ON THE TWO-PARAMETER FRACTIONAL BROWNIAN MOTION AND STIELTJES INTEGRALS FOR HÖLDER FUNCTIONS

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ABSTRACT. We extend the Stieltjes integral to Hölder functions of two variables and prove an existence and uniqueness result for the corresponding deterministic ordinary differential equations and also for stochastic equations driven by a two-parameter fractional Brownian motion.

1. INTRODUCTION

For functions of one variable the Stieltjes integral $\int_a^b f(t)dg(t)$ is well defined if f is α-Hölder, g is β-Hölder, $\alpha+\beta>1$ (see [2], [3], [8], [9]). This fact allows in particular to study the corresponding ordinary differential equations, to define the stochastic integral with respect to one-parameter fractional Brownian motion pointwise and consequently the study of the associated stochastic equations (see [5]).

In the present paper we use the Liouville space for functions of two variables in order to define the Stieltjes integral $\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)dg(t_1,t_2)$ if f is (α_1,α_2)-Hölder, g is (β_1,β_2)-Hölder, $\alpha_i+\beta_i>1$. Then we prove a general existence and uniqueness theorem for ordinary differential equations with Hölder continuous forcing and for stochastic equations with a two-parameter fractional Brownian motion as forcing term.

2. PRELIMINARIES

We fix $\alpha=(\alpha_1,\alpha_2)$, $\alpha_i\in(0,1]$ and let $T=[a_1,b_1]\times[a_2,b_2]$. For a function $f:T\to R$ we define *the Riemann-Liouville fractional integral of order* α by

$$\left(I^{\alpha}_{(a_1,a_2)+}f\right)(x_1,x_2)= \tag{2.1}$$

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$$\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_{a_1}^{x_1}\int_{a_2}^{x_2}\frac{f(t_1,t_2)}{(x_1-t_1)^{1-\alpha_1}(x_2-t_2)^{1-\alpha_2}}dt_1dt_2, (x_1,x_2)\in T.$$

The space $\Lambda_{\alpha,p}=\left(I^{\alpha}_{(a_1,a_2)+}\right)(L^p(T))$ is called the Liouville space (or the Besov space) and it becomes a separable Banach space with respect to the norm

$$\left\|I^{\alpha}_{(a_1,a_2)+}f\right\|_{\alpha,p}=\|f\|_p\,.$$

Remark 2.1 (see [6]) (a) (Semigroup property). For every α,β,

$$I^{\alpha}_{(a_1,a_2)+}I^{\beta}_{(a_1,a_2)+}=I^{\alpha+\beta}_{(a_1,a_2)+}.$$

(b) If $f\in C_b^2(T)$ and $f=0$ on $\partial_1 T=([a_1,b_1]\times\{b_1\})\cup(\{a_1\}\times[a_2,b_2])$, then the function

$$D^{\alpha}_{(a_1,a_2)+}f(x_1,x_2)= \tag{2.2}$$

$$\frac{1}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)}\int_{a_1}^{x_1}\int_{a_2}^{x_2}\frac{\partial^2 f(t_1,t_2)}{\partial t_1\partial t_2}\frac{dt_1dt_2}{(x_1-t_1)^{\alpha_1}(x_2-t_2)^{\alpha_2}},$$

is the unique function from $L_\infty(T)$ such that

$$I^{\alpha}_{(a_1,a_2)+}D^{\alpha}_{(a_1,a_2)+}f=f.$$

For a rectangle $D=[s_1,t_1]\times[s_2,t_2]\subset T$ we define the increment on D of the function $f:T\to R$ by

$$f(D)=f(t_1,t_2)-f(t_1,s_2)-f(s_1,t_2)+f(s_1,s_2).$$

We denote by $H_{[a_i,b_i],\alpha_i}$ the space of all α_i-Hölder functions on $[a_i,b_i]$ and

$$\|f\|_{[a_i,b_i],\alpha_i}=\sup_{u\neq v,a_i\le u,v\le b_i}\frac{|f(u)-f(v)|}{|u-v|^{\alpha_i}}$$

Also, we denote by H_{T,α_1,α_2} the space of all (α_1,α_2)-Hölder functions on T, i.e., $f\in H_{T,\alpha_1,\alpha_2}$ if f is continuous,

$$\|f(a_1,.)\|_{[a_2,b_2],\alpha_2}<\infty,\ \|f(.,a_2)\|_{[a_1,b_1],\alpha_1}<\infty,$$

and

$$\|f\|_{T,\alpha_1,\alpha_2}=\sup_{u_i\neq v_i}\frac{|f([u_1,v_1]\times[u_2,v_2])|}{|u_1-v_1|^{\alpha_1}|u_2-v_2|^{\alpha_2}}<\infty.$$

Remark 2.2 (see [2]). Assume that $0<\beta_1<\alpha_1, 0<\beta_2<\alpha_2$ and $p\geq 1$. Then we have the continuous inclusions

$$\Lambda_{\alpha,p}\subset\Lambda_{\beta,p},$$

$$\Lambda_{\alpha,p}\subset H_{\alpha_1-p^{-1},\alpha_2-p^{-1}},\ H_{\beta_1,\beta_2}\subset\Lambda_{\gamma,p}\text{ if }\alpha_i p>1,\beta_i>\gamma_i>0.$$

Remark 2.3 (see [2]). Assume that $f,\ g$ are $C^1([a,b])$-functions with $f(a)=0$. Let $\alpha,\beta\in(0,1]$ be such that $\alpha+\beta>1$ and let $\delta: a=t_0<...<t_n=b$ be a partition of $[a,b]$ with the norm $\|\delta\|=\max_j(t_{j+1}-t_j)$. Then for every $0<\varepsilon<\alpha+\beta-1$ the following estimates hold,

$$\left|\int_a^b f(t)dg(t)\right|\leq C(\alpha,\beta)\|f\|_{[a,b],\alpha}\|g\|_{[a,b],\beta}(b-a)^{1+\varepsilon},\tag{2.3}$$

$$\left|\int_a^b f(t)dg(t)-\sum_i f(t_i)\left[g(t_{i+1})-g(t_i)\right]\right|\leq \tag{2.4}$$

$$C(\alpha,\beta)\|f\|_{[a,b],\alpha}\|g\|_{[a,b],\beta}(b-a)^{\varepsilon}.$$

3. Main Results

The next result represents the essential step for extending the Stieltjes integral to Hölder functions of two variables.

Proposition 3.1. *Let $\alpha_i+\beta_i>1$, $\alpha_i,\beta_i\in(0,1]$, $f,g\in C^2(T)$ and let $0<\varepsilon_i<\alpha_i+\beta_i-1$. Then*

$$\left|\int_{a_1}^{b_1}\int_{a_2}^{b_2}f(t_1,t_2)dg(t_1,t_2)\right|\leq \tag{3.1}$$

$$C(\alpha_i,\beta_i)\|g\|_{T,\beta_1,\beta_2}\Big\{\|f\|_{T,\alpha_1,\alpha_2}(b_1-a_1)^{\alpha_1+\beta_1}(b_2-a_2)^{\alpha_2+\beta_2}+$$

$$\|f(.,a_2)\|_{[a_1,b_1],\alpha_1}(b_1-a_1)^{1+\varepsilon_1}(b_2-a_2)^{\beta_2}+$$

$$\|f(a_1,.)\|_{[a_2,b_2],\alpha_2}(b_1-a_1)^{\beta_1}(b_2-a_2)^{1+\varepsilon_2}+$$

$$|f(a_1,a_2)|(b_1-a_1)^{\beta_1}(b_2-a_2)^{\beta_2}\Big\}.$$

Moreover for every partition $\Delta=(s_i,t_j)_{i,j},\ a_1=s_1<...<s_{n_1}=b_1,\ a_2=t_1<...<t_{n_2}$,

$\|\Delta\| = \max_i (s_{i+1} - s_i) + \max_j (t_{j+1} - t_j)$, *we have*

$$\left| \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(u_1, u_2) dg(u_1, u_2) - \sum_{i=1}^{n_1-1} \sum_{j=1}^{n_2-1} f(s_i, t_j) g([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \right| \le C(\alpha_i, \beta_i) \|f\|_{T,\alpha_1,\alpha_2} \left[\|g\|_{T,\beta_1,\beta_2} \|\Delta\|^{\alpha_1+\beta_1+\alpha_2+\beta_2-2} + \left(\|g\|_{T,\beta_1,\beta_2} + \|g(a_1,.)\|_{[a_2,b_2],\beta_2} \right) \|\Delta\|^{\alpha_1} + \left(\|g\|_{T,\beta_1,\beta_2} + \|g(.,a_2)\|_{[a_1,b_1],\beta_1} \right) \|\Delta\|^{\alpha_2} \right]. \tag{3.2}$$

Proof. Assume first that $f = 0$ on $\partial_1 T$ and define

$$h(t_1, t_2) = g(b_1 - t_1, b_2 - t_2) - g(b_1 - a_1, b_2 - t_2) - g(b_1 - t_1, b_2 - a_2) + g(b_1 - a_1, b_2 - a_2).$$

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} f(t_1, t_2) \frac{\partial^2 g(t_1, t_2)}{\partial t_1 \partial t_2} dt_1 dt_2 = \frac{\partial^2 (f * h)(b_1, b_2)}{\partial t_1 \partial t_2}. \tag{3.3}$$

Choose $\varepsilon_i > 0, 0 < \alpha_i' < \alpha_i,\ 0 < \beta_i' < \beta_i$ with $\alpha_i' + \beta_i' = 1 + \varepsilon_i$.

By Remark 2.1-(b) the functions $f_1 = D_{(a_1,a_2)+}^{(\alpha_1',\alpha_2')} f,\ h_1 = D_{(a_1,a_2)+}^{(\beta_1',\beta_2')} h$ are in L_∞ and satisfy

$$I_{(a_1,a_2)+}^{(\alpha_1',\alpha_2')} f_1 = f, I_{(a_1,a_2)+}^{(\beta_1',\beta_2')} h_1 = h. \tag{3.4}$$

Then by Remark 2.1-(a), (3.3) and (3.4) we have

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} f(t_1, t_2) dg(t_1, t_2) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(t_1, t_2) \frac{\partial^2 g(t_1, t_2)}{\partial t_1 \partial t_2} dt_1 dt_2 =$$

$$\frac{\partial^2 (f * h)(b_1, b_2)}{\partial t_1 \partial t_2} = \frac{\partial^2}{\partial t_1 \partial t_2} \left[I_{(a_1,a_2)+}^{(\alpha_1',\alpha_2')} f_1 * I_{(a_1,a_2)+}^{(\beta_1',\beta_2')} h_1 \right] (b_1, b_2) =$$

$$\frac{\partial^2}{\partial t_1 \partial t_2} \left[I_{(a_1,a_2)+}^{(\alpha_1'+\beta_1',\alpha_2'+\beta_2')} (f_1 * h_1) \right] (b_1, b_2) =$$

$$\frac{\partial^2}{\partial t_1 \partial t_2}\left[I^{(1,1)}_{(a_1,a_2)+} I^{(\varepsilon_1,\varepsilon_2)}_{(a_1,a_2)+}(f_1 * h_1)\right](b_1,b_2) =$$

$$I^{(\varepsilon_1,\varepsilon_2)}_{(a_1,a_2)+}(f_1 * h_1)(b_1,b_2),$$

and then

$$\left|\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2) dg(t_1,t_2)\right| \leq \tag{3.5}$$

$$\frac{(b_1-a_1)^{\varepsilon_1}(b_2-a_2)^{\varepsilon_2}}{\varepsilon_1\varepsilon_2\Gamma(\varepsilon_1)\Gamma(\varepsilon_2)}\|f_1 * h_1\|_\infty \leq$$

$$\frac{(b_1-a_1)^{1+\varepsilon_1}(b_2-a_2)^{1+\varepsilon_2}}{\varepsilon_1\varepsilon_2\Gamma(\varepsilon_1)\Gamma(\varepsilon_2)}\|f_1\|_\infty \|h_1\|_\infty .$$

Next the integration by parts for functions of two variables (see [7]) yields

$$f_1(x_1,x_2) = \frac{1}{\Gamma(1-\alpha_1')\Gamma(1-\alpha_2')}\int_{a_1}^{x_1}\int_{a_2}^{x_2}\frac{df([t_1,x_1]\times[t_2,x_2])}{(x_1-t_1)^{\alpha_1'}(x_2-t_2)^{\alpha_2'}} =$$

$$\frac{1}{\Gamma(1-\alpha_1')\Gamma(1-\alpha_2')}\left\{\lim_{t_i\to x_i}\frac{f([t_1,x_1]\times[t_2,x_2])}{(x_1-t_1)^{\alpha_1'}(x_2-t_2)^{\alpha_2'}} -\right.$$

$$\lim_{t_2\to x_2}\int_{a_1}^{x_1}\frac{f([t_1,x_1]\times[t_2,x_2])dt_1}{(x_1-t_1)^{\alpha_1'}(x_2-t_2)^{\alpha_2'}} - \lim_{t_1\to x_1}\int_{a_2}^{x_2}\frac{f([t_1,x_1]\times[t_2,x_2])dt_2}{(x_1-t_1)^{\alpha_1'}(x_2-t_2)^{\alpha_2'}} +$$

$$\left.\alpha_1'\alpha_2'\int_{a_1}^{x_1}\int_{a_2}^{x_2}\frac{f([t_1,x_1]\times[t_2,x_2])dt_1dt_2}{(x_1-t_1)^{\alpha_1'+1}(x_2-t_2)^{\alpha_2'+1}}\right\} =$$

$$\frac{\alpha_1'\alpha_2'}{\Gamma(1-\alpha_1')\Gamma(1-\alpha_2')}\int_{a_1}^{x_1}\int_{a_2}^{x_2}\frac{f([t_1,x_1]\times[t_2,x_2])dt_1dt_2}{(x_1-t_1)^{\alpha_1'+1}(x_2-t_2)^{\alpha_2'+1}},$$

so that

$$\|f_1\|_\infty \leq \frac{\alpha_1'\alpha_2'}{\Gamma(1-\alpha_1')\Gamma(1-\alpha_2')}\|f\|_{T,\alpha_1,\alpha_2} \times \tag{3.6}$$

$$\times \int_{a_1}^{x_1}\int_{a_2}^{x_2}(x_1-t_1)^{\alpha_1-\alpha_1'-1}(x_2-t_2)^{\alpha_2-\alpha_2'-1}dt_1dt_2 \leq$$

$$c\left\|f\right\|_{T,\alpha_1,\alpha_2}(b_1-a_1)^{\alpha_1-\alpha_1'}(b_2-a_2)^{\alpha_2-\alpha_2'}.$$

Similarly

$$\left\|h_1\right\|_\infty \le c_1\left\|g\right\|_{T,\beta_1,\beta_2}(b_1-a_1)^{\beta_1-\beta_1'}(b_2-a_2)^{\beta_2-\beta_2'}. \tag{3.7}$$

By using (3.6) and (3.7) in (3.5) we obtain (3.1) if $f=0$ on $\partial_1 T$.

If f is not necessarily null on $\partial_1 T$ then we define

$$\bar{f}(t_1,t_2)=f\left([a_1,t_1]\times[a_2,t_2]\right).$$

Then $\bar{f}=0$ on $\partial_1 T$ and $f,\bar{f}$ have the same increments.

Then we have

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)\frac{\partial^2 g(t_1,t_2)}{\partial t_1\partial t_2}dt_1dt_2= \tag{3.8}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\bar{f}(t_1,t_2)\frac{\partial^2 g(t_1,t_2)}{\partial t_1\partial t_2}dt_1dt_2+\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(a_1,t_2)\frac{\partial^2 g(t_1,t_2)}{\partial t_1\partial t_2}dt_1dt_2+$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,a_2)\frac{\partial^2 g(t_1,t_2)}{\partial t_1\partial t_2}dt_1dt_2+f(a_1,a_2)g\left([a_1,b_1]\times[a_2,b_2]\right)=$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\bar{f}(t_1,t_2)\frac{\partial^2 g(t_1,t_2)}{\partial t_1\partial t_2}dt_1dt_2+$$

$$\int_{a_2}^{b_2} f(a_1,t_2)\left[\frac{\partial g(b_1,t_2)}{\partial t_2}-\frac{\partial g(a_1,t_2)}{\partial t_2}\right]dt_2+$$

$$\int_{a_1}^{b_1} f(t_1,a_2)\left[\frac{\partial g(t_1,b_2)}{\partial t_1}-\frac{\partial g(t_1,a_2)}{\partial t_1}\right]dt_1+$$

$$f(a_1,a_2)g\left([a_1,b_1]\times[a_2,b_2]\right)=\sum_{k=1}^{4}J_k.$$

From the previous reasoning we have

$$\left|\int_{a_1}^{b_1}\int_{a_2}^{b_2}\bar{f}(t_1,t_2)\frac{\partial^2 g(t_1,t_2)}{\partial t_1\partial t_2}dt_1dt_2\right|\le \tag{3.9}$$

$$C(\alpha_i,\beta_i)\left\|f\right\|_{T,\alpha_1,\alpha_2}\left\|g\right\|_{T,\beta_1,\beta_2}(b_1-a_1)^{\alpha_1+\beta_1}(b_2-a_2)^{\alpha_2+\beta_2}.$$

Next by using (2.3) we have

$$|J_2| \leq \left| \int_{a_2}^{b_2} [f(a_1,t_2) - f(a_1,a_2)] \left[\frac{\partial g(b_1,t_2)}{\partial t_2} - \frac{\partial g(a_1,t_2)}{\partial t_2} \right] dt_2 \right| +$$

$$|f(a_1,a_2) g([a_1,b_1] \times [a_2,b_2])| \leq$$

$$C(\alpha_i,\beta_i) \, \|f(a_1,.)\|_{[a_2,b_2],\alpha_2} \, \|g(b_1,.) - g(a_1,.)\|_{[a_2,b_2],\beta_2} \, (b_2 - a_2)^{1+\varepsilon_2} +$$

$$|f(a_1,a_2)| \, \|g\|_{T,\beta_1,\beta_2} \, (b_1 - a_1)^{\beta_1} \, (b_2 - a_2)^{\beta_2} \,,$$

so that

$$|J_2| \leq C(\alpha_i,\beta_i) \, \|g\|_{T,\beta_1,\beta_2} \left\{ \|f(a_1,.)\|_{[a_2,b_2],\alpha_2} \, (b_1 - a_1)^{\beta_1} \, (b_2 - a_2)^{1+\varepsilon_2} + \right.$$

$$\left. |f(a_1,a_2)| \, (b_1 - a_1)^{\beta_1} \, (b_2 - a_2)^{\beta_2} \right\}. \tag{3.10}$$

Similarly

$$|J_3| \leq C(\alpha_i,\beta_i) \, \|g\|_{T,\beta_1,\beta_2} \times \tag{3.11}$$

$$\times \left\{ \|f(.,a_2)\|_{[a_1,b_1],\alpha_1} \, (b_1 - a_1)^{1+\varepsilon_1} \, (b_2 - a_2)^{\beta_2} + \right.$$

$$\left. |f(a_1,a_2)| \, (b_1 - a_1)^{\beta_1} \, (b_2 - a_2)^{\beta_2} \right\}.$$

Replacing (3.10)-(3.11) in (3.8) we obtain (3.2).

Next we have

$$I_\Delta = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(u_1,u_2) dg(u_1,u_2) - \tag{3.12}$$

$$\sum_{i,j} f(s_i,t_j) g([s_i,s_{i+1}] \times [t_j,t_{j+1}]) =$$

$$\sum_{i,j} \int_{s_i}^{s_{i+1}} \int_{t_j}^{t_{j+1}} [f(u_1,u_2) - f(s_i,t_j)] \, dg(u_1,u_2) =$$

$$\sum_{i,j} \int_{s_i}^{s_{i+1}} \int_{t_j}^{t_{j+1}} [f(u_1,u_2) - f(u_1,t_j) - f(s_i,u_2) + f(s_i,t_j)] \, dg(u_1,u_2) +$$

$$\sum_{i,j}\int_{s_i}^{s_{i+1}}\int_{t_j}^{t_{j+1}}\left[f(u_1,t_j)-f(s_i,t_j)\right]dg(u_1,u_2)+$$

$$\sum_{i,j}\int_{s_i}^{s_{i+1}}\int_{t_j}^{t_{j+1}}\left[f(s_i,u_2)-f(s_i,t_j)\right]dg(u_1,u_2)=I_\Delta^1+I_\Delta^2+I_\Delta^3.$$

From (3.1) it follows that

$$\left|I_\Delta^1\right|\leq C\left\|f\right\|_{T,\alpha_1,\alpha_2}\left\|g\right\|_{T,\beta_1,\beta_2}\times \tag{3.13}$$

$$\times\sum_{i,j}\left(s_{i+1}-s_i\right)^{\alpha_1+\beta_1}\left(t_{j+1}-t_j\right)^{\alpha_2+\beta_2}\leq$$

$$C_1\left\|f\right\|_{T,\alpha_1,\alpha_2}\left\|g\right\|_{T,\beta_1,\beta_2}\left\|\Delta\right\|^{\alpha_1+\beta_1+\alpha_2+\beta_2-2}.$$

Next define

$$f_1(u_1,u_2)=f(u_1,u_2)-f(s_i,u_2)\text{ if }u_1\in\left[s_i,s_{i+1}\right).$$

Then (2.3), (2.4) imply

$$I_\Delta^2=\sum_{i,j}\int_{s_i}^{s_{i+1}}\int_{t_j}^{t_{j+1}}\left[f(u_1,t_j)-f(s_i,t_j)\right]\frac{\partial^2 g(u_1,u_2)}{\partial u_1\partial u_2}du_1du_2=$$

$$\sum_{i,j}\int_{s_i}^{s_{i+1}}\left[f(u_1,t_j)-f(s_i,t_j)\right]\left[\frac{\partial g(u_1,t_{j+1})}{\partial u_1}-\frac{\partial g(u_1,t_j)}{\partial u_1}\right]du_1=$$

$$\int_{a_1}^{b_1}\sum_j f_1(u_1,t_j)\left[\frac{\partial g(u_1,t_{j+1})}{\partial u_1}-\frac{\partial g(u_1,t_j)}{\partial u_1}\right]du_1,$$

$$\left|I_\Delta^2\right|\leq C\int_{a_1}^{b_1}\left\|f_1(u_1,.)\right\|_{[a_2,b_2],\alpha_2}\left\|g(u_1,.)\right\|_{[a_2,b_2],\beta_2}du_1. \tag{3.14}$$

Since $u_1\in[s_i,s_{i+1})$ we have

$$\left\|f_1(u_1,.)\right\|_{[a_2,b_2],\alpha_2}\leq\left\|f\right\|_{T,\alpha_1,\alpha_2}(u_1-s_i)^{\alpha_1}\leq\left\|f\right\|_{T,\alpha_1,\alpha_2}\left\|\Delta\right\|^{\alpha_1}, \tag{3.15}$$

$$\left\|g(u_1,.)\right\|_{[a_2,b_2],\beta_2}\leq(b_1-a_1)^{\beta_1}\left\|g\right\|_{T,\beta_1,\beta_2}+\left\|g(a_1,.)\right\|_{[a_2,b_2],\beta_2}, \tag{3.16}$$

it follows by replacing in (3.14) that

$$\left|I_{\Delta}^{2}\right| \leq C_1 \|f\|_{T,\alpha_1,\alpha_2} \|\Delta\|^{\alpha_1} \left(\|g\|_{T,\beta_1,\beta_2} + \|g(a_1,.)\|_{[a_2,b_2],\beta_2}\right). \tag{3.17}$$

Similarly

$$\left|I_{\Delta}^{3}\right| \leq C_1 \|f\|_{T,\alpha_1,\alpha_2} \|\Delta\|^{\alpha_2} \left(\|g\|_{T,\beta_1,\beta_2} + \|g(.,a_2)\|_{[a_1,b_1],\beta_1}\right). \tag{3.18}$$

Finally using (3.13)-(3.18) in (3.12) we obtain (3.2). ■

Next we define $H_{T,\alpha_1,\alpha_2,\infty}$ the space H_{T,α_1,α_2} endowed with the norm

$$\|x\|_{T,\alpha_1,\alpha_2,\infty} = \|x\|_{\infty} + \sup_{a_1 \leq t_1 \leq b_1} \|x(t_1,.)\|_{[a_2,b_2],\alpha_2} +$$

$$\sup_{a_2 \leq t_2 \leq b_2} \|x(.,t_2)\|_{[a_1,b_1],\alpha_1} + \|x\|_{T,\alpha_1,\alpha_2}.$$

The space $\left(H_{T,\alpha_1,\alpha_2,\infty}, \|.\|_{T,\alpha_1,\alpha_2,\infty}\right)$ is a Banach space.

Theorem 3.2. *Let* $T_0 = [a_1 - \varepsilon_0, b_1 + \varepsilon_0] \times [a_2 - \varepsilon_0, b_2 + \varepsilon_0]$, $\varepsilon_0 > 0$, *and let* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0,1]$ *be such that* $\alpha_i + \beta_i > 1$.

If $f \in H_{T_0,\alpha_1,\alpha_2}$, $g \in H_{T_0,\beta_1,\beta_2}$ *then there exists a unique real number* $\int_{a_1}^{b_1} \int_{a_2}^{b_2} f(u,v) dg(u,v)$ *such that for every sequence of partitions* $\Delta^n = \left(s_i^n, t_j^n\right)$, $a_1 = s_0 < ... < s_{k(n)} = b_1$, $a_2 = t_0 < ... < t_{k(n)} = b_2$, *with* $\|\Delta^n\| \to 0$, *the Riemann-Stieltjes summs*

$$S_{\Delta^n}^{f,g} = \sum_i \sum_j f(s_i^n, t_j^n) g\left(\left[s_i^n, s_{i+1}^n\right] \times \left[t_j^n, t_{j+1}^n\right]\right),$$

converge to $\int_{a_1}^{b_1} \int_{a_2}^{b_2} f(u,v) dg(u,v)$.

Moreover the following estimate holds

$$\left|\int_{a_1}^{b_1} \int_{a_2}^{b_2} f(u,v) dg(u,v)\right| \leq \tag{3.19}$$

$$\|f\|_{T,\alpha_1,\alpha_2,\infty} \|g\|_{T,\beta_1,\beta_2} (b_1 - a_1)^{\beta_1} (b_2 - a_2)^{\beta_2}.$$

Proof. It is enough to prove that for every $\delta > 0$ there exist $\eta > 0$ such that for every two partitions $(\Delta_i)_{i=1,2}$, $a_i = u_0^i < ... < u_{m(i)}^i = b_i$ with $\|\Delta_i\| < \eta$ we have

$$\left|S_{\Delta_1}^{f,g} - S_{\Delta_2}^{f,g}\right| \leq \delta.$$

Let $J \in C_c^\infty(R^2)$ be such that $J \geq 0,\ J(x) = 0$ if $\|x\| \geq 1$ and $\int_{R^2} J(x)dx = 1$ and define $J_\varepsilon(x) = \varepsilon^{-2} J(\frac{x}{\varepsilon})$. Consider the regularizations of f_ε, g_ε of f, g. Recall that

$$f_\varepsilon(x) = \int_{R^2} J_\varepsilon(x-y) f(y)dy = \int f(x-\varepsilon y)J(y)dy,$$

and g_ε similarly (as usual f, g are extended as 0 on $R^2 \setminus T_0$). It is well known that $f_\varepsilon, \to f, g_\varepsilon \to g$ uniformly on T. Also it is easily seen that $f_\varepsilon \in H_{T,\alpha_1,\alpha_2}$, $g_\varepsilon \in H_{T,\beta_1,\beta_2}$.

Next we show that if $0 < \alpha_i' < \alpha_i,\ 0 < \beta_i' < \beta_i$ then

$$f_\varepsilon \to f \text{ in } H_{T,\alpha_1',\alpha_2'},\ g_\varepsilon \to g \text{ in } H_{T,\beta_1',\beta_2'}, \tag{3.20}$$

$$f_\varepsilon(a_1,.) \to f(a_1,.) \text{ in } H_{[a_2,b_2],\alpha_2'}, g_\varepsilon(a_1,.) \to g(a_1,.) \text{ in } H_{[a_2,b_2],\beta_2'}, \tag{3.21}$$

$$f_\varepsilon(.,a_2) \to f(a_1,.) \text{ in } H_{[a_1,b_1],\alpha_1'}, g_\varepsilon(a_1,.) \to g(a_1,.) \text{ in } H_{[a_1,b_1],\beta_1'}. \tag{3.22}$$

We have

$$(f_\varepsilon - f)\left([s_1,t_1]\times[s_2,t_2]\right) =$$

$$\int_{B(0,1)} J(u,v)\left\{f([s_1-\varepsilon u, t_1-\varepsilon u]\times[s_2-\varepsilon v, t_2-\varepsilon v]) - \right.$$

$$\left. f([s_1,t_1]\times[s_2,t_2])\right\} dudv,$$

and then for every $\varepsilon, \delta > 0$,

$$\sup_{s_i \neq t_i} \frac{|(f_\varepsilon - f)\left([s_1,t_1]\times[s_2,t_2]\right)|}{|s_1-t_1|^{\alpha_1'}|s_2-t_2|^{\alpha_2'}} \leq$$

$$\sup\left\{\frac{|(f_\varepsilon - f)\left([s_1,t_1]\times[s_2,t_2]\right)|}{|s_1-t_1|^{\alpha_1'}|s_2-t_2|^{\alpha_2'}} : |s_i - t_i| > \delta,\ i = 1,2\right\} +$$

$$\sup\left\{\frac{|(f_\varepsilon - f)\left([s_1,t_1]\times[s_2,t_2]\right)|}{|s_1-t_1|^{\alpha_1'}|s_2-t_2|^{\alpha_2'}} : |s_1 - t_1| > \delta \text{ or } |s_2 - t_2| > \delta\right\} \leq$$

$$\frac{1}{\delta^{\alpha_1'+\alpha_2'}} \sup\left\{|f(u_1,v_1) - f(u_2,v_2)| : |u_i - v_i| < \varepsilon, u_i, v_i \in T_0, i = 1,2\right\} +$$

$$C\|f\|_{T,\alpha_1,\alpha_2} \max\left(\delta^{\alpha_1-\alpha_1'}, \delta^{\alpha_2-\alpha_2'}\right) \to 0 \text{ as } \varepsilon \to 0, \delta \to 0.$$

Similarly one prove (3.21), (3.22).

Next we choose $0 < \alpha_i' < \alpha_i,\ 0 < \beta_i' < \beta_i$ with $\alpha_i' + \beta_i' > 1$. Then from (3.20)-(3.22) and (3.12) we obtain

$$\left|S_{\Delta_1}^{f,g} - S_{\Delta_2}^{f,g}\right| \leq \left|S_{\Delta_1}^{f,g} - S_{\Delta_1}^{f_\varepsilon,g_\varepsilon}\right| + \left|S_{\Delta_2}^{f,g} - S_{\Delta_2}^{f_\varepsilon,g_\varepsilon}\right| +$$

$$\left|S_{\Delta_1}^{f_\varepsilon,g_\varepsilon} - \int_{a_1}^{b_1}\int_{a_2}^{b_2} f_\varepsilon)dg_\varepsilon\right| + \left|S_{\Delta_2}^{f_\varepsilon,g_\varepsilon} - \int_{a_1}^{b_1}\int_{a_2}^{b_2} f_\varepsilon dg_\varepsilon\right| \leq$$

$$\left|S_{\Delta_1}^{f,g} - S_{\Delta_1}^{f_\varepsilon,g_\varepsilon}\right| + \left|S_{\Delta_2}^{f,g} - S_{\Delta_2}^{f_\varepsilon,g_\varepsilon}\right| +$$

$$C\left(\|f_\varepsilon\|_{T,\alpha_1',\alpha_2'}, \|g_\varepsilon\|_{T,\beta_1',\beta_2'}\right)\left\{(\|\Delta_1\| + \|\Delta_2\|)^{\alpha_1'+\beta_1'+\alpha_2'+\beta_2'-2} +\right.$$

$$\left.(\|\Delta_1\| + \|\Delta_2\|)^{\alpha_1'} + (\|\Delta_1\| + \|\Delta_2\|)^{\alpha_2'}\right\} \leq$$

$$\left|S_{\Delta_1}^{f,g} - S_{\Delta_1}^{f_\varepsilon,g_\varepsilon}\right| + \left|S_{\Delta_2}^{f,g} - S_{\Delta_2}^{f_\varepsilon,g_\varepsilon}\right| +$$

$$C_1\left\{(\|\Delta_1\| + \|\Delta_2\|)^{\alpha_1'+\beta_1'+\alpha_2'+\beta_2'-2} +\right.$$

$$\left.(\|\Delta_1\| + \|\Delta_2\|)^{\alpha_1'} + (\|\Delta_1\| + \|\Delta_2\|)^{\alpha_2'}\right\} \to 0,$$

as $\varepsilon \to 0$ and than $\|\Delta_i\| \to 0$.

The previous computation also shows that

$$\lim_{\varepsilon\to 0}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f_\varepsilon dg_\varepsilon = \int_{a_1}^{b_1}\int_{a_2}^{b_2} f dg,$$

and this fact and (2.3) imply (3.19). ■

Next for $K > 0$ we define the closed sets

$$H_{[a,b],\gamma}(K) = \left\{\varphi \in H_{[a,b],\gamma} : \|\varphi\|_{[a,b],\gamma} \leq K\right\},$$

anf for $\varphi_i \in H_{[a_i,b_i],\alpha_i}$,

$$H_{T,\alpha_1,\alpha_2,\infty}(K,\varphi_1,\varphi_2) =$$

$$\left\{x \in H_{T,\alpha_1,\alpha_2,\infty} : x(a_1,.) = \varphi_1, x(.,a_2) = \varphi_2, \|x\|_{T,\alpha_1,\alpha_2} \leq K,\right.$$

$$\left.\sup_{a_1\leq t_1\leq b_1}\|x(t_1,.)\|_{[a_2,b_2],\alpha_2} \leq K, \sup_{a_2\leq t_2\leq b_2}\|x(.,t_2)\|_{[a_1,b_1],\alpha_1} \leq K\right\}.$$

Proposition 3.3. *Let $\beta_1, \beta_2 \in \left(\frac{1}{2}, 1\right]$ and α_1, α_2 be such that $\beta_i > \alpha_i > 1-\beta_i$. Let $g \in H_{R^2,\beta_1,\beta_2}$ and $b, \sigma : R \to R$ be such that b is bounded and Lipschitz and $\sigma \in C_b^2(R)$ with σ'' Lipschitz. Then for every $K > 0$ and $a_i, b_i \in R$, $a_i < b_i$, $i = 1, 2$, there esists $\varepsilon_0 > 0$ independent of a_i, b_i, such that for every $\varphi_i \in H_{[a_i,a_i+\varepsilon_0],\alpha_i}(K)$ the operator*

$$F : H_{D_\varepsilon,\alpha_1,\alpha_2,\infty}(2K, \varphi_1, \varphi_2) \to H_{D_\varepsilon,\alpha_1,\alpha_2,\infty}(2K, \varphi_1, \varphi_2),$$

defined by

$$(Fx)_{s,t} = \varphi_1(s) + \varphi_2(t) + \int_{a_1}^{s}\int_{a_2}^{t} b(x_{u,v})dudv + \int_{a_1}^{s}\int_{a_2}^{t} \sigma(x_{u,v})dg(u, v),$$

is a contraction, where $D_\varepsilon = [a_1, a_1 + \varepsilon] \times [a_2, a_2 + \varepsilon]$, $\varepsilon > 0$.

Proof. Clearly we have

$$\left\| \int_{a_1}^{\cdot}\int_{a_2}^{\cdot} b(x_{u,v})dudv \right\|_{T,\alpha_1,\alpha_2,\infty} \leq \tag{3.23}$$

$$\|b\|_\infty (b_1 - a_1)^{1-\alpha_1} (b_2 - a_2)^{1-\alpha_2} \left[(b_1 - a_1)^{\alpha_1} (b_2 - a_2)^{\alpha_2} + 1\right].$$

By using (3.19) it follows

$$\left\| \int_{a_1}^{\cdot}\int_{a_2}^{\cdot} \sigma(x_{u,v})dg(u, v) \right\|_{T,\alpha_1,\alpha_2,\infty} \leq \tag{3.24}$$

$$\|\sigma(x)\|_{T,\alpha_1,\alpha_2,\infty} \|g\|_{T,\beta_1,\beta_2} (b_1 - a_1)^{\beta_1-\alpha_1} (b_2 - a_2)^{\beta_2-\alpha_2} \times$$

$$\times \left[(b_1 - a_1)^{\alpha_1} (b_2 - a_2)^{\alpha_2} + 1\right].$$

Next

$$\sigma(x) \left([s_1, t_1] \times [s_2, t_2]\right) = \tag{3.25}$$

$$(x_{t_1,t_2} - x_{t_1,s_2}) \int_0^1 \sigma' \left(\lambda x_{t_1,t_2} + (1 - \lambda)x_{t_1,s_2}\right) d\lambda -$$

$$(x_{s_1,t_2} - x_{s_1,s_2}) \int_0^1 \sigma' \left(\lambda x_{s_1,t_2} + (1 - \lambda)x_{s_1,s_2}\right) d\lambda =$$

$$(x_{t_1,t_2} - x_{t_1,s_2} - x_{s_1,t_2} + x_{s_1,s_2}) \int_0^1 \sigma' \left(\lambda x_{t_1,t_2} + (1 - \lambda)x_{t_1,s_2}\right) d\lambda +$$

$$(x_{s_1,t_2} - x_{s_1,s_2}) \int_0^1 [\sigma' (\lambda x_{t_1,t_2} + (1-\lambda) x_{t_1,s_2})$$

$$-\sigma' (\lambda x_{s_1,t_2} + (1-\lambda) x_{s_1,s_2}) \, d\lambda.]$$

Then (3.25) implies

$$|\sigma(x) ([s_1,t_1] \times [s_2,t_2])| \leq$$

$$\Big\{ \|\sigma'\|_\infty \|x\|_{T,\alpha_1,\alpha_2} + \|\sigma'\|_L \|x(s_1,.)\|_{[a_2,b_2],\alpha_2} \times$$

$$\times \int_0^1 \Big[\lambda \|x(.,t_2)\|_{[a_1,b_1],\alpha_1} + (1-\lambda)) \|x(.,s_2)\|_{[a_1,b_1],\alpha_1} \Big] \, d\lambda \Big\} \times$$

$$\times (t_1 - s_1)^{\alpha_1} (t_2 - s_2)^{\alpha_1},$$

and hence if $x \in H_{T,\alpha_1,\alpha_2,\infty}(K, \varphi_1, \varphi_2)$ then

$$\|\sigma(x)\|_{T,\alpha_1,\alpha_2} \leq K (\|\sigma'\|_\infty + \|\sigma'\|_L) . \tag{3.26}$$

From (3.23), (3.24) and (3.26) it follows that $Fx \in H_{[a_1,b_1]\times[a_1,b_1],\alpha_1,\alpha_2,\infty}$ if $x \in H_{[a_1,b_1]\times[a_1,b_1],\alpha_1,\alpha_2,\infty}$ and also for $\varepsilon_1 > 0$ enough small we have $Fx \in H_{D_{\varepsilon_1},\alpha_1,\alpha_2,\infty}(2K, \varphi_1, \varphi_2)$ if $x \in H_{D_{\varepsilon_1},\alpha_1,\alpha_2,\infty}(2K, \varphi_1, \varphi_2)$.

Next we have

$$[\sigma(x) - \sigma(y)] ([s_1,t_1] \times [s_2,t_2]) =$$

$$(x_{t_1,t_2} - y_{t_1,t_2}) \int_0^1 \sigma' (\lambda x_{t_1,t_2} + (1-\lambda) y_{t_1,t_2}) \, d\lambda -$$

$$(x_{t_1,s_2} - y_{t_1,s_2}) \int_0^1 \sigma' (\lambda x_{t_1,s_2} + (1-\lambda) y_{t_1,s_2}) \, d\lambda -$$

$$(x_{s_1,t_2} - y_{s_1,t_2}) \int_0^1 \sigma' (\lambda x_{s_1,t_2} + (1-\lambda) y_{s_1,t_2}) \, d\lambda +$$

$$(x_{s_1,s_2} - y_{s_1,s_2}) \int_0^1 \sigma' (\lambda x_{s_1,s_2} + (1-\lambda) y_{s_1,s_2}) \, d\lambda =$$

$$(x - y) ([s_1,t_1] \times [s_2,t_2]) \int_0^1 \sigma' (\lambda x_{t_1,t_2} + (1-\lambda) y_{t_1,t_2}) \, d\lambda +$$

$$(x_{t_1,s_2} - y_{t_1,s_2}) \int_0^1 [\sigma' (\lambda x_{t_1,t_2} + (1-\lambda) y_{t_1,t_2}) -$$

$$\sigma' (\lambda x_{t_1,s_2} + (1-\lambda) y_{t_1,s_2}) \, d\lambda] + (x_{s_1,t_2} - y_{s_1,t_2}) \times$$

$$\times \int_0^1 \left[\sigma'\left(\lambda x_{t_1,t_2} + (1-\lambda) y_{t_1,t_2}\right) - \sigma'\left(\lambda x_{s_1 t_2} + (1-\lambda) y_{s_1,t_2}\right)\right] d\lambda -$$

$$\left(x_{s_1,s_2} - y_{s_1,s_2}\right) \int_0^1 \left[\sigma'\left(\lambda x_{t_1,t_2} + (1-\lambda) y_{t_1,t_2}\right) -\right.$$

$$\left.\sigma'\left(\lambda x_{s_1,s_2} + (1-\lambda) y_{s_1,s_2}\right) d\lambda\right] =$$

$$(x-y)\left([s_1,t_1] \times [s_2,t_2]\right) \int_0^1 \sigma'\left(\lambda x_{t_1,t_2} + (1-\lambda) y_{t_1,t_2}\right) d\lambda +$$

$$\left[\left(x_{s_1,t_2} - y_{s_1,t_2}\right) - \left(x_{s_1,s_2} - y_{s_1,s_2}\right)\right] \times$$

$$\times \int_0^1 \left[\sigma'\left(\lambda x_{t_1,t_2} + (1-\lambda) y_{t_1,t_2}\right) - \sigma'\left(\lambda x_{s_1,t_2} + (1-\lambda) y_{s_1,t_2}\right)\right] d\lambda +$$

$$\left(x_{s_1,s_2} - y_{s_1,s_2}\right) \int_0^1 \left[\sigma'\left(\lambda x_{t_1,t_2} + (1-\lambda) y_{t_1,t_2}\right) - \sigma'\left(\lambda x_{s_1,t_2} + (1-\lambda) y_{s_1,t_2}\right) +\right.$$

$$\left. + \sigma'\left(\lambda x_{s_1,s_2} + (1-\lambda) y_{s_1,s_2}\right) - \sigma'\left(\lambda x_{t_1,s_2} + (1-\lambda) y_{t_1,s_2}\right)\right] d\lambda =$$

$$(x-y)\left([s_1,t_1] \times [s_2,t_2]\right) \int_0^1 \sigma'\left(\lambda x_{t_1,t_2} + (1-\lambda) y_{t_1,t_2}\right) d\lambda +$$

$$\left[\left(x_{s_1,t_2} - y_{s_1,t_2}\right) - \left(x_{s_1,s_2} - y_{s_1,s_2}\right)\right] \times$$

$$\times \int_0^1 \left[\sigma'\left(\lambda x_{t_1,t_2} + (1-\lambda) y_{t_1,t_2}\right) - \sigma'\left(\lambda x_{s_1,t_2} + (1-\lambda) y_{s_1,t_2}\right)\right] d\lambda +$$

$$\left(x_{s_1,s_2} - y_{s_1,s_2}\right) \int_0^1 \left[\lambda\left(x_{t_1,t_2} - x_{s_1,t_2}\right) + (1-\lambda)\left(y_{t_1,t_2} - y_{s_1,t_2}\right)\right] \times$$

$$\times \int_0^1 \sigma''\left(\mu\left(\lambda x_{t_1,t_2} + (1-\lambda) y_{t_1,t_2}\right) + (1-\mu)\left(\lambda x_{s_1,t_2} + (1-\lambda) y_{s_1,t_2}\right)\right) d\mu d\lambda -$$

$$(x_{s_1,s_2}-y_{s_1,s_2})\int_0^1\left[\lambda\left(x_{t_1,s_2}-x_{s_1,s_2}\right)+(1-\lambda)\left(y_{t_1,s_2}-y_{s_1,s_2}\right)\right]\times$$

$$\times\int_0^1\sigma''\left(\mu\left(\lambda x_{t_1,s_2}+(1-\lambda)y_{t_1,s_2}\right)+(1-\mu)\left(\lambda x_{s_1,s_2}+(1-\lambda)y_{s_1,s_2}\right)\right)d\mu d\lambda.$$

Therefore

$$[\sigma(x)-\sigma(y)]\left([s_1,t_1]\times[s_2,t_2]\right)= \tag{3.27}$$

$$(x-y)\left([s_1,t_1]\times[s_2,t_2]\right)\int_0^1\sigma'\left(\lambda x_{t_1,t_2}+(1-\lambda)y_{t_1,t_2}\right)d\lambda+$$

$$\left[(x_{s_1,t_2}-y_{s_1,t_2})-(x_{s_1,s_2}-y_{s_1,s_2})\right]\times$$

$$\times\int_0^1\left[\sigma'\left(\lambda x_{t_1,t_2}+(1-\lambda)y_{t_1,t_2}\right)-\sigma'\left(\lambda x_{s_1,t_2}+(1-\lambda)y_{s_1,t_2}\right)\right]d\lambda+$$

$$(x_{s_1,s_2}-y_{s_1,s_2})\int_0^1\left[\lambda x\left([s_1,t_1]\times[s_2,t_2]\right)+(1-\lambda)y\left([s_1,t_1]\times[s_2,t_2]\right)\right]\times$$

$$\times\int_0^1\sigma''\left(\mu\left(\lambda x_{t_1,t_2}+(1-\lambda)y_{t_1,t_2}\right)+(1-\mu)\left(\lambda x_{s_1,t_2}+(1-\lambda)y_{s_1,t_2}\right)\right)d\mu d\lambda+$$

$$(x_{s_1,s_2}-y_{s_1,s_2})\int_0^1\left[\lambda\left(x_{t_1,s_2}-x_{s_1,s_2}\right)+(1-\lambda)\left(y_{t_1,s_2}-y_{s_1,s_2}\right)\right]\times$$

$$\times\int_0^1\left[\sigma''\left(\mu\left(\lambda x_{t_1,t_2}+(1-\lambda)y_{t_1,t_2}\right)+(1-\mu)\left(\lambda x_{s_1,t_2}+(1-\lambda)y_{s_1,t_2}\right)\right)-\right.$$

$$\left.\sigma''\left(\mu\left(\lambda x_{t_1,s_2}+(1-\lambda)y_{t_1,s_2}\right)+(1-\mu)\left(\lambda x_{s_1,s_2}+(1-\lambda)y_{s_1,s_2}\right)\right)d\mu d\lambda\right].$$

If $x,y\in H_{D_{\varepsilon_1},\alpha_1,\alpha_2,\infty}(K,\varphi_1,\varphi_2)$ then (3.27) yields

$$\|\sigma(x)-\sigma(y)\|_{T,\alpha_1,\alpha_2}\le C(K,\|\sigma'\|_\infty,\|\sigma'\|_L,\left\|\sigma''\right\|_L)\,\|x-y\|_{T,\alpha_1,\alpha_2}. \tag{3.28}$$

From (3.23), (3.24) and (3.28) it follows that there exists $\varepsilon_2 > 0$ enough small, independent of a_i, b_i, such that

$$\|Fx - Fy\|_{D_{\varepsilon_2},\alpha_1,\alpha_2,\infty} \le d\,\|x - y\|_{D_{\varepsilon_2},\alpha_1,\alpha_2,\infty}, \tag{3.29}$$

for some $0 < d < 1$, and hence, denoting $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$, we obtain that

$$F : H_{D_{\varepsilon_0},\alpha_1,\alpha_2,\infty}(2K, \varphi_1, \varphi_2) \to H_{D_{\varepsilon_0},\alpha_1,\alpha_2,\infty}(2K, \varphi_1, \varphi_2),$$

is a contraction. ■

Theorem 3.4. *Let $\beta_1, \beta_2 \in \left(\frac{1}{2}, 1\right]$ and α_1, α_2 be such that $\beta_i > \alpha_i > 1 - \beta_i$. Let $g \in H_{R^2,\beta_1,\beta_2}$ and $b, \sigma : R \to R$ be such that b is bounded and Lipschitz and $\sigma \in C_b^2(R)$ with σ'' Lipschitz.*

Then for every $a_1 < b_1, a_2 < b_2$ and $\varphi_i \in H_{[a_i,b_i],\alpha_i}$ with $\varphi_1(a_1) = \varphi_2(a_2)$, the equation

$$x_{s,t} = \varphi_1(s) + \varphi_2(t) - \varphi_1(a_1) + \int_{a_1}^{s}\int_{a_2}^{t} b(x_{u,v})du dv + \int_{a_1}^{s}\int_{a_2}^{t} \sigma(x_{u,v})dg(u,v),\ (s,t) \in T, \tag{3.30}$$

has a unique solution in $H_{T,\alpha_1,\alpha_2,\infty}$.

Proof. Let $K > 0$ be such that $\varphi_i \in H_{[a_i,b_i],\alpha_i}(K)$. Then from Proposition 3.3 we obtain the existence of the solution x of (3.30) on the rectangle $[a_1, a_1 + \varepsilon_0] \times [a_2, a_2 + \varepsilon_0]$, ε_0 independent of a_i, b_i (but dependent of K). If $a_1 + \varepsilon_0 < b_1$, let n_0 be the bigest integer such that $n_0\varepsilon < b_1$. Then $x \in H_{T,\alpha_1,\alpha_2,\infty}(2K)$ and inductively we obtain the existence of the solution on

$$[a_1 + \varepsilon_0, a_1 + 2\varepsilon_0] \times [a_2, a_2 + \varepsilon_0], ..., [a_1 + n_0\varepsilon_0, b_1] \times [a_2, a_2 + \varepsilon_0],$$

and then on

$$[a_1, a_1 + \varepsilon_0] \times [a_2 + \varepsilon_0, a_2 + 2\varepsilon_0], ..., [a_1 + n_0\varepsilon_0, b_1] \times [a_2 + \varepsilon_0, a_2 + 2\varepsilon_0],$$

and continuing again by induction we obtain the existence on T.

Let now $x_1,\ x_2$ be two solutions of (3.30). In particular there is $K > 0$ such that $x_1,\ x_2 \in H_{T,\alpha_1,\alpha_2,\infty}(K)$. From (3.29) we deduce the existence of a $\varepsilon_0 > 0$ (which does not depend on $a_i,\ b_i$) and $0 < d < 1$ such that

$$\|x_1 - x_2\|_{[a_1,a_1+\varepsilon_0]\times[a_2,a_2+\varepsilon_0]} \le d\,\|x_1 - x_2\|_{[a_1,a_1+\varepsilon_0]\times[a_2,a_2+\varepsilon_0]},$$

and therefore $x_1 = x_2$ on $[a_1, a_1 + \varepsilon_0] \times [a_2, a_2 + \varepsilon_0]$. Inductively (see the existence part) we obtain that $x_1 = x_2$ on T. ■

On a probability space $(\Omega, \mathcal{F}, P)$ we consider a two-parameter fractional Brownian motion $(B_t^\gamma)_{t\in[0,1]^2}$, with the Hurst parameter $\gamma = (\gamma_1, \gamma_2)$, $\gamma_i \in \left(\frac{1}{2}, 1\right)$, i.e., B^γ is a continuous centered Gaussian process vanishing on the axes and with the covariance function

$$R_\gamma^{(2)}(s,t) = R_{\gamma_1}^{(1)}(s_1,t_1) R_{\gamma_2}^{(1)}(s_2,t_2), \tag{3.31}$$

if $s = (s_1, s_2)$, $t = (t_1, t_2)$, where for $\gamma \in (0,1)$,

$$R_\gamma^{(1)}(u,v) = \frac{1}{2}\left(|u|^{2\gamma} + |v|^{2\gamma} - |u-v|^{2\gamma}\right), \ u, v \geq 0. \tag{3.32}$$

Theorem 3.5. *Let $(B_t^\gamma)_{t\in[0,1]^2}$ be as above and let α_i, $\beta_i > 0$ be such that $\frac{1}{2} < \beta_i < \gamma_i$, $\beta_i > \alpha_i > 1 - \beta_i$, $i = 1, 2$. Let $b, \sigma : R \to R$ be such that b is bounded and Lipschitz and $\sigma \in C_b^2(R)$ with σ'' Lipschitz and let be the processes $\{\varphi_i(t)\}_{t\in[0,1]}$ such that almost surely $\varphi_1(0) = \varphi_2(0)$ and $\varphi_i \in H_{[0,1],\alpha_i}$.*

Then with probability one the stochastic equation

$$X_{s,t} = \varphi_1(s) + \varphi_2(t) - \varphi_1(0) + \int_0^s \int_0^t b(X_{u,v})du dv + \int_0^s \int_0^t \sigma(X_{u,v}) dB_{u,v}^\gamma, \ (s,t) \in [0,1]^2, \tag{3.33}$$

has a unique solution $\{X_{u,v}\}_{(u,v)\in[0,1]^2}$ with the paths in $H_{[0,1]^2,\alpha_1,\alpha_2,\infty}$.

Proof. From the Kolmogorov criterium (see [1], [4]) it follows that B^γ has β-Hölder paths, i.e., there exists a random variable C such that for all $\omega \in \Omega$,

$$\left|\left(B^\gamma\left(\left[t^{(1)}, t^{(2)}\right)\right)\right)(\omega)\right| \leq C(\omega) \prod_{j=1}^{2} (t_j^{(2)} - t_j^{(1)})^{\beta_j}. \tag{3.34}$$

Therefore almost surely we have by Theorem 3.2 and (3.34) that the Stieltjes integral $\int_0^s \int_0^t f(u,v) dB_{u,v}^\gamma$ is well defined for $f \in H_{[0,1]^2,\alpha_1,\alpha_2}$.

Now the result is a consequence of Theorem 3.4 applied pointwise. ■

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