Pasik-Duncan [5], Hu and Oksendal [6] for fBm and by Huang and Cambanis [7] for Gaussian processes.

Recently, in Peréz-Abreu and Tudor [14], multiple fractional integrals are defined explicitly in terms of multiple Wiener-Itô integrals and the equality with the fractional integrals introduced by Dasgupta and Kallianpur [3], Duncan, Hu and Pasik-Duncan [5] is shown.

Also, by using the chaos decomposition method, we introduced an anticipating fractional integral.

In the present paper we consider the case of fBm with Hurst parameter $H > \frac{1}{2}$. By using a product formula for multiple fractional integrals, we get the chaos decomposition of the solution of anticipating bilinear equations with the drift in the first fractional chaos and initial condition in a finite fractional chaos.

Also, by using an extended contraction principle for large deviations (see [13]) and the large deviations for multiple Wiener-Itô integrals (see Ledoux [8] and Mayer-Wolf, Nualart and Peréz-Abreu [10]), we deduce

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the large deviations for the one-dimensional distributions of fractional bilinear equations perturbed by a small noise.

2. Preliminaries

Let $\frac{1}{2} < H < 1$ and $0 < t_0 < \infty$. We put $T = [0, t_0]$ or $T = R$.

Next we fix a normalized fBm $\left\{B_t^H\right\}_{t\in T}$ with Hurst parameter H, defined on a probability space $(\Omega, \mathcal{F}, P)$.

We assume that $\mathcal{F} = \mathcal{B}\left(B_t^H : t \in T\right)$. Recall that B^H is a continuous Gaussian process with:

(i) $B_0^H = 0$.

(ii) For every $s, t \in T$,

$$E(B_s^H B_t^H) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

Define

$$c_H = \left[\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\right]^{\frac{1}{2}}, \ d_H = c_H \Gamma(H + \frac{1}{2}).$$

Remark 2.1. It is known that there exists a standard Brownian motion $\left\{W_t^H\right\}_{t\in T}$ defined on the same probability space $(\Omega, \mathcal{F}, P)$ such that:

(j) $\left\{B_t^H\right\}_{t\in T}$ and $\left\{W_t^H\right\}_{t\in T}$ generate the same filtration $\mathcal{F}$.

(jj) (*Representation formula).* For each $t \in T$,

$$B_t^H = \int_T K_{H,T}(t,s) dW_s^H, \tag{2.1}$$

$$K_{H,R}(t,s) = c_H\left[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right],$$

$$K_{H,[0,t_0]}(t,s) = c_H\left[(t-s)^{H-\frac{1}{2}} + \right.$$

$$\left.\left(\frac{1}{2} - H\right)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right) du\right] 1_{[0,t)}(s) =$$

$$c_H (t-s)^{H-\frac{1}{2}} F(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}) 1_{[0,t)}(s),$$

where F is the Gauss hypergeometric function (see Theorems 4.3 and 4.5 of [1] or Theorem 4.8 of [4], or Theorems 3.2 and 5.2 and Remark 5.1 of [11]).

For $0 < \alpha < 1$ and $f : T^n \to R$ we define *the Liouville fractional integral*(see [17]) by

$$\left(I_{t_0-}^{\alpha,n} f\right)(x_1, ..., x_n) =$$

$$\frac{1}{[\Gamma(\alpha)]^n} \int_{x_1}^{t_0} ... \int_{x_n}^{t_0} \frac{f(t_1, ..., t_n)}{[(t_1 - x_1) ... (t_n - x_n)]^{1-\alpha}} dt_1...dt_n, x_i \in [0, t_0],$$

$$\left(I_{-}^{\alpha,n} f\right)(x_1, ..., x_n) =$$

$$\frac{1}{[\Gamma(\alpha)]^n} \int_{x_1}^{\infty} ... \int_{x_n}^{\infty} \frac{f(t_1, ..., t_n)}{[(t_1 - x_1) ... (t_n - x_n)]^{1-\alpha}} dt_1...dt_n, x_i \in R.$$

3. Multiple Fractional Integrals

We denote by $L_H^2([0, t_0]^n)$ the family of all measurable functions $f : [0, t_0]^n \to R$ such that

$$\int_{[0,t_0]^{2n}} \varphi(x_1, y_1)...\varphi(x_n, y_n) \left| f(x)f(y) \right| dxdy < \infty,$$

where

$$\varphi : R^2 \to R, \ \varphi(s, t) = H(2H - 1) \left| s - t \right|^{2H-2},$$

and $L_H^2(R^n)$ the class of all $f \in L^2(R^n)$ such that

$$\int_{R^{2n}} \varphi(x_1, y_1)...\varphi(x_n, y_n) \left| f(x)f(y) \right| dxdy < \infty,$$

For $f, g \in L_H^2(T^n)$ we define

$$\langle f, g \rangle_{H,T^n} = \int_{T^{2n}} \varphi(x_1, y_1)...\varphi(x_n, y_n) f(x)g(y) dxdy,$$

$$\left| f \right|_{H,T^n} = \langle f, f \rangle_{H,T^n}.$$

$L_{s,H}^2(T^n)$ is the subspace of all $f \in L_H^2(T^n)$, f symmetric.

Next for a function $f : T^n \to R$ we denote by $\mathrm{sym}(f)$ its symmetrization.

Remark 3.1 (see [15], [16], [19]) (a) We have the continuous and dense inclusions $L^2([0, t_0]^n) \subset L_H^2([0, t_0]^n) \subset L^1([0, t_0]^n)$. Also we have the inclusions

$$L^2(R^n) \cap L^1(R^n) \subset L_H^2(R^n) \subset L_{loc}^1(R^n),$$

the first one being dense.

(b) The space $\left(L_H^2(T^n), \langle .,. \rangle_{H,T^n}\right)$ is an incomplete separable pre-Hilbert space and the application

$$\Gamma_{H,T}^{(n)} : L_H^2(T^n) \to L^2(T^n) \ ,$$

$$\left(\Gamma_{H,T}^{(n)} f\right)(t_1, ..., t_n) = d_H^n \left(t_1 ... t_n\right)^{\frac{1}{2}-H} \times \tag{3.1}$$

$$\times \left(I_{t_0-}^{H-\frac{1}{2},n}\right)\left((x_1...x_n)^{H-\frac{1}{2}} f(x_1,...,x_n)\right)(t_1,...,t_n), \text{ if } T = [0, t_0] \, ,$$

$$\left(\Gamma_{H,T}^{(n)} f\right)(t_1, ..., t_n) = d_H^n \left(I_{-}^{H-\frac{1}{2},n} f\right)(t_1, ..., t_n) \text{ if } T = R, \tag{3.2}$$

is an isometry.

We shall denote by $I_p^{\frac{1}{2},T}(f_p)$ the multiple Wiener-Itô integral of the kernel $f_p \in L^2(T^p)$ with respect to the Wiener process W^H from (2.1).

Definition 3.2 If $f_p \in L_H^2(T^p)$ then we define the *multiple fractional integral of order p of f_p with respect to B^H* by

$$I_p^{H,T}(f_p) = I_p^{\frac{1}{2},T}(\Gamma_{H,T}^{(p)} f_p). \tag{3.3}$$

From the product(multiplication) formula for multiple Wiener-Itô integrals (see for example [12] or [18]) we have:

Remark 3.3 (*Product formula*). Let $f_m \in L_{s,H}^2(T^m), \ g_n \in L_{s,H}^2(T^n)$ be such that for any $1 \le r \le \min(m,n)$,

$$f_m(t_1,...,t_{m-r},.), \ g_n(s_1,...,s_{n-r},.) \in L_H^2(T^r), \text{ a.e. } t_1,...,t_{m-r}, s_1,...,s_{n-r},$$

$$t_1,...,t_{m-r} \to |f_m(t_1,...,t_{m-r},.)|_{H,T^r} \in L_H^2(T^{m-r}),$$

$$s_1,...,s_{n-r} \to |g_n(s_1,...,s_{n-r},.)|_{H,T^r} \in L_H^2(T^{n-r}).$$

Then

$$I_m^{H,T}(f_m) I_n^{H,T}(g_n) = \tag{3.4}$$

$$\sum_{r=0}^{\min(m,n)} r! C_m^r C_n^r I_{m+n-2r}^{H,T} \left(\langle f_m(t_1, ..., t_{m-r}, .), g_n(s_1, ..., s_{n-r}, .) \rangle_{H,T^r} \right),$$

(here $\langle f_m(t_1, ..., t_m), g_n(s_1, ..., s_n) \rangle_{H,T^0} = f_m \otimes g_n$).

We denote

$$L_H^2(\Omega, \mathcal{F}, P) = \oplus_{p=0}^{\infty} I_p^{H,T}(L_{s,H}^2(T^p)), \tag{3.5}$$

i.e., $F \in L_H^2(\Omega, \mathcal{F}, P)$ if F has an orthogonal fractional chaos decomposition of the form

$$F = E(F) + \sum_{p=1}^{\infty} I_p^{H,T}(f_p), \tag{3.6}$$

where $f_p \in L_{s,H}^2(T^p)$ and

$$\sum_{p=1}^{\infty} p! \, |f_p|_{H,T^p}^2 < \infty. \tag{3.7}$$

Remark 3.4. The chaos decomposition (3.6) of $F \in L_H^2(\Omega, \mathcal{F}, P)$ is unique and the subspace $L_H^2(\Omega, \mathcal{F}, P)$ is total in $L^2(\Omega, \mathcal{F}, P)$.

Moreover, for every $p \geq 1$, the fractional Wiener chaos of orden p, $I_p^{H,T}(L_{s,H}^2(T^p))$ is not closed. In particular $L_H^2(\Omega, \mathcal{F}, P)$ is not closed and it is strictly included in $L^2(\Omega, \mathcal{F}, P)$.

Next we denote by $L_{H,\infty}^2(\Omega, \mathcal{F}, P)$ the class of all functionals F with chaos decomposition of the form (3.6) such that for each $\lambda \geq 1$,

$$\|F\|_{H,\lambda}^2 = \sum_{p=1}^{\infty} p! \lambda^{2p} \, |f_p|_{H,T^p}^2 < \infty. \tag{3.8}$$

Every $F \in L_{H,\infty}^2(\Omega, \mathcal{F}, P)$ is called an *H-analytic functional.*

A functional F is of *H-exponential type* if for every $p \geq 1$,

$$|f_p|_{H,T^p}^2 \leq \frac{C^p}{(p!)^2}. \tag{3.9}$$

Every F of H-exponential type is analytic.

For $f \in L_H^2(T)$ we define the exponentials $\left\{e_t^H(f)\right\}_{t\in T}, \left\{e_t^H(if)\right\}_{t\in T}$ by

$$e_t^H(f) = \exp\left\{ I_1^{H,T}(f 1_{[0,t]}) - \frac{1}{2} \left| f 1_{[0,t]} \right|_{H,T}^2 \right\},$$

$$e_t^H(if) = \exp\left\{ I_1^{H,T}(if1_{[0,t]}) + \frac{1}{2}\left|f1_{[0,t]}\right|_{H,T}^2 \right\}.$$

Definition 3.5. For $F \in L_H^2(\Omega, \mathcal{F}, P)$ with the chaos decomposition (3.6) we introduce the *S-transform* as the mapping $S_F : L_H^2(T) \to C$, defined by

$$S_F(f) = \left\langle e_t^H(if), F \right\rangle_{L^2(\Omega,\mathcal{F},P)} = \sum_{p=0}^{\infty} i^p \left\langle f^{\otimes p}, f_p \right\rangle_{H,T^p}. \tag{3.10}$$

Since the family $\left\{e_1^H(if)\right\}_{t \in L_H^2(T)}$ is dense in $L_H^2(\Omega, \mathcal{F}, P)$ it follows that

$$S_F = S_G \text{ implies } F = G. \tag{3.11}$$

Definition 3.6. For $f_m \in L_{s,H}^2(T^m)$, $f_n \in L_{s,H}^2(T^n)$ we define the *Wick product*

$$I_m^{H,T}(f_m) \diamond I_n^{H,T}(f_n) = I_{m+n}^{H,T}(\mathrm{sym}(f_m \otimes f_n)). \tag{3.12}$$

By Lemma 2.2 of [2] we have, for each $\lambda \geq 1$,

$$\left\| I_m^{H,T}(f_m) \diamond I_n^{H,T}(f_n) \right\|_\lambda \leq \left\| I_m^{H,T}(f_m) \right\|_{2\lambda} \left\| I_n^{H,T}(f_n)) \right\|_{2\lambda},$$

and thus we can define by continuity the Wick product $F \diamond G$ for every $F, G \in L_{H,\infty}^2(\Omega, \mathcal{F}, P)$ and we have

$$F \diamond G \in L_{H,\infty}^2(\Omega, \mathcal{F}, P), \tag{3.13}$$

$$S_{F \diamond G} = S_F S_G. \tag{3.14}$$

Lemma 3.7. *Let $F \in L_H^2(\Omega, \mathcal{F}, P), F_1, F_2 \in L_{H,\infty}^2(\Omega, \mathcal{F}, P)$ with the chaos decompositions*

$$F = \sum_{p=0}^{\infty} I_p^{H,T}(f_p), \; F_i = \sum_{p=0}^{\infty} I_p^{H,T}(a_p^{(i)}). \tag{3.15}$$

Assume that for every $p \geq 0$,

$$p! f_p(t_1, ..., t_p) = \tag{3.16}$$

$$\sum_{j=0}^{p} \sum_{1 \leq i_1 < .. < i_j \leq p} j!(p-j)! a_j^{(1)}(t_{i_1}, ..., t_{i_j}) a_{p-j}^{(2)}(\hat{t}_{i_1}, ..., \hat{t}_{i_j}).$$

Then $F \in L_{H,\infty}^2(\Omega, \mathcal{F}, P)$ and

$$F = F_1 \diamond F_2. \tag{3.17}$$

Proof. We prove the equality of S-transforms in (3.17). We have by (3.14)

$$S_F(g)=\sum_{p=0}^{\infty}\frac{i^p}{p!}\sum_{j=0}^{p}\sum_{1\le i_1<..<i_j\le p} j!(p-j)!\left\langle a_j^{(1)},g^{\otimes j}\right\rangle_{H,T^j}\times$$

$$\times\left\langle a_{p-j}^{(2)},g^{\otimes(p-j)}\right\rangle_{H,T^{p-j}}=\sum_{p=0}^{\infty}i^p\sum_{j=0}^{p}\left\langle a_j^{(1)},g^{\otimes j}\right\rangle_{H,T^j}\times$$

$$\times\left\langle a_{p-j}^{(2)},g^{\otimes(p-j)}\right\rangle_{H,T^{p-j}}=S_{F_1}(g)S_{F_2}(g)=S_{F_1\diamond F_2}(g).\ \blacksquare$$

For $f\in L_H^2(T)$ we define

$$\|f\|_{H,T}^2=\langle|f|,|f|\rangle_{H,T}.$$

Definition 3.8. We define D_H^{ch} as the class of all measurable processes $\{u(t)\}_{t\in T}$ such that:

(1) $u\in L^2(\Omega,\mathcal{F},P,L_H^2(T))$ and $E\left(\|u\|_{H,T}^2\right)<\infty$.

(2) For a.a. $t\in T$, $u(t)\in L^2(\Omega,\mathcal{F},P)$ and $u(t)$ has the chaos decomposition

$$u(t)=\sum_{p=0}^{\infty}I_p^{H,T}(f_p(.,t)),\tag{3.18}$$

with

$$f_p(.,t)\in L_{s,H}^2(T^p),f_p\in L_H^2(T^{p+1}),\ \forall p\ge 1,\tag{3.19}$$

$$\sum_{p=1}^{\infty}p!\,\|f_p\|_{H,T^{p+1}}^2<\infty.\tag{3.20}$$

(3) The series

$$\delta_H^{ch}(u):=\sum_{p=0}^{\infty}I_{p+1}^{H,T}(\mathrm{sym}\,(f_p)),\tag{3.21}$$

converges in $L^2(\Omega,\mathcal{F},P)$.

Here $\mathrm{sym}(f_p)$ is the symmetrization of $f_p(t_1,...,t_p,t)$ with respect to all its $p+1$-variables, i.e.,

$$\mathrm{sym}\,(f_p)\,(t_1,...,t_p,t)=\frac{1}{p+1}\left\{f_p(t_1,...,t_p,t)+\right.$$

$$\left.\sum_{j=1}^{p} f_p(t_1, .., t_{j-1}, t, t_{j+1}, .., t_p, t_j)\right\}.$$

Remark 3.9 ([14])**.** The operator

$$\delta_H^{ch} : D_H^{ch} \subset L^2\left(\Omega, \mathcal{F}, P, L_H^2(T)\right) \to L^2\left(\Omega, \mathcal{F}, P\right)$$

is closable.

We denote again by δ_H^{ch} the closure of this operator and by $\mathrm{Dom}(\delta_H^{ch})$ its domain.

Therefore a measurable process $u \in \mathrm{Dom}(\delta_H^{ch})$ if there exists $u_n \in D_H^{ch}$, such that

$$E\left(|u_n - u|_{H,T}^2\right) \to 0 \text{ and } E\left(\left|\delta_H^{ch}(u_n) - v\right|^2\right) \to 0.$$

We have in this case $\delta_H^{ch}(u) = v$ and we call $\delta_H^{ch}(u)$ the *fractional stochastic integral of u with respect to B^H* and we denote it sometimes also by $\int_T u(t)dB_t^H$.

If $u1_{[a,b]} \in \mathrm{Dom}(\delta_H^{ch})$ then we define

$$\int_a^b u(t)dB_t^H = \delta_H^{ch}(u1_{[a,b]}).$$

An example. As consequence of the product formula (3.4) it follows the equality

$$\int_a^b B_t^H dB_t^H = \frac{\left(B_b^H - B_a^H\right)^2 - (b-a)^{2H}}{2}.$$

4. Fractional Bilinear Stochastic Equations

We consider the anticipating fractional bilinear equation

$$X_t = \eta + \int_0^t A(s)X_s ds + \int_0^t G(s)X_s dB_s^H, \ t \in T, \tag{4.1}$$

$A(t) = I_1^H(a^t)$, $\eta : (\Omega, \mathcal{F}, P) \longrightarrow R$ is a random variable and $a^t(s) : T^2 \longrightarrow R, G : T \longrightarrow R$ are measurable functions.

Definition 4.1. A measurable process $\{X_t\}_{t\in T}$ is a *strong solution* of (4.1) if:

(i) $1_{[0,t]}(.)A(.)X_. \in L^1(T)$, $1_{[0,t]}(.)G(.)X_. \in \mathrm{Dom}(\delta_H^{ch})$ for a.a. $t \in T$.

(ii) For a.a. $t \in T$, the equation (4.1) is satisfied $P-$a.s..

We introduce the following notation (later in Theorem 4.3 we give the hypotheses which guarantee that the quantities introduced below are well-defined):

Given $t_1, ..., t_p$ and a symmetric function of $p-j$ variables $(j<p)$, we denote by $f_{p-j}(\hat{t}_{i_1}, .., \hat{t}_{i_j})$ the function f_{p-j} evaluated in t*'s* other that $t_{i_1}, .., t_{i_j}$,

$$\Delta_{j,p} = \{(i_1, .., i_j) : i_k \neq i_l \text{ if } k \neq l, i_k = 1, ..., p\}, \; j \leq p,$$

$$h_t(u) = \int_0^t a^r(u)dr, \; Y_t = \exp\left\{\frac{1}{2}\left|h^t(.)\right|^2_{H,T}\right\},$$

$$C_t = \exp\left\{\int_0^t \left\langle a^s(.), 1_{[0,s]}(.)G(.)\right\rangle_{H,T} ds\right\},$$

and if

$$\eta = \sum_{k=0}^{M} I_k^{H,T}(\eta_k), \; M < \infty,$$

we define

$$S_0^t = \sum_{k=0}^{\infty} \left\langle \eta_k, h_t^{\otimes k}\right\rangle_{H,T^k},$$

$$S_p^t(t_1, ..., t_p) = \sum_{k=0}^{\infty} (k+1) ... (k+p) \left\langle \eta_{p+k}, h_t^{\otimes k}\right\rangle_{H,T^k}, \; p \geq 1.$$

Remark 4.2. We have the following straightforward equalities:

$$\frac{d}{dt} h_t^{\otimes p}(t_1, ..., t_p) = p\text{sym}\left(a^t(.) \otimes h_t^{\otimes(p-1)}\right)(t_1, ..., t_p), \tag{4.2}$$

$$\frac{dY_t}{dt} = \left\langle a^t(.), h_t(.)\right\rangle_{H,T} Y_t, \tag{4.3}$$

$$\frac{dC_t}{dt} = \left\langle a^t(.), 1_{[0,t]}G(.)\right\rangle_{H,T} C_t, \tag{4.4}$$

$$\frac{d}{dt} S_p^t(t_1, ..., t_p) = \left\langle a^t(.), S_{p+1}^t(t_1, ..., t_p, .)\right\rangle_{H,T}. \tag{4.5}$$

Theorem 4.3. *Assume that:*
(i) The initial condition

$$\eta = \sum_{p=0}^{M} I_p^{H,T}(\eta_p) \text{ for some } M < \infty, \tag{4.6}$$

is such that for all $1 \le p \le M$ and $1 \le r \le p-1$,

$$\eta_p(t_1, ..., t_r, .) \in L^2_H(T^{p-r}) \text{ for a.e. } t_1, ..., t_r, \tag{4.7}$$

$$(t_1, ..., t_r) \to \left|\eta_p(t_1, ..., t_r, .)\right|_{T^{p-r},H} \in L^2_H(T^r). \tag{4.8}$$

(ii) $a^r(.)$, $G \in L^2_H(T)$ *for any* r .

(iii) $r \to |a^r(.)|_{H,T}$, $r \to \left|1_{[0,r]}(.)G(.)\right|_{H,T} \in L^2(T)$ *if* $T = [0, t_0]$ *and* $r \to |a^r(.)|_{H,T}$, $r \to \left|1_{[0,r]}(.)G(.)\right|_{H,T} \in L^2_{loc}(T)$ *if* $T = R$.

Then the following statements hold.

(a) The equation (4.1) has a strong solution $\{X_t\}_{t\in T}$, *such that for every* t, $X_t \in L^2_{H,\infty}(\Omega, \mathcal{F}, P)$ *and* X_t *has the fractional chaos decomposition*

$$X_t = \sum_{p=0}^{\infty} I_p^{H,T}(f_p^t), \tag{4.9}$$

$$f_0^t = C_t Y_t S_0^t, \tag{4.10}$$

and for $p \ge 1$,

$$p! f_p^t(t_1, ..., t_p) = \sum_{j=1}^{p} \sum_{i_1<..<i_j} (-1)^{j-1} 1_{[0,t]^j}(t_{i_1}, ..., t_{i_j}) \times \tag{4.11}$$

$$\times G(t_{i_1})...G(t_{i_j})(p-j)! f_{p-j}^t(\hat{t}_{i_1}, ..., \hat{t}_{i_j}) + C_t Y_t \left\{ h_t^{\otimes p}(t_1, ..., t_p) S_0^t + \right.$$

$$\left. + \sum_{j=1}^{p} \sum_{i_1<..<i_j} h_t^{\otimes(p-j)}(\hat{t}_{i_1}, ..., \hat{t}_{i_j}) S_j^t(t_{i_1}, ..., t_{i_j}) \right\}.$$

(b) If we define the sequence of kernels $\left\{\hat{f}_p^t\right\}_{t\in T, p\ge 0}$ *by*

$$\hat{f}_0^t = f_0^t, \tag{4.12}$$

and for $p \ge 1$,

$$p! \hat{f}_p^t(t_1, ..., t_p) = C_t Y_t \left\{ h_t^{\otimes p}(t_1, ..., t_p) + \right. \tag{4.13}$$

$$\left. \sum_{j=1}^{p} \sum_{i_1<..<i_j} h_t^{\otimes(p-j)}(\hat{t}_{i_1}, ..., \hat{t}_{i_j}) S_j^t(t_{i_1}, ..., t_{i_j}) \right\},$$

then, for every $p \geq 1$,

$$p! f_p^t(t_1, ..., t_p) = \sum_{j=1}^{p} \sum_{i_1 < .. < i_j} (p-j)!_{[0,t]^j}(t_{i_1}, ..., t_{i_j}) \times \tag{4.14}$$

$$\times G(t_{i_1})...G(t_{i_j}) \hat{f}_{p-j}^t(\hat{t}_{i_1}, ..., \hat{t}_{i_j}) + \hat{f}_p^t(t_1, ..., t_p).$$

(c) For each $p \geq 1$ *and* $\bar{t} \in T$,

$$\sup_{t \in [0,\bar{t}]} \left| f_p^t \right|^2_{H,T^p} \leq \frac{C_1^p}{(p!)^2}. \tag{4.15}$$

Proof. (a) From the product formula (3.4) it follows that $X_t \in L_H^2(\Omega, \mathcal{F}, P)$ defined by (4.9), with the property that for all $p \geq 1$ and $1 \leq r \leq p-1$,

$$f_p^t(t_1, ..., t_r, .) \in L_H^2(T^{p-r}) \text{ for a.e. } \ t_1, ..., t_r, \tag{4.16}$$

$$(t_1, ..., t_r) \to \left| f_p^t(t_1, ..., t_r, .) \right|_{H,T^{p-r}} \in L_H^2(T^r), \tag{4.17}$$

is a solution of (4.1), if the sequence of kernels $\left\{ f_p^t \right\}_{p \geq 0}$ is solution of the infinite system of deterministic integral equations

$$f_0^t = \eta_0 + \int_0^t \langle a^s, f_1^s \rangle_{H,T} \, ds, \tag{4.18}$$

and for $p \geq 1$,

$$f_p^t(t_1, ..., t_p) = \eta_p(t_1, ..., t_p) + \frac{1}{p} \sum_{i=1}^{p} \int_0^t f_{p-1}^s(\hat{t}_i) a^s(t_i) ds + \tag{4.19}$$

$$(p+1) \int_0^t \left\langle a^s, f_{p+1}^s(t_1, ..., t_p, .) \right\rangle_{H,T} ds + \frac{1}{p} \sum_{i=1}^{p} 1_{t_i < t} G(t_i) f_{p-1}^{t_i}(\hat{t}_i).$$

It is easy to see that f_p^t given by (4.10), (4.11) satisfies (4.16), (4.17) and we prove by induction over p that f_p^t given by (4.10), (4.11) is a solution of (4.18), (4.19). From (4.11) we have

$$f_1^t(t_1) = f_0^t \left[1_{t_1 < t} G(t_1) + h_t(t_1) \right] + C_t Y_t S_1^t(t_1), \tag{4.20}$$

and from (4.3) and (4.4)

$$\frac{d}{dt}(C_t Y_t) = C_t Y_t \left\langle a^t(.), 1_{[0,t]}(.) G(.) + h_t(.) \right\rangle_{H,T}. \tag{4.21}$$

Then using (4.20) and (4.21) we deduce

$$\int_0^t \langle a^s, f_1^s(.)\rangle_{H,T}\, ds =$$

$$\int_0^t C_sY_s \left\langle a^s(.), 1_{[0,s]}(.)G + h_s\right\rangle_{H,T} ds + \int_0^t C_sY_s \left\langle a^s, S_1^s\right\rangle_{H,T} ds =$$

$$\int_0^t S_0^s \frac{d}{ds}(C_sY_s)ds + \int_0^t C_sY_s\frac{dS_0^s}{ds}ds =$$

$$\int_0^t \frac{d}{ds}(C_sY_sS_0^s)ds = C_tY_tS_0^t - \eta_0 = f_0^t - \eta_0.$$

Next from the form of f_1^t, f_2^t in (4.19) and using (4.2)-(4.5) and (4.18) we get

$$2\int_0^t \langle a^s, f_2^s(t_1,.)\rangle_{T,H}\, ds = 1_{t_1<t}G(t_1)\int_{t_1}^t \langle a^s, f_1^s\rangle_{H,T}\, ds +$$

$$\int_0^t f_1^s(t_1)\left\langle a^s, 1_{[0,s]}(.)G\right\rangle_{H,T} ds - 1_{t_1<t}G(t_1)\int_{t_1}^t f_0^s\left\langle a^s, 1_{[0,s]}G\right\rangle_{H,T} ds +$$

$$\int_0^t f_0^s h_s(t_1)\left\langle a^s, h_s\right\rangle_{H,T} ds + \int_0^t C_sY_sh_s(t_1)\left\langle a^s, S_1^s\right\rangle_{H,T} ds +$$

$$\int_0^t C_sY_sS_1^s(t_1)\left\langle a^s, h_s\right\rangle_{H,T} ds + \int_0^t C_sY_s)\left\langle a^s, S_2^s(t_1,.)\right\rangle_{H,T} ds =$$

$$1_{t_1<t}G(t_1)\left(f_0^t - f_0^{t_1}\right) + \int_0^t f_0^s h_s(t_1)\left\langle a^s, 1_{[0,s]}(.)G\right\rangle_{T,H} ds +$$

$$\int_0^t C_sY_sS_0^s(t_1)\left\langle a^s, 1_{[0,s]}(.)G\right\rangle_{T,H} ds + \int_0^t C_sS_0^s\frac{dY_s}{ds}ds +$$

$$\int_0^t C_sY_sh_s(t_1)\frac{dS_0^s}{ds}ds + \int_0^t C_s\frac{d(Y_sS_1^s)}{ds}ds =$$

$$1_{t_1<t}G(t_1)f_0^t - 1_{t_1<t}G(t_1)f_0^{t_1} + \int_0^t Y_sS_0^s h_s(t_1)\frac{dC_s}{ds}ds +$$

$$\int_0^t Y_sS_1^s(t_1)\frac{dC_s}{ds}ds + \int_0^t C_sS_0^{s`}\frac{d(Y_s)}{ds}ds +$$

$$\int_0^t C_s Y_s h_s(t_1) \frac{dS_0^s}{ds} ds + \int_0^t C_s \frac{d(Y_s S_0^{s'})}{ds} ds =$$

$$1_{t_1<t} G(t_1) f_0^t - 1_{t_1<t} G(t_1) f_0^{t_1} + \int_0^t \frac{d(C_s Y_s S_1^s(t_1))}{ds} ds +$$

$$\int_0^t Y_s h_s(t_1) \frac{d\left(C_s S_0^s\right)}{ds} ds = 1_{t_1<t} G(t_1) f_0^t - 1_{t_1<t} G(t_1) f_0^{t_1} +$$

$$\int_0^t C_s S_0^s \frac{dY_s}{ds} ds + C_t Y_t S_1^t(t_1) - \eta_1(t_1) +$$

$$\int_0^t Y_s h_s(t_1) \frac{d\left(C_s S_0^s\right)}{ds} ds = 1_{t_1<t} G(t_1) f_0^t - 1_{t_1<t} G(t_1) f_0^{t_1} +$$

$$\int_0^t h_s(t_1) \frac{d\left(C_s Y_s S_0^s\right)}{ds} ds + C_t Y_t S_1^t(t_1) - \eta_1(t_1).$$

Therefore we obtain the equality

$$2 \int_0^t \langle a^s, f_2^s(t_1,.)\rangle_{T,H} \, ds = \tag{4.22}$$

$$1_{t_1<t} G(t_1) f_0^t - 1_{t_1<t} G(t_1) f_0^{t_1} + C_t Y_t S_0^t h_t(t_1) -$$

$$\int_0^t f_0^s a^s(t_1) ds + C_t Y_t S_1^t(t_1) - \eta_1(t_1).$$

Now replacing (4.22) in (4.19) we see that (4.19) is satified for $p = 1$.

Assume now that $(f_n^t)_{n\le p}$ given by (4.11) satisfies (4.19) and prove that f_{p-1}^t, f_p^t, f_{p+1}^t with f_{p+1}^t given by (4.11) also satisfies (4.19).

We have

$$(p+1) \int_0^t \left\langle a^s, f_{p+1}^s(t_1, ..., t_p, .)\right\rangle_{H,T} ds = \sum_{k=1}^{10} I_k, \tag{4.23}$$

$$I_1 = \frac{1}{p!} \sum_{j=1}^{p-1} \sum_{(i_1,\dots,i_j)\in\Delta_{j,p}} (-1)^{j-1}(p+1-j)! 1_{t_{i_1}<\dots<t_{i_j}<t} \times$$

$$\times G(t_{i_1})...G(t_{i_j}) \int_{t_{i_j}}^t \left\langle a^s, f_{p+1-j}^s(\hat{t}_{i_1}, ..., \hat{t}_{i_j}, .)\right\rangle_{H,T} ds,$$

$$I_2 = \frac{(-1)^{p-1}}{p!} 1_{[0,t]^p}(t_1, ..., t_p) G(t_1)...G(t_p) \int_{\max(t_1,..,t_p)}^t \langle a^s, f_1^s \rangle_{T,H} \, ds,$$

$$I_3 = \int_0^t f_p^s(t_1, ..., t_p) \left\langle a^s, 1_{[0,s]}(.)G \right\rangle_{H,T} ds,$$

$$I_4 = \frac{1}{p!} \sum_{j=2}^p (-1)^{j-1}(p+1-j)! \sum_{(i_1,...,i_{j-1}) \in \Delta_{j-1,p}} 1_{t_{i_1} < \cdots < t_{i_{j-1}} < t} \times$$

$$\times G(t_{i_1})...G(t_{i_{j-1}}) \int_{t_{i_{j-1}}}^t f_{p+1-j}^s(\hat{t}_{i_1}, ..., \hat{t}_{i_{j-1}}, .) \left\langle a^s, 1_{[0,s]}(.)G \right\rangle_{H,T} ds,$$

$$I_5 = \frac{(-1)^p}{p!} 1_{[0,t]^p}(t_1, ..., t_p) G(t_1)...G(t_p) \int_{\max(t_1,..,t_p)}^t f_0^s \left\langle a^s, 1_{[0,s]}G \right\rangle_{H,T} ds,$$

$$I_6 = \frac{1}{p!} \int_0^t f_0^s h_s^{\otimes p}(t_1, ..., t_p) \langle a^s, h_s \rangle_{H,T} \, ds,$$

$$I_7 = \frac{1}{p!} \sum_{j=1}^p \sum_{i_1 <, ... < i_j \le p} \int_0^t C_s Y_s S_j^s(t_{i_1}, ..., t_{i_j}) \times$$

$$\times h_s^{\otimes(p-j)}(\hat{t}_{i_1}, ..., \hat{t}_{i_{j-1}}, \hat{t}_{p+1}) \langle a^s, h_s \rangle_{H,T} \, ds,$$

$$I_8 = \int_0^t C_s Y_s h_s^{\otimes p}(t_1, ..., t_p) \langle a^s, S_1^s \rangle_{H,T} \, ds,$$

$$I_9 = \sum_{j=2}^p \sum_{i_1 < ... < i_{j-1} \le p} \int_0^t C_s Y_s h_s^{\otimes(p+1-j)}(\hat{t}_{i_1}, ..., \hat{t}_{i_j}) \times$$

$$\times \left\langle a^s, S_j^s(t_{i_1}, ..., t_{i_{j-1}}, .) \right\rangle_{H,T} ds,$$

$$I_{10} = \int_0^t C_s Y_s \left\langle a^s, S_{p+1}^s(t_1, ..., t_p, .) \right\rangle_{H,T} ds.$$

Replacing in I_1 the term $\int_{t_{i_j}}^t \left\langle a^s, f^s_{p+1-j}(\hat{t}_{i_1}, ..., \hat{t}_{i_j}, .)\right\rangle_{H,T} ds$ given by (4.19) we obtain

$$I_1 = \frac{1}{p!}\sum_{j=1}^{p-1}\sum_{(i_1,...,i_j)\in\Delta_{j,p}}(-1)^{j-1}(p-j)! 1_{t_{i_1}<\cdots<t_{i_j}<} \times$$

$$\times G(t_{i_1})...G(t_{i_j})\left\{f^s_{p-j}(\hat{t}_{i_1}, ..., \hat{t}_{i_j}) - f^{t_{i_j}}_{p-j}(\hat{t}_{i_1}, ..., \hat{t}_{i_j}) - \right.$$

$$\frac{1}{p-j}\sum_{r\neq i_1,...,i_j}\int_{t_{i_j}}^t f^s_{p-1-j}(\hat{t}_{i_1}, ..., \hat{t}_{i_j}, \hat{t}_r)a^s(r)ds -$$

$$\left.\frac{1}{p-j}\sum_{r\neq i_1,...,i_j} 1_{t_{i_j}<t_r<t}G(t_r)f^{t_r}_{p-j-1}(\hat{t}_{i_1}, ..., \hat{t}_{i_j}, \hat{t}_r)\right\},$$

and thus

$$I_1 = \frac{1}{p!}\sum_{j=1}^{p-1}\sum_{(i_1,...,i_j)\in\Delta_{j,p}}(-1)^{j-1}(p-j)! \times \tag{4.24}$$

$$\times 1_{t_{i_1}<\cdots<t_{i_j}}G(t_{i_1})...G(t_{i_{j-1}})f^t_{p-j}(\hat{t}_{i_1}, ..., \hat{t}_{i_j}) -$$

$$\frac{1}{p!}\sum_{r=1}^{p} 1_{t_r<t}G(t_r)(p-1)!f^{t_r}_{p-1}(\hat{t}_r) - \frac{1}{p!}\sum_{r=1}^{p}G(t_r)1_{t_r<t} \times$$

$$\times\sum_{j=1}^{p-2}\sum_{(i_1,...,i_j)\in\Delta_{j,p}}(-1)^p(p-1-j)! 1_{t_{i_1}<\cdots<t_{i_j}<t_r} \times$$

$$\times G(t_{i_1})...G(t_{i_j})f^{t_r}_{p-j-1}(\hat{t}_{i_1}, ..., \hat{t}_{i_j}, \hat{t}_r).$$

On the other hand from (4.18) we have

$$I_2 = \frac{(-1)^{p-1}}{p!}1_{[0,t]^p}(t_1, ..., t_p)G(t_1)...G(t_p)f^t_{\mathbf{0}} - \tag{4.25}$$

$$\frac{(-1)^p}{p!}G(t_1)...G(t_p)1_{[0,t]^p}(t_1, ..., t_p)f^{\max(t_1,...,t_p)}_{\mathbf{0}}.$$

Now if we multiply (4.14) (for $t = s$) with $\left\langle a^s, 1_{[0,s]}G\right\rangle_{H,T}$ and then we integrate from 0 to t we get

$$I_3 = \frac{1}{p!}\sum_{j=1}^{p}(-1)^{j-1}(p-j)! \sum_{(i_1,\ldots,i_j)\in\Delta_{j,p}} 1_{t_{i_1}<\cdots<t_{i_j}<t} \times \tag{4.26}$$

$$\times G(t_{i_1})\ldots G(t_{i_{j-1}}) \int_{t_{i_j}}^{t} f^s_{p-j}(\hat{t}_{i_1},....,\hat{t}_{i_j}) \left\langle a^s, 1_{[0,s]}G\right\rangle_{H,T} ds +$$

$$\frac{1}{p!}\int_0^t Y_s \frac{dC_s}{ds} S_0^s h_s^{\otimes p}(t_1,..,t_p)ds +$$

$$\frac{1}{p!}\sum_{j=1}^{p}\sum_{i_1<\ldots<i_j}\int_0^t Y_s \frac{dC_s}{ds} h_s^{\otimes(p-j)}(\hat{t}_{i_1},....,\hat{t}_{i_j}) S_j^s(t_{i_1},...,t_{i_j})ds,$$

$$I_4 = \frac{1}{p!}\sum_{j=1}^{p-1}(-1)^{j}(p-j)! \sum_{(i_1,\ldots,i_j)\in\Delta_{j,p}} 1_{t_{i_1}<\cdots<t_{i_j}<t} \times \tag{4.27}$$

$$\times G(t_{i_1})\ldots G(t_{i_j}) \int_{t_{i_j}}^{t} f^s_{p-j}(\hat{t}_{i_1},....,\hat{t}_{i_j}) \left\langle a^s, 1_{[0,s]}G\right\rangle_{H,T} ds.$$

From (4.25)-(4.27) we have

$$I_3 + I_4 + I_5 = \int_0^t Y_s \frac{dC_s}{ds} S_0^s h_s^{\otimes p}(t_1,..,t_p)ds + \tag{4.28}$$

$$\sum_{j=1}^{p}\sum_{i_1<\ldots<i_j}\int_0^t Y_s \frac{dC_s}{ds} h_s^{\otimes(p-j)}(\hat{t}_{i_1},....,\hat{t}_{i_j}) S_j^s(t_{i_1},...,t_{i_j})ds.$$

Next

$$I_6 = \int_0^t \frac{dY_s}{ds} C_s S_0^s h_s^{\otimes p}(t_1,..,t_p)ds, \tag{4.29}$$

(4.30)

$$I_7 = \sum_{j=1}^{p}\sum_{i_1<\ldots<i_j\le p}\int_0^t \frac{dY_s}{ds} C_s h_s^{\otimes(p-j)}(\hat{t}_{i_1},....,\hat{t}_{i_j},\hat{t}_{p+1}) S_j^s(t_{i_1},...,t_{i_j})ds,$$

$$I_8 = \int_0^t Y_s C_s \frac{dS_0^s}{ds} h_s^{\otimes p}(t_1,..,t_p)ds, \tag{4.31}$$

(4.32)
$$I_9 = \sum_{j=1}^{p} \sum_{i_1<..<i_j\le p} \int_0^t Y_s C_s h_s^{\otimes(p-j)}(\hat{t}_{i_1},..,\hat{t}_{i_j},\hat{t}_{p+1}) \frac{dS_j^s}{ds}(t_{i_1},..,t_{i_j})ds,$$

$$I_{10} = \int_0^t Y_s C_s \frac{dS_p^s}{ds}(t_1,..,t_p)ds. \tag{4.33}$$

Now (4.28)-(4.33) imply

$$\sum_{j=3}^{10} I_j = \frac{1}{p!}\int_0^t \frac{d\left(Y_s C_s S_0^s\right)}{ds} h_s^{\otimes p}(t_1,..,t_p)ds +$$

$$\frac{1}{p!}\sum_{j=1}^{p-1} \sum_{i_1<...<i_j\le p} \int_0^t h_s^{\otimes(p-j)}(\hat{t}_{i_1},....,\hat{t}_{i_j}) \frac{d}{ds}\left(Y_s C_s S_j^s\right)(t_{i_1},...,t_{i_j})ds +$$

$$\frac{1}{p!}\int_0^t \frac{d}{ds}\left(Y_s C_s S_p^s\right)(t_1,...,t_p)ds,$$

or equivalently,

$$\sum_{j=3}^{10} I_j = \frac{1}{p!} Y_t C_t S_0^t h^{\otimes p}(t_1,...,t_p) - \tag{4.34}$$

$$\frac{1}{(p-1)!}\int_0^t f_0^s \mathrm{sym}\left(a^s \otimes h_s^{\otimes(p-1)}\right) ds +$$

$$\frac{1}{p!}\sum_{j=1}^{p-1} \sum_{i_1<...<i_j\le p} C_t Y_t S_j^s(t_{i_1},...,t_{i_j}) h_t^{\otimes(p-j)}(\hat{t}_{i_1},....,\hat{t}_{i_j}) -$$

$$\frac{1}{p!}\sum_{j=1}^{p-1} \sum_{i_1<...<i_j\le p} (p-j)\int_0^t C_s Y_s S_j^s(t_{i_1},...,t_{i_j}) \mathrm{sym}\left(a^s \otimes h_s^{\otimes(p-j-1)}\right) ds +$$

$$\frac{1}{p!} C_t Y_t S_p^t(t_1,...,t_p) - \frac{1}{p!}\eta_p(t_1,...,t_p).$$

Next by using the form of f_{p-1}^s given by (4.11) we obtain

$$\frac{1}{p}\sum_{r=1}^{p}\int_0^t a^s(t_r) f_{p-1}^s(\hat{t}_r)ds = \tag{4.35}$$

$$\frac{1}{p!}\sum_{j=1}^{p-1}(-1)^{j-1}(p-j)!\sum_{(i_1,\ldots,i_j)\in\Delta_{j,p}}1_{t_{i_1}<\cdots<t_{i_j}<t}\times$$

$$\times G(t_{i_1})...G(t_{i_j})\int_0^t \operatorname{sym}\left(a^s\otimes f^s_{p-j-1}(\hat{t}_{i_1},....,\hat{t}_{i_j},.)\right)ds+$$

$$\frac{1}{(p-1)!}\int_0^t f^s_0\operatorname{sym}\left(a^s\otimes h_s^{\otimes(p-1)}\right)ds-$$

$$\frac{1}{p!}\sum_{j=1}^{p-1}(-1)^{j-1}(p-j)!\sum_{(i_1,\ldots,i_j)\in\Delta_{j,p}}1_{t_{i_1}<\cdots<t_{i_j}<t}\times$$

$$\times G(t_{i_1})...G(t_{i_j})\int_0^{t_{i_j}} \operatorname{sym}\left(a^s\otimes f^s_{p-j-1}(\hat{t}_{i_1},....,\hat{t}_{i_j},.)\right)ds+$$

$$\frac{1}{p!}\sum_{j=1}^{p-1}(p-j)\sum_{i_1<\ldots<i_j}\int_0^t C_sY_sS^s_j(t_{i_1},...,t_{i_j})\times$$

$$\times\operatorname{sym}(\left(a^s(.)\otimes h_s^{\otimes(p-j-1)}(\hat{t}_{i_1},....,\hat{t}_{i_j})\right)ds.$$

Finally from (4.24), (4.25) , (4.34), (4.35) we get (4.14).

(b) It follows from (4.11) by induction.

(c) First it is easy to see that for some positive constant α,

$$p!\sup_{t\in[0,\bar{t}]}\left|S^t_p\right|_{H,T^p}\leq\alpha^p,\ \forall p\geq 1,$$

which implies that for some positive constant β,

$$p!\sup_{t\in[0,\bar{t}]}\left|\hat{f}^t_p\right|_{H,T^p}\leq\beta^p,\ \forall p\geq 1,$$

and by (4.14) we obtain (4.15). ■

Remark 4.4. (a) The problem of uniqueness of the solution (4.1) is still open for $H\neq\frac{1}{2}$, but for $H=\frac{1}{2}$ the uniqueness hold (see [2]).

(b) Formally the form of X_t in the previous theorem is similar to that for $H=\frac{1}{2}$ (see [9]). In fact in the definition of C_t, Y_t, S^t_p for general H, the L^2-norm is replaced by the norm $|.|_{H,T}$.

In general for zero drift we have the following:

Corollary 4.5. *Let η and G be as in Theorem 4.3. Then the equation*

$$X_t = \eta + \int_0^t G(s) X_s dB_s^H,\ t \in T, \tag{4.36}$$

has the unique solution

$$X_t = \eta + \sum_{p=p}^{\infty} I_p^{H,T}(g_p^t) = e_t^H(G) \diamond \eta, \tag{4.37}$$

where

$$p! g_p^t(t_1, ..., t_p) = \sum_{j=1}^{p} \sum_{i_1 < .. < i_j} (-1)^{j-1} 1_{[0,t]^j}(t_{i_1}, ..., t_{i_j}) \times \tag{4.38}$$

$$\times G(t_{i_1}) ... G(t_{i_j})(p-j)! g_{p-j}^t(\hat{t}_{i_1}, ..., \hat{t}_{i_j}) + \eta_p(t_1, ..., t_p).$$

Next for $A \in L_H^2(T), p \geq 1, f_p \in L_{s,H}^2(T^p)$ and $s \leq t$, we define

$$T_{s,t}^{A,H}\left(I_p^{H,T}(f_p)\right) = \tag{4.39}$$

$$\sum_{k=0}^{p} C_p^k \left\langle \left(\Gamma_{H,T}^{(1)}\left(1_{[s,t]} A\right)\right)^{\otimes k}, I_{p-k}^{\frac{1}{2},T}\left(\left(\Gamma_{H,T}^{(p)} f_p\right)(t_1, .., t_k, .)\right)\right\rangle_{L^2(T^k)}.$$

We put $T_t^{A,H} = T_{0,t}^{A,H}$. If we denote by $T_{s,t}^{A,\frac{1}{2}}$ the transformation defined in [2] for $H = \frac{1}{2},\ T = [0,1]$, i.e.,

$$T_{s,t}^{A,\frac{1}{2}}\left(I_p^{\frac{1}{2},T}(g_p)\right) = \sum_{k=0}^{p} C_p^k \left\langle 1_{[s,t]^k} A^{\otimes k}, I_{p-k}^{\frac{1}{2},T}(g_p(t_1, .., t_k, .))\right\rangle_{L^2(T^k)},$$

then it is clear that we have

$$T_{s,t}^{A,H}\left(I_p^{H,T}(f_p)\right) = T_1^{\Gamma_{H,T}^{(1)}\left(1_{[s,t]}A\right), \frac{1}{2}}\left(I_p^{\frac{1}{2},T}(\Gamma^{(p)}(f_p)\right). \tag{4.40}$$

Since from (4.40) and [2] we have

$$\left\| T_{s,t}^{A,H}\left(I_p^{H,T}(f_p)\right)\right\|_{H,\lambda}^2 \leq C_1 \left\| I_p^{H,T}(f_p)\right\|_{H,2\lambda}^2, \tag{4.41}$$

we can extend $T_{s,t}^{A,H} : L_{H,\infty}^2(\Omega, \mathcal{F}, P) \to L_{H,\infty}^2(\Omega, \mathcal{F}, P)$ and we have

$$\left\| T_{s,t}^{A,H} F\right\|_{H,\lambda}^2 \leq C_1 \left\| F \right\|_{H,2\lambda}^2. \tag{4.42}$$

Remark 4.6. From [2] and (4.40) it follows that

$$T_{s,t}^{A,H}(T_{s,t}^{B,H}F)) = T_{s,t}^{A+B,H}F, \tag{4.43}$$

$$e_t^H(A) \diamond F = e_t^H(A)T_t^{-A,H}(F),\ \forall F \in L^2_{H,\infty}(\Omega, \mathcal{F}, P)\,. \tag{4.44}$$

We have the following representation of the solution of (4.1) in terms of Wick products (see [2] for the Brownian case).

Proposition 4.7. *Assume $T = [0,1]$ and let η and G be as in Theorem 4.3. If we define*

$$\hat{Y}_t = \sum_{p=0}^{\infty} \frac{1}{p!} I_p^{H,T}(S_p^t), t \in T, \tag{4.45}$$

then the solution $\{X_t\}$ of (4.1) given by (4.9) has the representation

$$X_t = e_t^H(G) \exp\left\{\int_0^t T_{s,t}^{-G,H}(I_1^{H,T}(a^s))ds\right\} \diamond \hat{Y}_t = \tag{4.46}$$

$$e_t^H(G) \exp\left\{\int_0^t T_{s,t}^{-G,H}(I_1^{H,T}(a^s))ds\right\} \diamond T_1^{h_t,H}\eta.$$

Proof. Define

$$\hat{X}_t = \sum_{p=0}^{\infty} I_p^{H,T}(\hat{f}_p^t),$$

with $\hat{f}_p^t$ given by (4.12), (4.13). Then from (4.14) and Lemma 3.7 it follows

$$\hat{X}_t = C_t Y_t e_1^H(h_t) \diamond \hat{Y}_t, \tag{4.47}$$

$$X_t = e_t^H(G) \diamond \hat{X}_t. \tag{4.48}$$

Replacing (4.47) in (4.48) and using (4.44) we obtain

$$X_t = C_t Y_t e_1^H(1_{[0,t]}G + h_t) T_t^{-(G1_{[0,t]}+h_t),H}(\hat{Y}_t). \tag{4.49}$$

On the other hand

$$T_{s,t}^{-G,H}(I_1^{H,T}(a^s)) = I_1^{H,T}(a^s) - \left\langle a^s(.), 1_{[s,t]}G \right\rangle_{H,T},$$

so that

$$\exp\left\{\int_0^t T_{s,t}^{-G,H}(I_1^{H,T}(a^s))ds\right\} = \tag{4.50}$$

$$C_t Y_t \exp\left\{- \left\langle h_t, 1_{[0,t]}G \right\rangle_{H,T}\right\} e_1^H(h_t).$$

If Z_t is the right-hand side of (4.46) then using (4.43), (4.44) and (4.50) we deduce

$$Z_t = C_t Y_t e_1^H \left(h_t + 1_{[0,t]} G\right) T_t^{-G,H}(\eta). \tag{4.51}$$

From (4.49) and (4.51) we see that we have to prove that

$$T_1^{-(1_{[0,t]}G+h_t),H}(\hat{Y}_t) = T_t^{-G,H}(\eta),$$

or equivalently (by (4.43)),

$$T_t^{-G,H}(T_1^{-h_t,H}\hat{Y}_t) = T_1^{-(1_{[0,t]}G,H}(T_1^{-h_t,H}\hat{Y}_t) = T_t^{-G,H}(\eta),$$

and since $T_t^{A,H}$ is one to one,

$$\hat{Y}_t = T_1^{h_t,H}\eta. \tag{4.52}$$

The equality (4.52) it is easily seen for $\eta = I_p^{H,T}(\eta_p)$. ■

Next we denote by K_∞ the class of all absolutely continuous functions $\varphi : T \to R$ such that

$$\|\dot{\varphi}\|^2_{L^2(T)} = \int_T |\dot{\varphi}(t)|^2 \, dt < \infty.$$

Proposition 4.8. *Assume that A, G are as in Theorem 4.3 and for any $\varepsilon > 0$,*

$$\eta_\varepsilon = \sum_{p=0}^{\infty} \varepsilon^{\frac{p}{2}} I_p^{H,T}(\eta_p),$$

is of $\frac{1}{2}$-exponential type.

Consider the perturbed fractional bilinear stochastic equation

$$X_t^\varepsilon = \eta_\varepsilon + \int_0^t \varepsilon^{\frac{1}{2}} A(s) X_s^\varepsilon ds + \varepsilon^{\frac{1}{2}} \int_0^t G(s) X_s^\varepsilon dB_s^H, \; t \in T. \tag{4.53}$$

Let $\{X_t^\varepsilon\}_{t\in T}$ be the solution of (4.53) obtained in Theorem 4.3, i.e.,

$$X_t^\varepsilon = \sum_{p=0}^{\infty} \varepsilon^{\frac{p}{2}} I_p^{H,T}(f_p^t),$$

with $\{f_p^t\}_{t\in T, p\geq 0}$ given by (4.10) and (4.11).

Then for each $t \in T$, the family of random variables $\{X_t^\varepsilon\}_{\varepsilon>0}$ satisfies the large deviations principle on R with the good rate function

$$\Lambda(x) = \inf\left\{\frac{1}{2}\|\dot{\varphi}\|^2_{L^2(T)} : F(\varphi) = x\right\}, \; x \in R, \tag{4.54}$$

where $F : K_\infty \to \bar{R}_+$ *is defined by*

$$F(\varphi) = \sum_{p=0}^{\infty} \int_{T^p} \left(\Gamma_{H,T}^{(p)} f_p^t\right)(t_1, .., t_p)\, \dot{\varphi}^{\otimes p}(t_1, .., t_p)\, dt_1 ... dt_p. \tag{4.55}$$

Proof. Since by Theorem 4.3 the kernels $\left\{f_p^t\right\}_{t\in T, p\geq 0}$ are of H-exponential type we can apply Theorem 3.1 of [13] in order to get the result. ■

References

[1] R.J. Barton and H. Vincent Poor. Signal detection in fractional Gaussian noise. *IEEE Transactions on Information Theory,* Vol. **34**, Nr. **5** (1988), 943-959.

[2] R. Buckdahn and D. Nualart. Linear stochastic differential equations and Wick products. *Prob. Theory Rel. Fields* **99** (1994), 501-526.

[3] A. Dasgupta and G. Kallianpur. Multiple fractional integrals. *Prob. Theory Rel. Fields* **115** (1999), 505-525.

[4] L. Decreusefond and A.S. Üstunel. Stochastic Analysis with respect to fractional Brownian motion. *Potential Analysis* **10(2)** (1999), 177-214.

[5] T. Duncan, Y. Hu and B. Pasik-Duncan. Stochastic calculus for fractional Brownian motion I.Theory. *SIAM J. Control Optim.* **38(2)** (2000), 582-612.

[6] Y. Hu and B. Oksendal. Fractional white noise calculus and applications to finance. Preprint, 1999.

[7] S.T. Huang and S. Cambanis. Stochastic and multiple Wiener integrals for Gaussian processes. *Ann. Probab.* **6** (1978), 585-614.

[8] M. Ledoux. A note on large deviations for Wiener chaos. In: *Séminaire de Probab. XXIV.* Springer-Verlag, New York. Lecture Nores in Math. **1426** (1990), 1-14.

[9] J.A. Leòn and V. Pérez-Abreu. Strong solutions of stochastic bilinear equations with anticipating drift in the first Wiener chaos. In Cambanis, S., Ghosh, J.K., Karandikar, R. and Sen, P.K. (eds), *Stochastic Processes: A Festschrift in Honor of Gopinath Kallianpur.* Springer Verlag, 1993, 235-243.

[10] E. Mayer-Wolf, D. Nualart and V. Pérez-Abreu. Large deviations for multiple Wiener-Itô integral processes. In: *Séminaire de Probab. XXVI.* Springer-Verlag, New York. Lecture Nores in Math. **1526** (1992), 11-31.

[11] I. Norros, E. Valkeila and J. Virtamo. An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli* **5(4)** (1999), 571-587.

[12] D. Nualart, *The Malliavin Calculus and Related Topics.* Springer-Verlag, 1995.

[13] V. Pérez-Abreu, C. Tudor. Large deviations for a class of chaos expansions. *J. Theoretical Probability* **7(4)** (1994),757-765.

[14] V. Pérez-Abreu, C. Tudor. A transfer principle for multiple stochastic fractional integrals. Preprint CIMAT, 2001.

[15] V. Pipiras and M.S. Taqqu. Integration questions related to fractional Brownian motion. *Prob. Theory Rel. Fields* **118(2)** (2000), 251-291.

[16] V. Pipiras and M.S. Taqqu. Are classes of deterministic integrands for fractional Brownian motion on an interval complete? *Bernoulli* **7(6)** (2001), 873-897.

[17] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives.* Gordon and Breach Science, 1993.

[18] I. Shigekawa. Derivatives of Wiener functionals and absolute continuity of induced measures. *J. Math. Kyoto Univ.* **26** (1986), 263-289.

[19] C. Tudor. On the Wiener integral with respect to the fractional Brownian motion. *Bol. Mex. Mat. Soc.* **1(8)** (2002).

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