

**A NEW FAMILY OF LIFE DISTRIBUTIONS
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DISTRIBUTION**

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A New Family of Life Distributions Based on Birnbaum-Saunders Distribution

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Abstract

The Birnbaum-Saunders distribution was originally developed to model the rupture time of metals exposed to fatigue. This article presents an extension of this distribution, generated from an Elliptically Contoured distribution, in which the density and some of its properties are obtained. Explicit expressions for the density are found for a large number of specific Elliptic distributions, such as Pearson Type VII, t , Cauchy, Kotz Type, Normal, Bessel, Laplace and Logistic.

Keywords: Birnbaum-Saunders distribution, life distributions, material fatigue, reliability analysis, Elliptic distributions.

AMS 2000 Subject Classification: Primary 62E15
Secondary 62N05

1 Introduction

Fatigue is a class of structural damage that occurs when a material is exposed to fluctuating stress and tension. The first such problem was identified in the axles of carriages and wagons in the early 1800s [21]. Fatigue was also responsible for aeroplane crashes affecting British cargo planes in the 1950s [5], and has been observed in many other cases.

Statistical models for processes of fatigue enable us to describe the random variation of failure times associated with materials exposed to fatigue as a result of different patterns of cyclic forces. Such materials can be characterised by the values presented by the parameters of associated empirical laws. These characterisations are important in forecasting the behavior of vulnerable materials under different conditions. For example, we are often interested in predicting the time elapsing before fatigue occurs under various low levels of stress. Nevertheless, performing a life fatigue test under stress conditions requires a great deal of time. To avoid this problem, we can observe the failure times of materials at high levels of stress and then use the empirical characterisations of

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material properties to predict the corresponding times for lower levels of stress. This type of test is termed an accelerated life test.

The Birnbaum-Saunders distribution (see [3] and [4]) was derived from a model that showed how material failure is due to the development and growth of a dominant crack. Desmond (see [8] and [9]) showed that the Birnbaum-Saunders distribution provides a total description of the failure T that occurs when an accumulated damage $D(t)$ exceeds a threshold value, such that $T = \inf\{t : D(t) > \text{threshold}\}$. Although this distribution is known as Birnbaum-Saunders, it was in fact obtained previously by Freudental and Shinokuza (see [15], cited in [16, p. 654]).

The Birnbaum-Saunders distribution is the distribution of

$$S = \beta \left[\frac{\alpha}{2} V + \sqrt{\left(\frac{\alpha}{2} V\right)^2 + 1} \right]^2,$$

where $V \sim N(0, 1)$, $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter and the median of the distribution. We shall use the notation $S \sim BS(\alpha, \beta)$. The density of $S \sim BS(\alpha, \beta)$ is

$$f_S(s) = \frac{\exp(\alpha^{-2})}{2\alpha(2\pi\beta)^{1/2}} s^{-3/2} (s + \beta) \exp\left[-\frac{1}{2\alpha^2} \left(\frac{s}{\beta} + \frac{\beta}{s}\right)\right], \quad s > 0.$$

It is straightforward to show that if S fits a Birnbaum-Saunders distribution, then

$$V = \alpha^{-1} \left[\sqrt{\frac{S}{\beta}} - \sqrt{\frac{\beta}{S}} \right] \sim N(0, 1). \quad (1)$$

Recent proposals have discussed a family of multivariate distributions whose density contours have the same elliptic shape as that of the Normal distribution. These, however, also include distributions with tails that are weighted more and less than those of the Normal distribution. Moreover, the Normal distribution is a particular case within this family. Such distributions are termed Elliptic Contour or simply Elliptic distributions.

Elliptic distributions have been studied by many authors, such as [17], [7] and [6]. Although the use of these distributions began in the 1970s, pioneering studies were made by such as [20] and [19]. At present, a large body of Normal theory is being reconstructed using elliptic distributions (see, for example, the books [1], [12], [13], and [14], and the more recent studies [2], [18], [10] and [11]).

For the case of a random variable (one-dimensional case), Elliptic distributions correspond to all the symmetric distributions in \mathbb{R} . Specifically, a random variable X fits an elliptic distribution if its characteristic function is

$$\psi_X(t) = \exp(it\mu)\phi(t^2\sigma^2), \quad (2)$$

with $\phi : \mathbb{R} \rightarrow \mathbb{R}$; or if the density of X is given by

$$f_X(x) = c g \left[\frac{(x - \mu)^2}{\sigma^2} \right]; \quad x \in \mathbb{R}, \quad (3)$$

where $g(u)$, with $u > 0$, is a real function and corresponds to the kernel of the density of X and c is the normalisation constant such that $f_X(x)$ is a density. This is denoted as $X \sim EC(\mu, \sigma^2; \phi)$ or $X \sim EC(\mu, \sigma^2; g)$, respectively. In general, μ is the position parameter, and coincides with the mean if the first moment of the distribution exists. σ^2 is the scale parameter, the variance of which, if the first two moments of the distribution exist, is $\sigma_0^2 = c_0\sigma^2$, where $c_0 = -2\phi'(0)$, and ϕ' is the derivative of ϕ , given in (2). Some specific Elliptic distributions are presented in Table 1.

Table 1: Explicit forms of the kernels and normalisation constants of the densities of the Elliptic distributions indicated, following (3).

Elliptic law	Notation ($\mu \in \mathbb{R}$ and $\sigma^2 > 0$)	Constant (c)	Kernel ($g(\cdot)$)
Pearson VII	$P_{VII}(\mu, \sigma^2; q, r)$, $r > 0$ and $q > 1/2$	$\frac{\Gamma(q)}{(r\pi)^{1/2}\Gamma(q-1/2)\sigma}$	$\left(1 + \frac{(x-\mu)^2}{r\sigma^2}\right)^{-q}$
Type Kotz	$K(\mu, \sigma^2; q, r, s)$ $r, s > 0$ and $q > 1/2$	$\frac{sr^{\frac{2q-1}{2s}}}{\Gamma\left(\frac{2q-1}{2s}\right)\sigma}$	$\frac{\left(\frac{(x-\mu)^2}{\sigma^2}\right)^{q-1}}{\exp\left(r\left[\frac{(x-\mu)^2}{\sigma^2}\right]^s\right)}$
Bessel ^a	$B(\mu, \sigma^2, r, q)$ $r > 0$ and $q > -1/2$	$\frac{1}{2^q r^{q+1} \pi^{1/2} \Gamma(q+1/2) \sigma}$	$\frac{\left(\frac{(x-\mu)^2}{\sigma^2}\right)^{q/2}}{\left(K_q\left(\frac{1}{r}\left(\frac{(x-\mu)^2}{\sigma^2}\right)^{1/2}\right)\right)^{-1}}$
Special Case	$SC(\mu, \sigma^2)$	$\frac{2^{1/2}}{\pi\sigma}$	$\frac{1}{\left(1 + \left(\frac{x}{\sigma}\right)^4\right)}$
Logistic	$Log(\mu, \sigma^2)$	$\frac{1}{\sigma \int_0^\infty z^{-1/2} \frac{\exp(-z)}{1 + \exp(-z)} dz}$	$\frac{\exp\left(-\frac{(x-\mu)^2}{\sigma^2}\right)}{\left(1 + \exp\left(-\frac{(x-\mu)^2}{\sigma^2}\right)\right)^2}$

^aWhere

$$K_q(z) = \frac{\pi}{2} \frac{I_{-q}(z) - I_q(z)}{\sin(q\pi)}, \quad |\arg(z)| < \pi,$$

with q an integer, this is the third-class modified Bessel function and

$$I_q(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k+q+1)} \left(\frac{z}{2}\right)^{q+2k}, \quad |z| < \infty, \quad |\arg(z)| < \pi.$$

Remark 1

- i) The t distribution (denoted by $t(\nu)$), where ν represents its degrees of freedom, is a particular case of the Pearson Type VII distribution when $q = (\nu + 1)/2$.*
- ii) The Cauchy distribution is a particular case of the $t(\nu)$ distribution when $\nu = 1$. An important characteristic of this Elliptic distribution is that it has no moments.*
- iii) The distribution termed "Special Case" is cited in [14, p.70] and also has no moments.*
- iv) The Normal distribution is a particular case of the Kotz Type distribution when $q = s = 1$ and $r = 1/2$.*
- iv) The Laplace distribution is a particular case of the Bessel distribution when $q = 0$ and $r = \sigma/\sqrt{2}$.*

Strictly speaking, the expression given in (1) is fulfilled only asymptotically (see Equation 2.3 in [3]). In the same source, (1) is taken as an equality, based on a heuristic, non-mathematical argument, based specifically on a physical support.

In this article, also on the basis of a heuristic argument, (1) is replaced by the supposition

$$U = \alpha^{-1} \left[\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}} \right] \sim EC(0, 1; g), \quad \alpha, \beta > 0, \quad (4)$$

in order to determine the distribution of T . This supposition is based on the search for faster-growing life distributions, with a greater or lesser kurtosis and/or with left tails that are more or less weighted than those of the Birnbaum-Saunders distribution, based on the Normal distribution, among other interesting properties such as the absence of moments in the life distribution. By these means, it is possible to generalise the Birnbaum-Saunders distribution on the basis of an Elliptic distribution. We find the density and other interesting properties of the generalised Birnbaum-Saunders distribution, as well as explicit expressions for the density of a large number of specific Elliptic distributions, including Pearson Type VII, t , Cauchy, Kotz Type, Normal, Bessel, Laplace, Special Case and Logistic distributions. Finally, for the sake of illustration, we include some figures showing these densities.

2 A New Family of Life Distributions

This section describes how the density of a Birnbaum-Saunders distribution is obtained on the basis of an Elliptic distribution.

From (4), we find that

$$T = \beta \left[\frac{\alpha}{2}U + \sqrt{\left(\frac{\alpha}{2}U\right)^2 + 1} \right]^2,$$

fits a generalised Birnbaum-Saunders distribution, denoted as $T \sim GBS(\alpha, \beta; g)$.

Theorem 1 Let $T \sim GBS(\alpha, \beta; g)$. Then, the density of T is given by

$$f_T(t) = \frac{c}{2\alpha\beta^{1/2}} t^{-3/2}(t+\beta) g\left(\frac{1}{\alpha^2} \left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right), \quad t > 0,$$

where c is given in (3).

Proof. Let $f_U(u) = c h(u^2)$. Taking $u = \frac{1}{\alpha} \left[\sqrt{\frac{t}{\beta}} + \sqrt{\frac{\beta}{t}} \right]$, for which the Jacobian is $\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}$, we obtain

$$\begin{aligned} f_T(t) &= f_U\left(\frac{1}{\alpha} \left[\sqrt{\frac{t}{\beta}} + \sqrt{\frac{\beta}{t}} \right]\right) \left| \frac{du}{dt} \right| \\ &= c h\left(\frac{1}{\alpha^2} \left[\sqrt{\frac{t}{\beta}} + \sqrt{\frac{\beta}{t}} \right]^2\right) \frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}} \\ &= \frac{c}{2\alpha\beta^{1/2}} t^{-3/2}(t+\beta) g\left(\frac{1}{\alpha^2} \left[\frac{t}{\beta} + \frac{\beta}{t} - 2 \right]\right). \end{aligned}$$

■

The following are explicit expressions for the density of a generalised Birnbaum-Saunders distribution obtained from Pearson Type VII, t , Cauchy, Kotz Type, Normal, Bessel, Laplace, Special Case and Logistic distributions, the proofs of which are immediate from Theorem 1. These are accompanied by figures showing these densities (see Appendix A).

Corollary 1 (Pearson Type VII distribution) Let $T \sim GBS(\alpha, \beta; g)$, with $g(\cdot)$ as given in Table 1. Then, the density of T is given by

$$f_T(t) = \frac{\Gamma(q)}{2\alpha(r\pi\beta)^{1/2}\Gamma(q-1/2)} t^{-3/2}(t+\beta) \left(1 + \frac{1}{r\alpha^2} \left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right)^{-q},$$

with $t, \alpha, \beta, r > 0$ and $q > 1/2$.

Corollary 2 (t distribution) Let $T \sim GBS(\alpha, \beta; g)$, with $g(\cdot)$ as given implicitly in Table 1 (see Remark 1 i)). Then, the density of T is given by

$$f_T(t) = \frac{\Gamma((\nu+1)/2)}{2\alpha(\nu\pi\beta)^{1/2}\Gamma(\nu/2)} t^{-3/2}(t+\beta) \left(1 + \frac{1}{\nu\alpha^2} \left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right)^{-\frac{\nu+1}{2}},$$

with $t, \alpha, \beta, \nu > 0$ and $q > 1/2$.

Corollary 3 (Cauchy distribution) Let $T \sim GBS(\alpha, \beta; g)$, with $g(\cdot)$ as given implicitly in Table 1 (see Remark 1 ii)). Then, the density of T is given by

$$f_T(t) = \frac{1}{2\pi\alpha\beta^{1/2}} t^{-3/2}(t+\beta) \left(1 + \frac{1}{\alpha^2} \left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right)^{-1},$$

with $t, \alpha, \beta > 0$.

Corollary 4 (Special Case distribution) Let $T \sim GBS(\alpha, \beta; g)$, with $g(\cdot)$ as given in Table 1.

Then, the density of T is given by

$$f_T(t) = \frac{t^{-3/2} (t + \beta)}{\pi\alpha(2\beta)^{1/2}} \left(1 + \frac{1}{\alpha^4} \left[\frac{t}{\beta} + \frac{\beta}{t} - 2 \right]^2 \right)^{-1},$$

with $t, \alpha, \beta > 0$.

Corollary 5 (Kotz Type distribution) Let $T \sim GBS(\alpha, \beta; g)$, with $g(\cdot)$ as given in Table 1.

Then, the density of T is given by

$$f_T(t) = \frac{sr^{(2q-1)/(2s)} t^{-3/2}(t + \beta)}{2\alpha^{2q-1}\beta^{1/2}\Gamma\left(\frac{2q-1}{2s}\right)} \left(\frac{t}{\beta} + \frac{\beta}{t} - 2 \right)^{q-1} \exp\left(-\frac{r}{\alpha^{2s}} \left[\frac{t}{\beta} + \frac{\beta}{t} - 2 \right]^s\right),$$

with $t, \alpha, \beta, r, s > 0$ and $q > 1/2$.

Corollary 6 (Bessel distribution) Let $T \sim GBS(\alpha, \beta; g)$, with $g(\cdot)$ as given in Table 1. Then,

the density of T is given by

$$f_T(t) = \frac{\left(\frac{t}{\beta} + \frac{\beta}{t} - 2 \right)^{q/2} t^{-3/2}(t + \beta)}{(2\alpha)^{q+1}(\pi\beta)^{1/2}\Gamma(q + 1/2)r^{q+1}} K_q \left(\frac{1}{r\alpha^2} \left[\frac{t}{\beta} + \frac{\beta}{t} - 2 \right]^{1/2} \right),$$

where $K_q(z)$ is given in Table (1) and with $t, \alpha, \beta, r > 0$ and $q > -1/2$.

Corollary 7 (Logistic distribution) Let $T \sim GBS(\alpha, \beta; g)$, with $g(\cdot)$ as given in Table 1. Then,

the density of T is given by

$$f_T(t) = \frac{t^{-3/2}(t + \beta)}{2\alpha\beta^{1/2} \int_0^\infty z^{-1/2} \frac{\exp(-z)}{1 + \exp(-z)} dz} \frac{\exp\left[-\frac{1}{\alpha^2} \left[\frac{t}{\beta} + \frac{\beta}{t} - 2 \right]\right]}{\left[1 + \exp\left[-\frac{1}{\alpha^2} \left(\frac{t}{\beta} + \frac{\beta}{t} - 2 \right)\right]\right]^2}, \quad t > 0.$$

3 Some Properties of the Generalised Birnbaum-Saunders

Distribution

In this section, we discuss some properties of the generalised Birnbaum-Saunders distribution.

In the Normal case, clearly aT , with $a > 0$, this is also a Birnbaum-Saunders distribution, with $a\beta$ and α parameters. Moreover, the distribution of T^{-1} is the same as that of T , replacing β with β^{-1} , while the value of the α parameter does not change (see [4]). This property is invariant for all generalised Birnbaum-Saunders distributions, in the sense that $aT \sim GBS(\alpha, a\beta; g)$ and $T^{-1} \sim GBS(\alpha, \beta^{-1}; g)$, as shown in the following theorem.

Theorem 2 Let $T \sim GBS(\alpha, \beta; g)$. Then,

i) As $a > 0$, $Y = aT \sim GBS(\alpha, a\beta; g)$ and

ii) $Y = T^{-1} \sim GBS(\alpha, \beta^{-1}; g)$.

Proof.

i) Let $a > 0$ and $Y = aT$. Then $T = Y/a$, and so $dT = (1/a)dY$. Therefore, the Jacobian is $|J| = 1/a$. Then, and given that

$$f_T(t) = \frac{c}{2\alpha\beta^{1/2}} t^{-3/2}(t + \beta) g\left(\frac{1}{\alpha^2} \left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right),$$

we find that

$$\begin{aligned} f_Y(y) &= f_T(y/a) |J| \\ &= \frac{c}{2\alpha\beta^{1/2}} \left(\frac{y}{a}\right)^{-3/2} \left(\frac{y}{a} + \beta\right) g\left(\frac{1}{\alpha^2} \left[\frac{y/a}{\beta} + \frac{\beta}{y/a} - 2\right]\right) \frac{1}{a} \\ &= \frac{c}{2\alpha(a\beta)^{1/2}} y^{-3/2} (y + a\beta) g\left(\frac{1}{\alpha^2} \left[\frac{y}{a\beta} + \frac{a\beta}{y} - 2\right]\right) \end{aligned}$$

■

ii) Now let $Y = T^{-1}$. Then $T = Y^{-1}$, and so $dT = -Y^{-2}dY$. Thus, the Jacobian is $|J| = Y^{-2}$, and hence

$$\begin{aligned} f_Y(y) &= f_T(y^{-1}) |J| \\ &= \frac{c}{2\alpha\beta^{1/2}} (y^{-1})^{-3/2} (y^{-1} + \beta) g\left(\frac{1}{\alpha^2} \left[\frac{y^{-1}}{\beta} + \frac{\beta}{y^{-1}} - 2\right]\right) y^{-2}. \end{aligned}$$

Note that

$$\frac{1}{\alpha^2} \left[\frac{y^{-1}}{\beta} + \frac{\beta}{y^{-1}} - 2\right] = \frac{1}{\alpha^2} \left[\frac{\beta^{-1}}{y} + \frac{y}{\beta^{-1}} - 2\right]$$

and that

$$\frac{y^{-3/2}}{\beta^{1/2}} (y^{-1} + \beta)y^{-2} = \frac{y^{-1/2}}{\beta^{1/2}} \left(\frac{1}{y} + \beta\right) = \beta^{1/2} y^{-3/2} \left(\frac{1}{\beta} + y\right).$$

Thus, finally,

$$f_Y(y) = \frac{c}{2\alpha(\beta^{-1})^{1/2}} y^{-3/2} (y + \beta^{-1}) g\left(\frac{1}{\alpha^2} \left[\frac{y}{\beta^{-1}} + \frac{\beta^{-1}}{y} - 2\right]\right).$$

■

Theorem 3 *The generalised Birnbaum-Saunders distribution possesses moments if and only if the corresponding Elliptic distribution that generates it possesses moments.*

Proof.

$$\mathbb{E} \left(\begin{bmatrix} T \\ \beta \end{bmatrix}^r \right) = \mathbb{E} \left(\left[\frac{\alpha}{2} U + \sqrt{\left(\frac{\alpha}{2} U\right)^2 + 1} \right]^{2r} \right).$$

From the Binomial Theorem, $(a + b)^m = \sum_{k=0}^m \binom{m}{k} a^{m-k} b^k$, we obtain

$$\mathbb{E} \left(\left[\frac{T}{\beta} \right]^r \right) = \sum_{k=0}^{2r} \binom{2r}{k} \mathbb{E} \left(\left[\left(\frac{\alpha}{2} U \right)^2 + 1 \right]^{k/2} \left[\frac{\alpha}{2} U \right]^{2r-k} \right).$$

Now, note that if s is odd $\mathbb{E} \left(\left[\left(\frac{\alpha}{2} U \right)^2 + 1 \right]^t \left[\frac{\alpha}{2} U \right]^s \right) = 0$. Then,

$$\mathbb{E} \left(\left[\frac{T}{\beta} \right]^r \right) = \sum_{k=0}^r \binom{2r}{2k} \mathbb{E} \left(\left[\left(\frac{\alpha}{2} U \right)^2 + 1 \right]^k \left[\frac{\alpha}{2} U \right]^{2(r-k)} \right).$$

By expanding the binomial $(\cdot)^k$, we finally obtain that

$$\begin{aligned} \mathbb{E} \left(\left[\frac{T}{\beta} \right]^r \right) &= \sum_{k=0}^r \binom{2r}{2k} \mathbb{E} \left(\sum_{s=0}^k \binom{k}{s} \left[\left(\frac{\alpha}{2} U \right)^2 + 1 \right]^s \left(\frac{\alpha}{2} \right)^{2(r-k)} [U]^{2(r-k)} \right) \\ &= \sum_{k=0}^r \binom{2r}{2k} \sum_{s=0}^k \binom{k}{s} \mathbb{E} \left[(U)^{2(r+s-k)} \right] \left(\frac{\alpha}{2} \right)^{2(r+s-k)}. \end{aligned}$$

Therefore, $\mathbb{E} \left(\left[\frac{T}{\beta} \right]^r \right)$ exists if and only if $\mathbb{E} \left[(U)^{2(r+s-k)} \right]$ exists. ■

Corollary 8 *If $T \sim GBS(\alpha, \beta; g)$ and this has moments, then*

$$\mathbb{E} \left(\left[\frac{T}{\beta} \right]^r \right) = \sum_{k=0}^r \binom{2r}{2k} \frac{(2(r+s-k))!}{(r+s-k)!} \phi^{(r+s-k)}(0) \left(\frac{\alpha}{2} \right)^{2(r+s-k)},$$

where ϕ is given in (2).

Proof. From (2) we have $\psi_U(t) = \phi(t^2)$. Thus,

$$\psi_U^{(m)}(0) = \begin{cases} \frac{m!}{(m/2)!} \phi^{(m/2)}(0); & \text{if } m \text{ is even,} \\ 0; & \text{if } m \text{ is odd,} \end{cases}$$

(see Theorem 3.2.1, [14, pp.91-92]). Then, if the moments exist,

$$\mathbb{E}(U^m) = \begin{cases} \frac{1}{i^m} \psi_U^{(m)}(0); & \forall m, \\ \psi_U^{(m)}(0); & \text{if } m \text{ is even.} \end{cases}$$

And so, finally,

$$\mathbb{E} \left(\left[\frac{T}{\beta} \right]^r \right) = \sum_{k=0}^r \binom{2r}{2k} \frac{(2(r+s-k))!}{(r+s-k)!} \phi^{(r+s-k)}(0) \left(\frac{\alpha}{2} \right)^{2(r+s-k)}.$$

■

Corollary 9 *If $T \sim GBS(\alpha, \beta; \phi)$ and this has moments, then*

$$i) \mathbb{E}(T) = \beta(1 + \phi^{(1)}(0)\alpha^2) y$$

$$ii) \text{Var}(T) = \beta^2[(6\phi^{(2)}(0)\alpha^4 + 4\phi^{(1)}(0)\alpha^2 + 1) - (1 + \phi^{(1)}(0)\alpha^2)^2].$$

Proof.

i) Here $r = 1$. Therefore

$$\begin{aligned} \mathbb{E}\left(\frac{T}{\beta}\right) &= \sum_{k=0}^1 \binom{2}{2k} \frac{(2(s-k+1))!}{(s-k+1)!} \phi^{(s-k+1)}(0) \left(\frac{\alpha}{2}\right)^{2(s-k+1)} \\ &= \frac{\phi^{(1)}(0)\alpha^2}{2} + \left(1 + \frac{\phi^{(1)}(0)\alpha^2}{2}\right) \\ &= 1 + \phi^{(1)}(0)\alpha^2. \end{aligned}$$

And then, $\mathbb{E}(T) = \beta(1 + \phi^{(1)}(0)\alpha^2)$.

ii) As $\text{Var}(T) = \mathbb{E}(T^2) - (\mathbb{E}(T))^2$, we have

$$\begin{aligned} \mathbb{E}\left(\left[\frac{T}{\beta}\right]^2\right) &= \sum_{k=0}^2 \binom{4}{2k} \sum_{s=0}^k \binom{k}{s} \frac{(2(s-k+2))!}{(s-k+2)!} \\ &\quad \phi^{(s-k+2)}(0) \left(\frac{\alpha}{2}\right)^{2(s-k+2)} \\ &= \frac{3\phi^{(2)}(0)\alpha^4}{4} + 6 \left(\frac{\phi^{(1)}(0)\alpha^2}{2} + \frac{3\phi^{(2)}(0)\alpha^4}{4} \right) + \\ &\quad \left(1 + \phi^{(1)}(0)\alpha^2 + \frac{3\phi^{(2)}(0)\alpha^4}{4} \right) \\ &= 6\phi^{(2)}(0)\alpha^4 + 4\phi^{(1)}(0)\alpha^2 + 1. \end{aligned}$$

Then, $\mathbb{E}(T^2) = \beta^2(6\phi^{(2)}(0)\alpha^4 + 4\phi^{(1)}(0)\alpha^2 + 1)$.

Finally,

$$\begin{aligned} \text{Var}(T) &= \beta^2(6\phi^{(2)}(0)\alpha^4 + 4\phi^{(1)}(0)\alpha^2 + 1) - \beta^2(1 + \phi^{(1)}(0)\alpha^2)^2 \\ &= \beta^2[(6\phi^{(2)}(0)\alpha^4 + 4\phi^{(1)}(0)\alpha^2 + 1) - (1 + \phi^{(1)}(0)\alpha^2)^2]. \end{aligned}$$

■

4 Conclusions

In this article, we have discussed a generalisation of the Birnbaum-Saunders life distribution, from the basis of an Elliptic distribution, thus obtaining its density and some important properties. We have thus obtained a new family of life distributions in different and wider-ranging situations, such as life distributions that do not have moments, for example, when the generalised Birnbaum-Saunders distribution is obtained from the Cauchy and Special Case distributions. Figures 1-8 show the following, for the specified parameters: life distributions with left tails that are more weighted or less

weighted, when these are generated on the basis of Pearson Type VII, t , Cauchy, Kotz Type, Bessel and Laplace distributions. Life distributions with a more accelerated initial growth are obtained from Pearson Type VII, t and Kotz Type distributions. Life distributions with bimodality are obtained from Cauchy, Special Case and Kotz Type distributions. The latter may seem somewhat unusual among life data, but this situation is found, for example in a marine aquarium, in how the aquarium evolves and matures. With these results, we generate a family of life distributions with a large coverage of life situations.

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A Density Graphs of a Generalised Birnbaum-Saunders Distribution

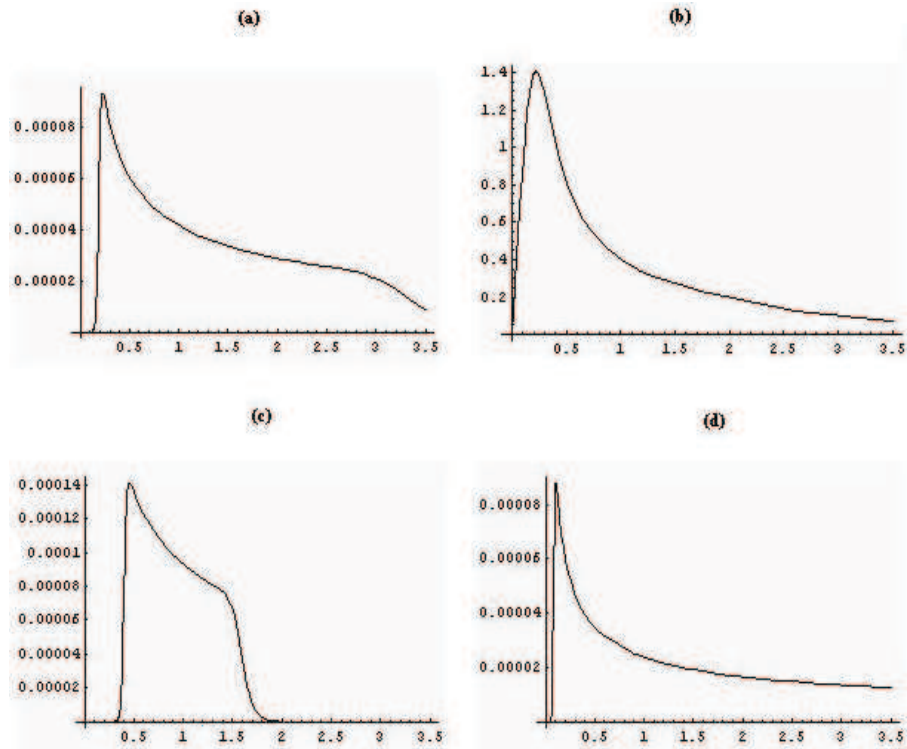


Figure 1: Density graphs of a generalised Birnbaum-Saunders life distribution for $\alpha = 0.5$ and $\beta = 0.8$, obtained from a Pearson Type VII distribution with: (a) $m = 10$ and $s = 10$, (b) $m = 2$ and $s = 10$, (c) $m = 10$ and $s = 2$, and (d) $m = 10$ and $s = 30$.

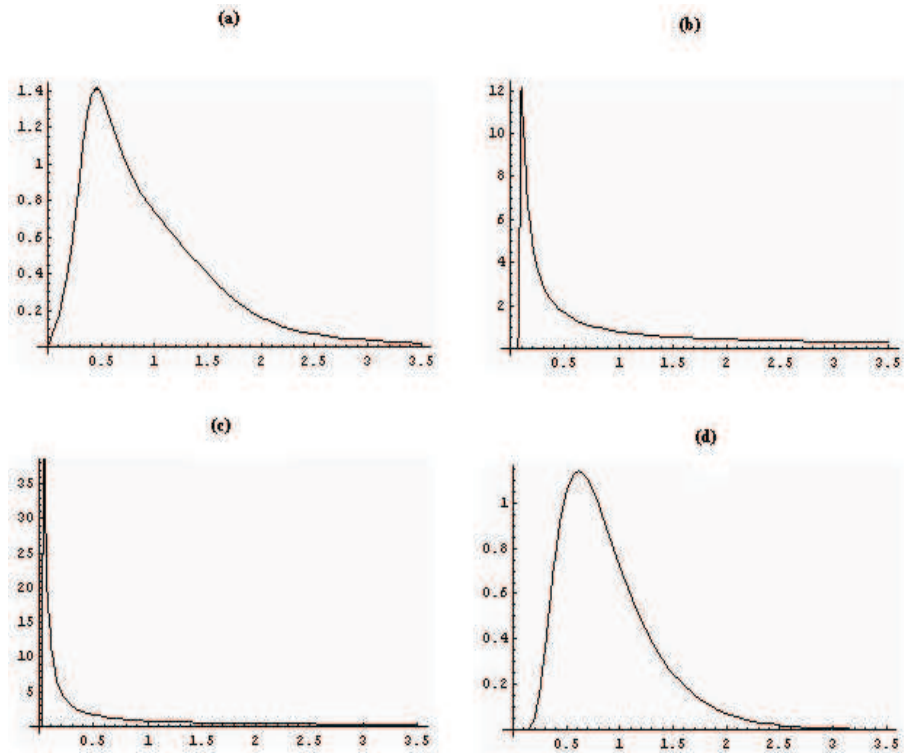


Figure 2: Density graphs of a generalised Birnbaum-Saunders life distribution for $\alpha = 0.5$ and $\beta = 0.8$, obtained from a distribution: (a) t with $n = 3$ degrees of freedom, (b) t with $n = 30$ degrees of freedom, (c) t with $n = 10$ degrees of freedom, and (d) Normal.

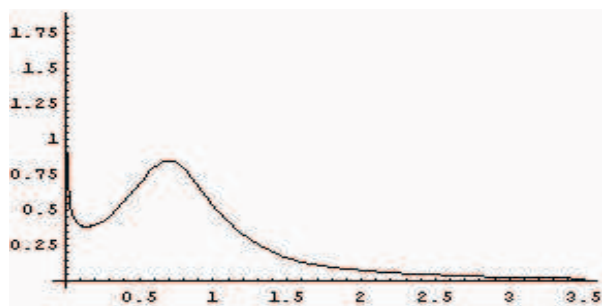


Figure 3: Density graphs of a generalised Birnbaum-Saunders life distribution for $\alpha = 0.5$ and $\beta = 0.8$, obtained from a Cauchy distribution.

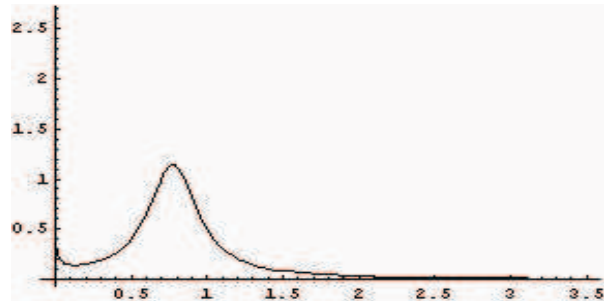


Figure 4: Density graphs of a generalised Birnbaum-Saunders life distribution for $\alpha = 0.5$ and $\beta = 0.8$, obtained from a Special Case distribution.

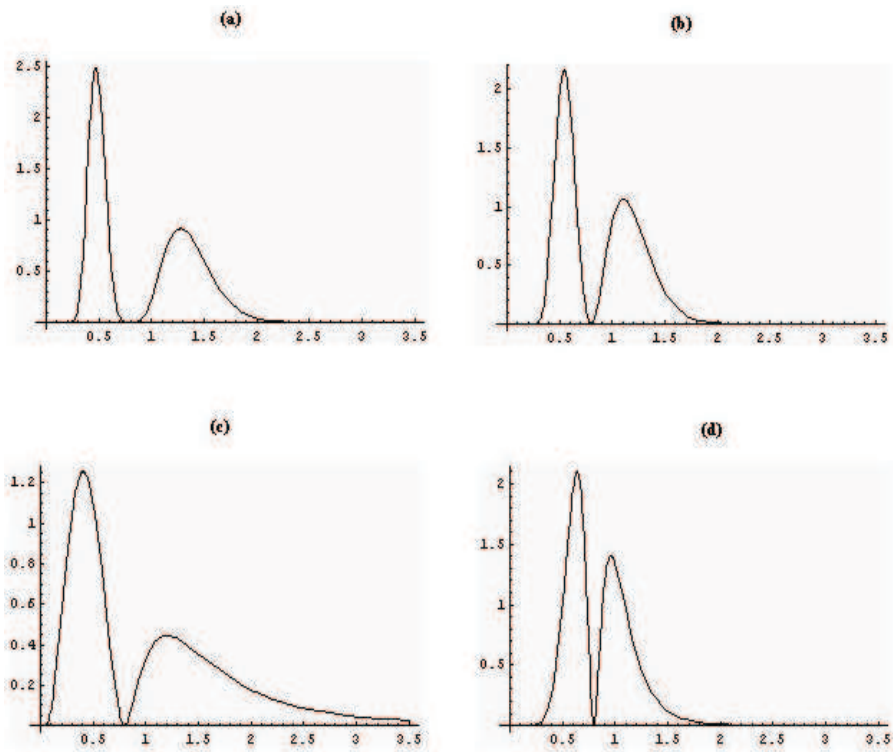


Figure 5: Density graphs of a generalised Birnbaum-Saunders life distribution for $\alpha = 0.5$ and $\beta = 0.8$, obtained from a Kotz Type distribution with: (a) $r = 2$ and $s = 1$ and $q = 3$, (b) $r = 2$ and $s = 1$ and $q = 2$, (c) $r = 2$ and $s = 1/2$ and $q = 2$, and (d) $r = 5$ and $s = 1/2$ and $q = 2$.

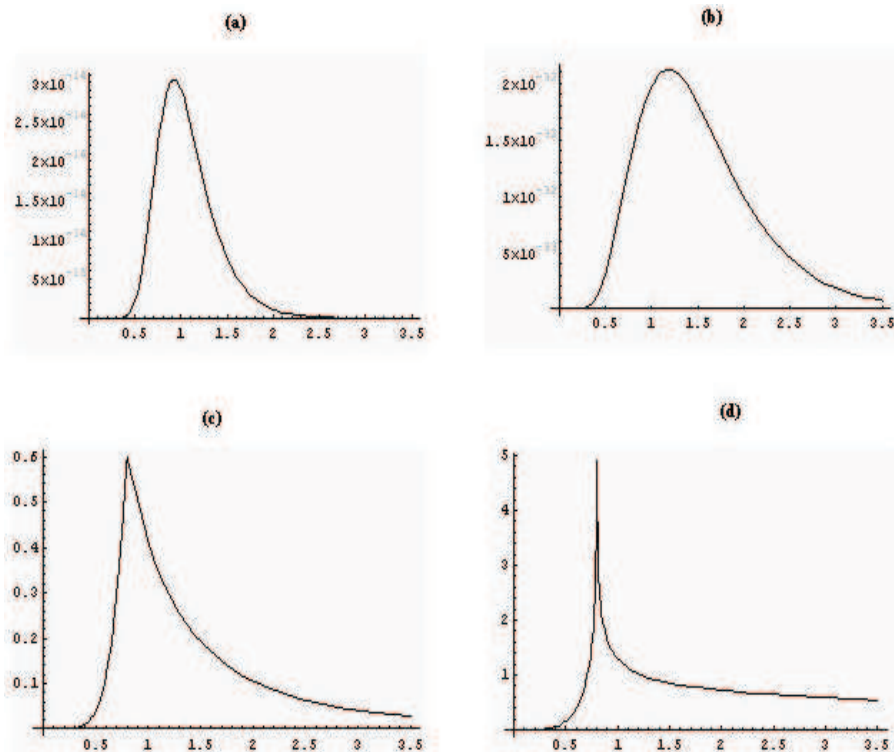


Figure 6: Density graphs of a generalised Birnbaum-Saunders life distribution for $\alpha = 0.5$ and $\beta = 0.8$, obtained from a Bessel distribution with: (a) $r = 1/4$ and $q = 10$, (b) $r = 1/4$ and $q = 25$, (c) $r = 1$ and $q = 1/2$, and (d) $r = 2$ and $q = 0$.

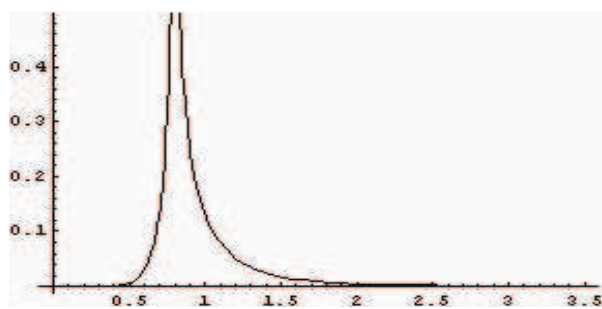


Figure 7: Density graphs of a generalised Birnbaum-Saunders life distribution for $\alpha = 0.5$ and $\beta = 0.8$, obtained from a Laplace distribution.

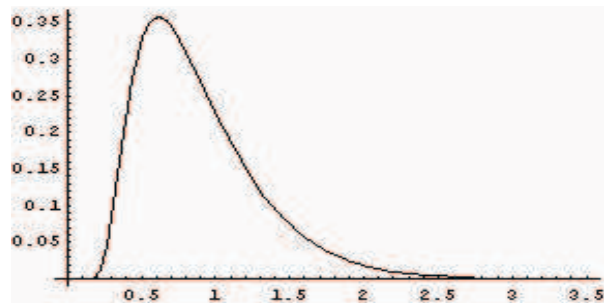


Figure 8: Density graphs of a generalised Birnbaum-Saunders life distribution for $\alpha = 0.5$ and $\beta = 0.8$, obtained from a Logistic distribution.